Solution to Problem Set Five - Demand: Consumer Choice

Lecture Notes in Microeconomic Theory by Ariel Rubinstein

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1. *Calculate the demand function for the utility function $\sum_k \alpha_k \ln(x_k)$.*

   The consumer's problem is:

   $$\max_{\{x_k\}} \sum_k \alpha_k \ln(x_k) \qquad \text{st } \sum_k p_k x_k \leq w \text{ and } x_k \geq 0 \text{ for all } k \in \{1, \ldots, K\}.$$

   Since $x_k = 0 \Rightarrow u(x) = -\infty$ for every $k = 1, ..., K$, the non-negativity constraints will never bind since. The FOCs of the problem imply

   $$\frac{\alpha_k}{p_k x_k} = \frac{\alpha_l}{p_l x_l} \Rightarrow p_l x_l = \frac{\alpha_l}{\alpha_k} p_k x_k \text{ for all } k, l \in \{1, \ldots, K\}.$$

   By substituting this result into the budget constraint for $l \neq k$, it follows

   $$\frac{p_k x_k}{\alpha_k} \sum_i \alpha_i = w \Rightarrow x_k(p, w) = \frac{w}{p_k} \frac{\alpha_k}{\sum_i \alpha_i} \text{ for all } k \in \{1, \ldots, K\}.$$

2. *Verify that when preferences are continuous, the demand function $x(p, w)$ is continuous in prices and in wealth (and not only in p).*

   Choose an arbitrary convergent sequence of price/wealth pairs $(p^n, w^n) \rightarrow (p, w)$ such that $(p, w) \gg (0, 0)$. Since $x(p, w)$ is homogeneous of degree zero, this implies

   $$x(p^n, w^n) = x\left(\frac{p^n}{w^n}, 1\right).$$

   Since demand is continuous in p, then

   $$x\left(\frac{p^n}{w^n}, 1\right) \rightarrow x\left(\frac{p}{w}, 1\right) = x(p, w),$$

   where the second equality follows from demand being homogeneous of degree zero.

3. *Show that if a consumer has a homothetic preference relation, then his demand function is homogeneous of degree one in w.*

   Assume that $\succsim$ is homothetic, and let $\lambda > 0$ and $(p, w) \in \Re^{K+1}_{++}$. Let $y^* \in B(p, \lambda w)$, ie $py^* \leq \lambda w$. Clearly, $\frac{1}{\lambda} y^* \in B(p, w)$, and thus $x(p, w) \succsim \frac{1}{\lambda} y^*$. By homotheticity, it follows $\lambda x(p, w) \succsim y^*$.

   Since y^* was arbitrarily chosen from $B(p, \lambda w)$, it follows that $\lambda x(p, w) \succsim y$ for every $y \in B(p, \lambda w)$. Moreover, $\lambda x(p, w) \in B(p, \lambda w)$, and therefore $\lambda x(p, w) = x(p, \lambda w)$ since it is the optimal bundle in $B(p, \lambda w)$.

4. *Consider a consumer in a world with $K = 2$, who has a preference relation that is quasi-linear in the first commodity. How does the demand for the first commodity change with w?*

   I will further assume that preferences satisfy strict convexity. Let p be a price vector where p_1 is normalized to 1. There exists an $\alpha \in \Re_+ \cup \infty$ such that

$$x_1(p, w) = \begin{cases} 0 & \text{if } w \leq \alpha \\ w - p_2 x_2(p, \alpha) & \text{otherwise.} \end{cases}$$

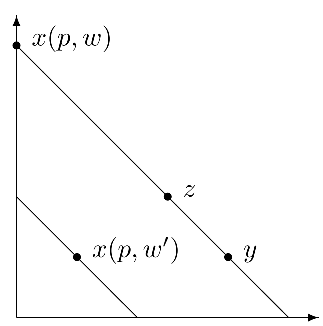
   In words, after a certain wealth threshold α, a change in wealth is absorbed by commodity 1.

   - $x_1(p, w) = 0$, $w' < w \Rightarrow x_1(p, w') = 0$.

     Proof: By contradiction, assume $w' < w$ and $x_1(p, w') > 0$. Define another bundle

$$y = \Big(x_1(p, w') + [w - w'], x_2(p, w') \Big).$$

     Note that y is on the frontier of $B(p, w)$. Consequently, there exists a convex combination of $x(p, w)$ and y, say z, where $z = (w - w', \frac{w'}{p_2})$. Graphically, we have



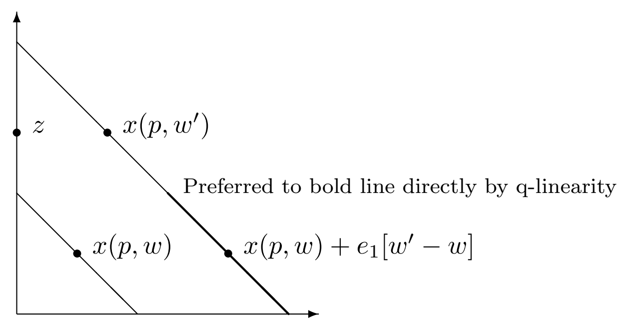
     Clearly, $x(p, w) \succsim y$, and thus by strict convexity $z \succ y$. But then $(0, \frac{w'}{p_2}) \succ x(p, w')$ by quasi-linearity, a contradiction.

