Debreu's Theorem

Debreu's theorem, which states that continuous preferences have a *continuous* utility representation, is one of the classical results in economic theory. For a proof of the theorem, in a more general setting, see Debreu (1954, 1960).

In what follows, we will need the mathematical concept of a dense set. A set Y is said to be dense in X if every non-empty open set $B \subset X$ contains an element in Y. Any set $X \subseteq \mathbb{R}^m$ has a *countable dense subset*. (The standard topology in $\mathbb{R}^n$ has a countable base, that is, any open set is the union of subsets of the countable collection of open sets: $\{Ball(a, 1/m)|\ a \in \mathbb{R}^m$ and all its componenets are rational numbers; m is a natural number$\}$. For every set $Ball(q, 1/m)$ that intersects X, pick a point $y_{q,m} \in X \cap Ball(q, 1/m)$. The set that contains all of the points $\{y_{q,m}\}$ is a countable dense set in X.)

Proposition (Debreu):

Let $\succsim$ be a continuous preference relation on X, which is a convex subset of $\mathbb{R}^n$. Then $\succsim$ has a continuous utility representation.

Proof:

(fn: Oren Danieli and Luke Levy-Moore assisted me in formulating the current version of the proof.)

For the case in which the relation is the total indifference $\succsim$, the proof is trivial. From here on, assume that $\succsim$ is not the total indifference.

Lemma 1:

If $x \succ y$, then there exists z in X such that $x \succ z \succ y$.

Proof:

Assume not. Let I be the interval between x and y. By the convexity of X, $I \subseteq X$. Construct inductively two sequences of points in I, $\{x_t\}$ and $\{y_t\}$, in the following manner: First, define $x_0 = x$ and $y_0 = y$. Assume that the two points x_t and y_t are defined, belong to I, and satisfy $x_t \succsim x$ and $y \succsim y_t$. Consider m, the middle point between x_t and y_t. Either $m \succsim x$ or $y \succsim m$. In the former case, define $x_{t+1} = m$ and $y_{t+1} = y_t$, and in the latter case define $x_{t+1} = x_t$ and $y_{t+1} = m$. The sequences $\{x_t\}$ and

$\{y_t\}$ are converging, and they must converge to the same point z because the distance between x_t and y_t converges to zero. By the continuity of $\succsim$, we have $z \succsim x$ and $y \succsim z$ and thus, by transitivity, $y \succsim x$, which contradicts the assumption that $x \succ y$.

Another simple proof would fit the more general case, in which the assumption that the set X is convex is replaced by the weaker assumption that X is a connected subset of $\mathbb{R}^n$: If there is no z such that $x \succ z \succ y$, then X is the union of two disjoint sets $\{a|a \succ y\}$ and $\{a|x \succ a\}$, which are open by the continuity of the preference relation. This contradicts the connectedness of X (a connected set cannot be covered by two nonempty disjoint open sets).

Lemma 2:

Let Y be dense in X. Then, for every $x, y \in X$, if $x \succ y$ there exists $z \in Y$ such that $x \succ z \succ y$.

Proof:

By Lemma 1, there exists $z \in X$ such that $x \succ z \succ y$. By continuity, there is a ball around z such that any point in the ball is sandwiched between x and y and, by the denseness of Y, the ball contains an element of Y.

Lemma 3:

Let E be the set of $\succsim$ -maxima and $\succsim$ -minima in X. Let Y be a countable dense set in $X - E$. Then, $\succsim$ has a utility representation on Y, u with a range that consists of all dyadic rational numbers in $(0,1)$ (namely all numbers that can be expressed as $k/2^l$ where k and l are natural numbers and $k < 2^l$).

Proof:

By Lemma 1, $X - E$ is an infinite set and therefore Y is as well. Let $Y = \{y_n\}$. Construct u by induction as follows: Start with $u(y_1) = 1/2$. Let $P(y_n) = \{y_1, .., y_{n-1}\}$, i.e., the set of elements that precedes y_n in the enumeration of Y. If $y_n \sim y_m$ for some $y_m \in P(y_n)$, let $u(y_n) = u(y_m)$. If $y_n \succ y_k$ where y_k is maximal in $P(y_n)$, set $u(y_n) = (1 + u(y_k))/2$. If $y_k \succ y_n$ where y_k is minimal in $P(y_n)$, set $u(y_n) = u(y_k)/2$. Otherwise, there are $y_i, y_j \in P(y_n)$ such that y_i is minimal among the elements in $P(y_n)$

that are preferred to y_n and y_j is maximal among the elements in $P(y_n)$ that are inferior to y_n. Let $u(y_n) = (u(y_i) + u(y_j))/2$. Note that by Lemma 2, for every element in the sequence there will always eventually be one element in the sequence that is above it and one that is below it and for every two elements in the sequence there will eventually be an element in the sequence that is sandwiched between the two. Therefore, the range of u is exactly all dyadic numbers in $(0, 1)$.

Completing the Proof:

Let Y be a countable dense set in $X - E$. Define u on Y according to Lemma 3. The function u can be extended to X by: (i) assigning the value 1 to all maxima points in X and the value 0 to all minima points and (ii) defining $u(x) = sup\{u(y) \mid x \succ y \ and \ y \in Y\}$ for all $x \notin Y \cup E$. This function represents the preference relation since by definition if $x \sim z$ we have $u(x) = u(z)$ and if $x \succ z$ then by Lemma 2 there are y_1 and y_2 in Y such that $x \succ y_1 \succ y_2 \succ z$ and thus $u(x) \geq u(y_1) > u(y_2) \geq u(z)$.

In order to prove the continuity of u, consider a point $x \notin E$ (a similar proof applies to extreme points). Let $\varepsilon > 0$. By Lemma 3, there are y_1 and y_2 in Y such that $u(x) - \varepsilon < u(y_1) < u(x) < u(y_2) < u(x) + \varepsilon$. By twice applying the definition of the continuity of $\succsim$, we obtain a ball B around x that is between y_1 and y_2 with respect to the preference relation. By definition, elements in this ball receive u values between $u(y_1)$ and $u(y_2)$ and thus are not further than ε from $u(x)$.