Microeconomics I Midterm (Fall 2013) Solutions
Ariel Rubinstein
October 17, 2013

Question 1:

Consider the following procedure which yields a choice function C over subsets of a finite set X:

The decision maker has in mind a set $\{\succ_i\}_{i=1,..n}$ of orderings over X and a set of weights $\{\alpha_i\}_{i=1,..n}$. Facing a choice set $A \subseteq X$, the decision maker counts for each alternative $x \in A$ a score: the sum of the weights of those orderings which rank x first from among the members of A and then chooses the alternative which gets the highest score.

To Warm Up:

a) Explain why a rational choice function is consistent with this procedure.

b) Give an example to show that the procedure can produce a choice function which is not rationalizeable.

And now to the main question:

c) Show that for $|X| = 3$ all choice functions are consistent with the procedure.

d) Explain why it is not generally true that a choice function C which is derived from this procedure satisfies the condition that if $x = C(A) = C(B)$ then $x = C(A \cup B)$.

e) (Take Home) Can you find any non-trivial property which is satisfied by choice functions which are derived by this procedure and not by the other? Is there any choice function which cannot be explained by this procedure?

Solution:

a) Suppose that the decision maker is rational i.e. he chooses the $\succsim$ maximal element for some preference relation $\succsim$. Then set $\alpha_1 = 1, \alpha_i = 0\ \forall i > 1$ and take $\succ_1 = \succsim$.

b) Consider the following situation: $X = \{a, b, c\}$ and

$$\begin{array}{ll} a \succ_1 b \succ_1 c & \alpha_1 = 5 \\ b \succ_2 c \succ_2 a & \alpha_2 = 4 \\ c \succ_3 b \succ_3 a & \alpha_3 = 3 \end{array}$$

Then $C(X) = a, C(\{a, b\}) = b \in X$, so C violates condition α.

c) Let $X = \{a, b, c\}$. If the choice function is rationalizeable, then we are done by part a). Suppose that C is not rationalizeable. Then condition α is violated so WLOG we can assume $C(X) = a, C(\{a, b\}) = b$. Define $R(\{x, y\})$ to be the element which is rejected from $\{x, y\}$ and define the orderings as

$$\begin{aligned} \alpha_1 &= 5 \quad a \succ_1 C(\{b, c\}) \succ_1 R(\{b, c\}) \\ \alpha_2 &= 4 \quad b \succ_2 C(\{a, c\}) \succ_2 R(\{a, c\}) \\ \alpha_3 &= 3 \quad c \succ_3 b \succ_3 a \end{aligned}$$

d) Our example in part b) violates this condition since $C(\{b, c\}) = b$ so $b = C(\{a, b\}) = C(\{b, c\})$, but $b \neq C(\{a, b, c\})$. The issue here is that while b is preferred to a and to c in two preference rankings each (whose sum of weights is greater than 5), it is only the maximal element of one ranking whose weight is not the highest. Condition α is not satisfied here since adding new elements can make other elements have a higher relative score.

e) One cannot find such a non-trivial property since any choice function can be explained by this procedure. We can prove by induction on $|X| = m$. The statement holds for $m = 2$ and by part c), it holds for $m = 3$. For the inductive step, suppose that this holds for all $m < M$. Let $|X| = M$. For each $x_i \in X$, create some vector of rankings $\{\succ_j^i\}_{j=1,\ldots,n_i}$ and a vector of weights $\{\alpha_j^i\}_{j=1,..,n_i}$ such that

- x_i is the top element in each ranking $\succ_j^i$
- These rankings and weights can explain our choice function on $X \backslash \{x_i\}$.
- $\sum_j \alpha_j^i = 1$.

This is possible by the inductive hypothesis and a renormalization of the weights.

Let Δ be the minimum score difference on the rankings using $\{\alpha_j^i\}_{j=1,..,n_i}$ and let $\epsilon < \frac{\Delta}{M+1}$ (we want ϵ small enough such that it is a tie-breaker on the the procedure over the X but will not affect the final outcome of the procedure on $A \subset X$). WLOG let $C(X) = x_1$ and distribute ϵ equally across α_j^1.