- $x_1(p,w) > 0,\ w < w' \Rightarrow x_1(p,w') = x_1(p,w) + [w' - w]$.

  Proof: By contradiction, assume $x(p,w') \neq x(p,w) + e_1[w' - w]$. Since $x(p,w) \succsim y$ for all $y \in B(p,w)$, then

  $$\Big(x_1(p,w) + [w' - w], x_2(p,w)\Big) \succsim \Big(y_1 + [w' - w], y_2\Big)$$

  for all $y \in B(p,w)$ by quasi-linearity. Consequently, it must be that $x_2(p,w') > \frac{w}{p_2}$. Graphically, we have



  By quasi-linearity, it must be that $z = (0, x_2(p,w'))$ is optimal in $B(p, w' - x_1(p,w'))$. But $w < w' - x_1(p,w')$, and thus by the first claim $x_1(p,w) = 0$, a contradiction.

5. *Let $\succsim$ be a continuous preference relation (not necessarily strictly convex) and w a number. Consider the set $G = \{(p,z) \in \Re^K_{++} \times \Re^K_{+} \mid z$ is optimal in $B(p,w)\}$. Note that for some price vectors, there could be multiple $(p,x) \in G$. Calculate G for the case of $K = 2$ and preferences represented by $x_1 + x_2$.*

   Note that the agent is indifferent between consuming α units of x_1 and α units of x_2. He will thus allocate his entire wealth to the cheapest good:

   $$G(w) = \begin{cases} \big(p, \big(\frac{w}{p_1}, 0\big)\big) & \text{if } p_1 < p_2 \\ \big(p, \big(\alpha, \frac{w - p_1\alpha}{p_2}\big)\big) \text{ for any } \alpha \in \big[0, \frac{w}{p_1}\big] & \text{if } p_1 = p_2 \\ \big(p, \big(0, \frac{w}{p_2}\big)\big) & \text{if } p_1 > p_2 \end{cases}$$

   Show that, in general, G is a closed set.

   Let $w > 0$ and the sequence $\{(p^n, x^n)\}$ be such that $(p^n, x^n) \in G(w)$ for every n, and let $(p^n, x^n) \to (p, x)$. We must show $(p,x) \in G(w)$ to prove that G is closed. In other words, we must show that $x \in B(p,w)$ and that $x \succsim y$ for every $y \in B(p,w)$.

Since $(p^n, x^n) \in G(w)$, then $p^n x^n \leq w$ for every n. It readily follows that

$$px = \lim_{n\to\infty} p^n x^n \leq w \Rightarrow x \in B(p, w).$$

By way of contradiction, assume x is not optimal in $B(p, w)$. Then there exists a $y \in B(p, w)$ such that $y \succ x$. Then there exists an $\epsilon > 0$ such that $B_\epsilon(y) \succ B_\epsilon(x)$ by continuity, and thus there exists a bundle $z \in B_\epsilon(y)$ such that $z < y$ and $z \succ x$. Moreover, since $z < y$ and $p^n \to p$, then $p^n z \leq w$ for n large enough. But since $x^n \to x$, then $x^n \in B_\epsilon(x)$ for n large enough, and thus $z \succ x^n$, a contradiction to x^n being the optimal bundle in $B(p^n, w)$. Therefore $x \succsim y$ for every $y \in B(p, w)$, and thus $(p, x) \in G(w)$.

6. *Determine whether the following behavior patterns are consistent with the consumer model:*

   (a) *The consumer's demand function is* $x(p, w) = \left(\frac{2w}{2p_1+p_2}, \frac{w}{2p_1+p_2}\right)$.

   Yes, $x(p, w)$ can be rationalized by the monotonic preference relation represented by $u(x) = \min\{x_1, 2x_2\}$. Since $\succsim$ is monotonic, the consumer will always set

   i. $p_1x_1 + p_2x_2 = w$ by Walras' Law, and
   ii. $x_1 = 2x_2$ by the functional form of $u(x)$.

   Substituting (ii) into (i), it follows

   $$2p_1x_2 + p_2x_2 = w \Rightarrow x_2(p, w) = \frac{w}{2p_1 + p_2} \Rightarrow x_1(p, w) = \frac{2w}{2p_1 + p_2}.$$

   (b) *The consumer consumes up to quantity 1 of* x_1 *and spends his excess wealth on* x_2.

