Solution to Problem Set One - Preferences

Lecture Notes in Microeconomic Theory by Ariel Rubinstein

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1. *Let $\succsim$ be a preference relation on a set X. Define $I(x)$ to be the set of all $y \in X$ for which $y \sim x$. Show that the set (of sets!) $\{I(x)|x \in X\}$ is a partition of X, ie,*

   (a) $\forall x \in X$, $I(x) \neq \emptyset$.

   (b) $\forall x \in X$, $\exists y \in X$ *such that* $x \in I(y)$.

   (c) $\forall x, y \in X$, *either* $I(x) = I(y)$ *or* $I(x) \cap I(y) = \emptyset$.

   Proof of (a) and (b) Choose any $x \in X$. By the reflexivity of $\succsim$,[1] it follows $x \sim x \Rightarrow x \in I(x)$. $\square$

   Proof of (c) Choose any $x, y \in X$, and assume that $I(x) \cap I(y) \neq \emptyset$, ie $\exists z \in I(x) \cap I(y)$. Choose any $a \in I(x) \Rightarrow a \sim x$. Moreover, since $x \sim z$, the transitivity of $\sim$[2] implies that $a \sim z$.

   In addition, note that $z \in I(y) \Rightarrow z \sim y \Rightarrow a \sim y$, again by transitivity. Therefore, $I(x) \subseteq I(y)$, A symmetric argument can be used to show that $I(y) \subseteq I(x)$. Consequently, if $I(x) \cap I(y) \neq \emptyset$, then $I(x) = I(y)$. $\blacksquare$

2. *Kreps (1990) introduces another formal definition for preferences. His primitive is a binary relation P interpreted as "strictly preferred." He requires P to satisfy:*

   Asymmetry *For no x, y do we have both xPy and yPx.*

   Negative-Transitivity $\forall x, y, z \in X$, *if xPy, then either xPz or zPy (or both).*

   Explain the sense in which Kreps' formalization is equivalent to the traditional definition.

   Kreps' formalization and the traditional formalization are equivalent. This proof will closely follow the one presented on pages 6–8 of the lecture notes. The following steps are required to complete the proof:

[1] Reflexivity of $\succsim$ is implied by definition: By the completeness of $\succsim$, we know $x \succsim x \Rightarrow x \sim x$.

[2] Transitivity of $\sim$ follows directly from the transitivity of $\succsim$; try proving this as an exercise.

(a) Construct a candidate correspondence T that maps from the possible responses to $P \rightarrow$ possible responses to R *and* preserves the interpretation of the two formalizations.

(b) Verify T is well defined.

(c) Verify T maps to responses that satisfy the definition of preferences in the traditional sense.

(d) Verify T is one-to-one.

(e) Verify T maps onto all possible responses to R.

Let's go through steps (a)–(e):

(a) Consider the following candidate correspondence T, which maps left to right on the table:

A response to xPy and yPx	A response to $R(x, y)$ and $R(y, x)$
Yes, No	Yes, No
No, No	Yes, Yes
No, Yes	No, Yes

T preserves our interpretation - if "x is strictly preferred to y" according to Kreps' formalization, then T maps to "x is at least as good as y, but y is not at least as good as x" in the traditional sense, and so on.

(b) From Kreps' asymmetry property, note that xPy, yPx can never be "Yes, Yes." For every $x, y \in X$, therefore, the answer to xPy, yPx will be one of the three rows in the left-hand side of the table. Consequently, the responses to $R(x, y)$ and $R(y, x)$ are well defined.

(c) **Completeness:** Note that in each of the three rows, the answer to either $R(x, y)$ or $R(y, x)$ is "Yes." Therefore, T satisfies completeness.

Transitivity: Choose any $x, y, z \in X$ such that $R(x, y)$ and $R(y, z)$ are both "Yes." We need to show $R(x, z)$ is "Yes" as well.

By way of contradiction, assume not, ie the answer to $R(x, z)$ is "No," which implies zPx. By negative transitivity, it follows either zPy or yPx. Note, however, that $zPy \Rightarrow R(y, z)$ is "No" and $yPx \Rightarrow R(x, y)$ is "No," $\bigotimes$.