Let $A \subseteq X$ and $C(A) = a$. If $A = X$, then by our choice of weights, the procedure yields a as the result.

Suppose $A \neq X$ and let $b \in A, b \neq a$. The for the rankings with $y \notin A$ as the top element, we will have $score(b) < score(a)$ since $A \subseteq X \backslash \{y\}$. For rankings with $y \in A, y \neq a, b$, the $score(a) = score(b) = 0$. For rankings with a, b as the top ranking, the maximum possible difference $score(b) - score(a)$ is ϵ. By our choice of ϵ, this will be less than the $score(a) - score(b)$ on the rankings with $y \notin A$ as the top element.

Question 2:

Imagine a consumer who operates in two stages when he faces a budget set $B(p, w)$ in a world with the commodities 1,...,K split into two exclusive non-empty groups A and B :

Stage 1: He allocates w to the two groups by maximizing a function v on the set of pairs (w_A, w_B).

Stage 2: He chooses an A-bundle maximizing a function u_A defined over the A-bundles given w_A, and separately he chooses a B-bundle maximizing a function u_B defined over the B-bundles given w_B.

a) Show that if the consumer is interested to choose at the end a bundle (over the K commodities) which maximizes the (ridiculous) utility function $\Pi_{k=1,..,K} x_k^{\alpha_k}$ (where $\alpha_k > 0\ \forall k$ and $\sum_{k=1}^{K} \alpha_k = 1$) then he can attain his goal by following the procedure above with some functions (v, u_A, u_B).

b) Show that the claim in (a) is not true in general. For example, you might (but don't have to) look at the case $K = 4$, $A = \{1, 2\}$, $B = \{3, 4\}$ and the utility function $max\{x_1x_3, x_2x_4\}$. (Note that this is the max, not min function)

c) (Take Home) Show that if the consumer follows the above procedure, then it might be that his overall choice cannot be rationalized (For the first stage, you can choose a simple function like $v = min\{w_A, w_B\}$.

Solution

a. Let $A = \{1, ..., L\}, B = \{L+1, .., K\}$. This Cobb-Douglas utility function yields the demand function $x_k(p, w) = \frac{w\alpha_k}{p_k}$. Consider the following procedure:

- For stage 1: Any function whose maximization sets $w_A = w \sum_{i=1}^{L} \alpha_i$ and $w_B = w \sum_{j=L+1}^{K} \alpha_j$ will work. For example, take

$$v(w_A, w_B) = min\{w_A \sum_{j=L+1}^{K} \alpha_j, w_B \sum_{i=1}^{L} \alpha_i\} \quad s.t. w_A + w_B \leq w$$

- For stage 2: Choose

$$u_A(p, w_A) = \prod_{i=1}^{n} x_i^{\alpha_i} \Rightarrow x_{A,i}(p, w_A) = \frac{w_A \alpha_i}{p_i \sum_{i=1}^{n} \alpha_j}$$

$$u_B(p, w_B) = \prod_{j=n+1}^{K} x_j^{\alpha_j} \Rightarrow x_{B,j}(p, w_A) = \frac{w_B \alpha_j}{p_j \sum_{j=n+1}^{K} \alpha_j}$$

Since the first maximization yields $w_A = w\sum_{i=1}^{n}\alpha_i$ and $w_B = w\sum_{j=n+1}^{K}\alpha_j$, we get

$$x_{A,i}(p,w_A) = \frac{w\alpha_i}{p_i\alpha}$$
$$x_{B,j}(p,w_A) = \frac{w\alpha_j}{p_j\alpha}$$

which yields the same demand as our original problem.

b. Looking at $U(x) = max\{x_1x_3, x_2x_4\}$, the consumer will purchase only good $1,3$ if $p_2p_4 > p_1p_3$ and will purchase only good $2,4$ if $p_1p_3 > p_2p_4$.

Assume there are utility functions v, u_A, u_B such that the outcome of the procedure coincides with the maximization of U. Let x_A, x_B be demand functions of u_A, u_B respectively.