   Yes, the behavior is rationalizable by the utility function

   $$u(x) = \begin{cases} x_1 & \text{if } x_1 < 1 \\ 1 + x_2 & \text{if } x_1 \geq 1 \end{cases}$$

   (c) *The consumer chooses a bundle* (x_1, x_2) *which satisfies* $\frac{x_1}{x_2} = \frac{p_1}{p_2}$ *and costs* w. *Does the utility function* $u(x) = x_1^2 + x_2^2$ *rationalize the consumer's behavior?*

   No, $u(x) = x_1^2+x_2^2$ does not rationalize the behavior. Since $u(x)$ is not quasi-concave, the maximization approach that we used in Question 1 is not appropriate. If $0 < p_2 < p_1$, then a consumer maximizing $u(x)$ would set $x(p, w) = (0, \frac{w}{p_2})$. Nevertheless, the consumer actually chooses

   $$\frac{x_1}{x_2} = \frac{p_1}{p_2} > 0 \Rightarrow x_1(p, w) > 0.$$

Moreover, the behavior violates the WA and therefore is not rationalizable. Consider the choices from the following budget sets:

$$x((2,1),5) = (2,1) \qquad \text{and} \qquad x((1,2),5) = (1,2).$$

Note that each of the bundles is affordable at the other bundle's prices:

$$(1,2) \cdot x((2,1),5) = (1,2) \cdot (2,1) = 4 < 5$$
$$(2,1) \cdot x((1,2),5) = (2,1) \cdot (1,2) = 4 < 5,$$

which is a violation of the WA.

7. *In this question, we consider a consumer who behaves differently from the classic consumer we talked about in the lecture. Once again we consider a world with K commodities. The consumers choice will be from budget sets. The consumer has in mind a preference relation that satisfies continuity, monotonicity, and strict convexity; for simplicity, assume it is represented by a utility function u.*

   The consumer maximizes utility up to utility level u^0. If the budget set allows him to obtain this level of utility, he chooses the bundle in the budget set with the highest quantity of commodity 1 subject to the constraint that his utility is at least u^0.

   (a) *Formulate the consumer's problem.*

   The agent's objective is

   $$\begin{array}{lll} \max_{x \in B(p,w)} u(x) & & \text{if } \max_{x \in B(p,w)} u(x) < u^0, \text{ and} \\ \max_{x \in B(p,w)} x_1 & \text{s.t. } u(x) \geq u^0 & \text{if } \max_{x \in B(p,w)} u(x) \geq u^0. \end{array}$$

   (b) *Show that the consumer's procedure yields a unique bundle.*

   Case 1: $\max_{x \in B(p,w)} u(x) < u^0$

   In this instance, the consumer acts as in the standard framework. Since preferences are continuous, monotonic and strictly convex, then the problem has a unique solution (see the lecture notes).

   Case 2: $\max_{x \in B(p,w)} u(x) \geq u^0$

   Suppose, by way of contradiction, that x and y both solve the problem. Then $x, y \in B(p,w)$, $u(x), u(y) \geq u^0$ and $x_1 = y_1$. Define $z = \alpha x + (1-\alpha)y$ for some $\alpha \in (0,1)$. Note that $z_1 = x_1 = y_1$, $z \in B(p,w)$ and $u(z) > \min\{u(x), u(y)\} \geq u^0$.

   Note there exists a $j = 2, ..., K$ such that $z_j > 0$, and by continuity there exists an $\epsilon > 0$ such that $u(z - \epsilon e_j) > u^0$. With the ϵp_j the consumer is saving, he can afford to purchase $\frac{\epsilon p_j}{p_1}$ more units of commodity 1. Define $z' = z - \epsilon e_j + \frac{\epsilon p_j}{p_1} e_1$, and note that $z' \in B(p,w)$, $u(z') > u^0$ and $z'_1 > x_1$, a contradiction to x being a solution to the problem.

(c) *Is this demand procedure rationalizable?*

Yes, it can be rationalized by the monotonic utility function

$$v(x) = \begin{cases} u(x) & \text{if } u(x) < u^0 \\ u^0 + x_1 & \text{if } u(x) \geq u^0. \end{cases}$$

(d) *Does the demand function satisfy Walras Law?*

Yes. If $\max_{x \in B(p,w)} u(x) < u^0$, Walras Law is implied by monotonicity. If $\max_{x \in B(p,w)} u(x) \geq u^0$, then $px = w$; otherwise, the consumer could purchase more of x_1 and obtain a better bundle.