(d) Next, we must show T is one to one. Consider two different responses to P. Since the two responses are different, there exists an x, y such that xPy in one response but not in the other. Since the corresponding responses for $R(x, y)$ and $R(y, x)$ must differ according to our table, this implies that T maps the two different responses to $P \rightarrow$ two different responses to R. Therefore, T is one to one.

(e) To complete the proof, we now must check that T is onto, ie the range of T contains all possible responses to R. We will show that the function ϕ that maps from right to left on the table maps every response to $R \rightarrow$ a response to P.

By the completeness assumption of the traditional formalization, the response to $R(x,y)$, $R(y,x)$ cannot be "No, No." The table thus exhausts the possible responses to $R(x,y), R(y,x)$ which implies that the mapping ϕ is well defined.

Asymmetry: Note that in the three rows, the response to xPy, yPx is never "Yes, Yes."

Negative Transitiviey: Choose any $x, y, z \in X$ such that xPy. We must show that either xPz or zPy.

By the completeness of $\succsim$, one of two cases holds: (1) $x \succsim z$, not $z \succsim x$ or (2) $z \succsim x$.

Case 1: $x \succsim z$, not $z \succsim x$: Our correspondence ϕ directly implies that the answer to xPz is "Yes."

Case 2: $z \succsim x$: By contradiction, let's assume the answer to zPy is "No," which implies $y \succsim z$. By the transitivity property of the traditional formalization, it follows $y \succsim x$, which is a contradiction since our initial assumption was that xPy. Therefore, zPy.

We've thus shown that the range of T contains the set of all possible responses to R.

And we're done. ■

3. *Let Z be a finite set and let X be the set of all nonempty subsets of Z. Let $\succsim$ be a preference relation on X (not Z). Consider the following two properties of preference relations on X:*

   (a) *If $A \succsim B$ and C is disjoint to both A and B, then $A \cup C \succsim B \cup C$, and*
   if $A \succ B$ and C is disjoint to both A and B, then $A \cup C \succ B \cup C$.

   (b) *If $x \in Z$ and $\{x\} \succ \{y\}\ \forall y \in A$, then $A \cup \{x\} \succ A$, and*
   if $x \in Z$ and $\{y\} \succ \{x\}\ \forall y \in A$, then $A \succ A \cup \{x\}$.

   Discuss the plausibility of the properties in the context of interpreting $\succsim$ as the attitude of the individual toward sets from which he will have to make a choice at a "second stage."

   In this problem, an agent first chooses from possible "menus" (elements of the set of sets X) that restrict the agent's choice from Z in the second stage. If we assume that the agent has well defined preferences $\succsim^*$ over the items in Z, the second conjectures of both (a) and (b) seem implausible.

   Since Z has a finite number of elements, this implies that each menu $A \in X$ has a finite number of elements and consequently contains a $\succsim^*$-maximal element. A rational agent, therefore, will prefer menu A to menu B if and only if the $\succsim^*$-maximal element in A is preferred to the $\succsim^*$-maximal element in B.

Consider the following counterexample to the second conjecture in (a): Let A and B be two menus whereby the best element in A is better than the best element in B, and consider a set C, disjoint to both A and B, that contains the best element in Z. A rational agent will be indifferent to $A \cup C$, $B \cup C$ since both contain the best element in Z, which violates the second part of (a).

Next, consider the following counterexample to the second conjecture in (b): Let z be the worst element in Z, and let A be any menu such that $z \notin A$. It follows that a rational agent will be indifferent between A and $A \cup \{z\}$, which violates the second part of (b).

Provide an example of a preference relation that satisfies:

- *Both properties.* Consider a preference relation over X where $A \succsim B$ iff $|A| \geq |B|$, where $|A|$ is the cardinality of A (ie the number of elements in A).

  Proof of (a) First, note that if two sets A and C are disjoint, then $|A \cup C| = |A| + |C|$.
  Choose any $A, B, C \in X$ such that C is disjoint to both A and B. It readily follows that $A \succsim B \iff |A| \geq |B| \iff |A \cup C| \geq |B \cup C| \iff A \cup C \succsim B \cup C$, and analogously for the strict case.