If $p = (1,1,2,1)$, then since $p_1p_3 > p_2p_4$ it must be that $x_B((2,1),w_B)$ contains only good four.

Now let $p' = (1,3,2,1)$. Then since $p_1p_3 < p_2p_4$, it must be that $x_B((2,1),w_B)$ contains only good three. But this cannot be since the optimization problem over B-bundles has not changed, so x_B will not change.

c. (More Difficult) Let $v(w_A, w_B) = min\{w_A, w_B\}$ (so that maximization of v leads to $w_A = w_B = \frac{w}{2}$. Consider the following two utility functions for $A = \{1,2\}, B = \{3,4\}$:

$$u_A(p,w) = \begin{cases} x_1 & if\ x_1 < 1 \\ 1 + x_2 & if\ x_1 \geq 1 \end{cases}$$
$$u_B(p,w) = \begin{cases} x_3 & if\ x_3 < 1 \\ 1 + x_3 & if\ x_3 \geq 1 \end{cases}$$

Let $(p,w) = ((1.1, 0, 1.1, 0), 2)$ and $(p', w') = ((1,1,1,1), 3)$. Then he will get the demand functions

$$x(p,w) = (x_A(p, \frac{w}{2}), x_B(p, \frac{w}{2})) = (\frac{10}{11}, 0, \frac{10}{11}, 0)$$
$$x(p'w') = (x_A(p', \frac{w'}{2}), x_B(p', \frac{w'}{2})) = (1, \frac{1}{2}, 1, \frac{1}{2})$$

This violates the weak axiom since $x(p',w')$ is feasible in $B(p,w)$ and $x(p,w)$ is feasible in $B(p',w')$ as

$$p \cdot x(p', w') = \frac{10}{11} + 0 + \frac{10}{11} + 0 = \frac{20}{11} < 2 = w$$
$$p' \cdot x(p,w) = \frac{10}{11} + 0 + \frac{10}{11} + 0 = \frac{20}{11} < 3 = w'$$

Question 3:

An agent makes a binary comparisons of pairs of numbers. His real interest is to maximize the sum $x_1 + x_2$. When he compares (x_1, x_2) and (y_1, y_2) he makes always the right comparison if one of the pairs dominates the other. When this is not the case he might make a mistake. The technology of mistake is characterized by a function $\alpha(G, L)$ with the interpretation that if the gain in one dimension is $G \geq 0$ and the loss in the other dimension is $L \geq 0$, then the probability of mistake is $\alpha(G, L)$.

a) Suggest reasonable and workable assumptions for the function α (such as $\alpha(G, L) \leq 1/2$ for all G and L).

b) Suggest a notion expressing the phrase "agent 1 is more accurate than agent 2".

c) Show that with the notion you defined in b) the probability that three binary comparisons on the triple $(7, 2)$, $(3, 10)$, $(0, 6)$ yields a cycle is smaller for the agent who is more accurate.

d) Show that the probability the binary comparisons will yield cycle on a general triple of pairs is not necessarily smaller for the agent who is more accurate.

e) (Take Home): Show that if agent 1 is characterized by a mistake function $\alpha(G, L)$ and agent 2 is characterized by the mistake function $\lambda\alpha(G, L)$ for $0 < \lambda < 1$, then for a triple of pairs, the probability that the agent 2's answers exhibit a cycle is less than the probability that agent 1's answers exhibit a cycle.

Solution:

a) Some possible assumptions on the function:

- 1) $\alpha(G, L) \leq \frac{1}{2}$
- 2) $\alpha(G, 0) = \alpha(0, L) = 0$
- 3) $\alpha(G, L) = 0$ if $G = L$ (this expresses the fact that there can be no mistake here).
- 4) $\alpha(G, L)$ is decreasing in G and decreasing in L (i.e. as the gain gets larger compared to the loss, the agent is more likely to recognize the better item)
- 5) $\alpha(G, L) \leq \alpha(G', L')$ if $|G - L| \geq |G' - L'|$; the agent is less likely to make a mistake when the difference between the optimal pair and suboptimal pair is larger.

- 6) $\alpha(G,L) \leq \alpha(G',L')$ when $|G-L| = |G'-L'|$ and $G' \geq G, L' \geq L$ so that the agent is more likely to make a mistake when he is dealing with larger numbers.

b) We can say that agent 1 makes less mistakes than agent 2 if for any G, L, we have $\alpha_1(G,L) \leq \alpha_2(G,L)$ and for some (G,L), the inequality is strict.

c) First, note that $(3,10)$ strictly dominates $(0,6)$ so each agent will always make the correct choice $(3,10) \succ (0,6)$.

The only cycle we can obtain is

$$(7,2) \succ (3,10) \succ (0,6) \succ (7,2)$$

The probability of this cycle is $\alpha_i(4,8) \cdot \alpha_i(7,4)$. If $\alpha_1(G,L) \leq \alpha_2(G,L)$, then the probability of cycle for $1 = \alpha_1(4,8) \cdot \alpha_1(7,4) \leq \alpha_2(4,8) \cdot \alpha_2(7,4) =$ probability of cycle for 2.

d) Consider the tripe $(7,4), (1,7), (0,6)$. Since $(1,7)$ dominates $(0,6)$, each agent will always choose $(1,7) \succ (0,6)$ correctly. Thus the only possible cycle is

$$(7,4) \succ (1,7) \succ (0,6) \succ (7,4)$$

The probability of this cycle is $(1-\alpha_i(6,3))\alpha_i(7,2)$. We can then set the probabilities of the two agents to be

Table 1: Mistake Techologies

Agent	$\alpha_i(6,3)$	$\alpha_i(7,2)$
$i=1$	$\frac{1}{4}$	$\frac{1}{4}$
$i=2$	$\frac{1}{2}$	$\frac{1}{4}$

Then the probability that agent 2's answers exhibit a cycle is $(1-\frac{1}{2})\frac{1}{4} = \frac{1}{8}$ and the probability that agent 1's answers exhibit a cycle is $(1-\frac{1}{4})\frac{1}{4} = \frac{3}{16}$.

e) If the agents answers exhibit a cycle $a \succ b \succ c \succ a$ then he must exhibit either one or two mistakes.

- Suppose the cycle requires only one mistake. Then the probability that agent 2 exhibits a mistake is $p(\lambda\alpha) = (1-\lambda\alpha_1)(1-\lambda\alpha_2)\lambda\alpha_3$. Taking derivatives, we get

$$\frac{\partial p(\lambda\alpha)}{\partial\lambda} = (1-\lambda\alpha_1-\lambda\alpha_1+\lambda^2\alpha_1\alpha_2)\lambda\alpha_3$$

  We can see that this derivative is positive since $\alpha_3 \geq 0$ and

$$1-\lambda\alpha_1-\lambda\alpha_2+\lambda^2\alpha_1\alpha_2 \geq \lambda^2\alpha_1\alpha_2 \geq 0$$

  Thus the probability of agent 2's answers displaying a cycle is smaller in this case since $p(\lambda\alpha)$ is decreasing for $\lambda \in [0,1]$.

- Now suppose the cycle requires two mistakes. Then the probability that agent 2 exhibits a mistake is $p(\lambda\alpha) = (1-\lambda\alpha_1)\lambda^2\alpha_2\alpha_3$. Taking derivatives, we get

  $$\frac{\partial p(\lambda\alpha)}{\partial\lambda} = (2 - 3\lambda^2\alpha_1)\lambda\alpha_2\alpha_3$$

  which is positive since $\lambda\alpha_2\alpha_3 \geq 0$ and

  $$\lambda\alpha_1 \leq \frac{1}{2} < \frac{2}{3} \Rightarrow 3\lambda\alpha_1 < 2$$

  So again the probability of agent 2's answers displaying a cycle is smaller in this case since $p(\lambda\alpha)$ is decreasing for $\lambda \in [0, 1]$.

Since this holds for arbitrary cycles on the triple, even if there is more than one way to get a cycle, agent 2 will always have a smaller probability of his answers displaying a cycle than agent 1.