  Proof of (b) This property is vacuously true, since the "if" condition of (b) never holds. To see this, note that $\forall x, y \in Z$, $|\{x\}| = 1 = |\{y\}| \Rightarrow \{x\} \sim \{y\}$. Thus (b) is trivially true.

- *The first but not the second property.* Let $z^* \in Z$ denote a particular element that the agent strictly prefers to all other elements in Z. Define a preference relation over X whereby

  $A \succ B \iff z^* \in A, z^* \notin B$; and
  $A \sim B$ if $z^* \in A, B$ or $z^* \notin A, B$.

  Proof of (a) Choose any $A, B \in X$ such that $A \succsim B$, and let C denote a menu that is disjoint to both A and B. One of the two cases holds:

  Case 1: $A \succ B$ In this case, A contains z^* while B does not. It follows that z^* is in $A \cup C$ and not in $B \cup C$, and thus $A \cup C \succ B \cup C$.

  Case 2: $A \sim B$ There are two cases to consider here. First, if z^* is in both A and B, it follows $A \cup C \sim B \cup C$. Second, if z^* is not in A or B, again it follows that $A \cup C \sim B \cup C$.

  Thus, in either case, $A \cup C \succsim B \cup C$.

  Counterexample of (b) Let $A = \{z^*\}$. Choose any $x \in Z \setminus \{z^*\}$. Clearly, $\{z^*\} \succ \{x\}$, but note that $A \sim A \cup \{x\}$, which violates the second part of (b).

- *The second but not the first property.* Again, consider an agent who has preferences $\succsim^*$ over the elements in Z. Denote a^* (a_*) the $\succsim^*$-maximal (minimal) element of A. Since A is finite, we know such elements exist. Consider the following variation of lexicographic preferences over X:[3]

  $$A \succsim B \iff a^* \succ^* b^* \text{ or } a^* \sim^* b^*, a_* \succsim^* b_*.$$

  In other words, A is preferred to B if the best element in A is strictly preferred than the best element in B, or if the best elements in A and B are equally preferred and the least preferred element in A is at least as good as the least preferred element in B.

  Counterexample of (a) Consider two sets such that $A \succ B$, and consider a set C disjoint to both A, B such that c^* is strictly better than a^*, b^*; and c_* is strictly worse than a_*, b_*. Here, it follows $A \cup C \sim B \cup C$, contradicting the second part of (a).

  Proof for (b) Take any set A and an element $z \in Z$ such that z is better than all the elements in A. Clearly, $A \cup \{z\} \succ A$.
  Next, choose any set A and an element $z \in Z$ such that z is worse than all the elements in A. Again, it readily follows that $A \succ A \cup \{z\}$ by the definition of the preference relation.

Show that if there are $x, y, z \in Z$ such that $\{x\} \succ \{y\} \succ \{z\}$, then there is no preference relation satisfying both properties.

By way of contradiction, assume there does exist a preference relation satisfying properties (a) and (b). From (b), we have:

$$\{x\} \succ \{x, y\} \qquad \{y, z\} \succ \{z\}$$

Applying (a) to the above, it follows:

$$\{x, z\} \succ \{x, y, z\} \qquad \{x, y, z\} \succ \{x, z\}$$

which yields the contradiction.

4. *Listen to the illusion called the Shepard Scale. (You can find it on the internet. Currently, it is available at http://asa.aip.org/demo27.html.) Can you think of any economic analogies?*

   An economic analogy to the Shepard Scale is transitivity (and the violation of it). In the recording, the tones sound as if the pitch in the recording is becoming higher and higher in frequency. At some point which is indistinguishable for the listener, however, the frequencies begin to repeat themselves. This "circular" pattern is analogous to an agent that violates transitivity, ie a case where $a \succsim b$, $b \succsim c$ and $c \succsim a$. We'll explore violations of rationality in general, and transitivity in particular, as the course progresses.

[3] We'll learn more about lexicographic preferences in Chapter 2.