(e) *Show that in the domain of (p, w) for which there is a feasible bundle yielding utility of at least u^0 the consumer's demand function for commodity 1 is decreasing in p_1 and increasing in w.*

Let (p, w) be such that $\max_{x \in B(p,w)} u(x) \geq u^0$, and define $p' = p - \gamma e_1$ for some $\gamma > 0$. Since $p'x(p, w) < w$, then $x_1(p, w) < x_1(p', w)$, as the agent can afford strictly more of commodity 1 under $B(p', w)$ while maintaining utility u^0. Consequently, $x_1(p.w)$ is decreasing in p_1.

Similarly, define $w' = w + \gamma$, and note $px(p, w) < w'$. As before, it must be that $x_1(p, w) < x_1(p.w')$, as the agent can afford strictly more of commodity 1 under $B(p, w')$, and thus $x_1(p.w)$ is increasing in w.

(f) *Is the demand function continuous?*

Yes. Since demand is homogeneous of degree zero in (p, w), it is sufficient to show that $x(p, w)$ is continuous in p by Question 2. By contradiction, assume that $x(p, w)$ is not continuous in p, ie there exists a sequence of prices $\{p^n\}$ such that $p^n \to p$ and $\|x(p, w) - x(p^n, w)\| \geq \epsilon$ for every n. For notational ease, define $x^n = x(p^n, w)$ and $x = x(p, w)$. First, we must show that $\{x^n\}$ converges. As in the lecture notes, define

$$m = \inf \left\{ p_i^n \mid i \in \{1, \ldots, K\} \text{ and } n \in \mathbb{N} \right\} > 0,$$

and note that $x_i^n \leq \frac{w}{m}$ for every $i = 1, ..., K$ and every n. Consequently, $\{x^n\}$ is contained in the compact hypercube $[0, \frac{w}{m}]^K$. Therefore, without loss of generality we can assume that $x^n \to y$. Moreover, note that $py = \lim_{n \to \infty} p^n x^n \leq w$, and thus $y \in B(p, w)$. There are two cases to consider:

Case 1: $u(y) < u^0$

Since x is unique by (b), then $u(y) < u(x)$. By continuity, there exists a point $z \ll x$ such that $pz < w$ and $u(y) < u(z)$. Then for n large enough, $p^n z \leq w$ and $u(x^n) < u(z)$, a contradiction to x^n being the optimal bundle in $B(p^n, w)$.

Case 2: $u(y) \geq u^0$

Since x is unique by (b), then $u(x) \geq u^0$ and $x_1 > y_1$. Define $z = \frac{1}{2}x + \frac{1}{2}y$, and note that $z_1 > y_1$, $z \in B(p, w)$ and $u(z) > u^0$ by strict convexity. By continuity, there exists an $\epsilon > 0$ such that $z_1 - \epsilon > y_1$, $p[z - \epsilon e] < w$ and $u(z - \epsilon e) > u^0$. Then for n large enough, $p^n[z - \epsilon e] \leq w$ and $z_1 - \epsilon > x_1^n$, a contradiction to x^n being the optimal bundle in $B(p^n, w)$.

8. *A common practice in economics is to view aggregate demand as being derived from the behavior of a "representative consumer." Give two examples of "well-behaved" consumer preference relations that can induce average behavior that is not consistent with maximization by a "representative consumer." (That is, construct two "consumers," 1 and 2, who choose the bundles x_1 and x_2 out of the budget set A and the bundles y_1 and y_2 out of the budget set B so that the choice of the bundle $\frac{x_1+x_2}{2}$ from A and of the bundle $\frac{y_1+y_2}{2}$ from B is inconsistent with the model of the rational consumer.)*

   Consider the following sets of preferences:

$$u_1(x) = \begin{cases} x_1 & \text{if } x_1 < 4 \\ 4 + x_2 & \text{if } x_1 \geq 4. \end{cases} \qquad u_2(x) = \begin{cases} x_2 & \text{if } x_2 < 4 \\ 4 + x_1 & \text{if } x_2 \geq 4. \end{cases}$$

   and consider the price/wealth pairs $(p^A, w^A) = ((1,2), 8)$ and $(p^B, w^B) = ((2,1), 8)$. Note that

$$x_1^A = (4,2) \qquad x_1^B = (4,0) \qquad x_2^A = (0,4) \qquad x_2^B = (2,4).$$

   Taking averages across the price/wealth pairs, note that

$$\overline{x^A} = (2,3) \Rightarrow p^B \overline{x^A} = 7 < 8 \Rightarrow x^B \succ x^A, \text{ and}$$

$$\overline{x^B} = (3,2) \Rightarrow p^A \overline{x^B} = 7 < 8 \Rightarrow x^A \succ x^B, \text{ a contradiction.}$$

9. *Let $\succ$ be an acyclic binary relation on a finite set X. Show that there is a complete, asymmetric and transitive relation $\succ^*$ which extends $\succ$ (that is, if $a \succ b$ then $a \succ^* b$.)*

   Lemma: If X is finite and $\succ$ is an acyclic relation on X, then there exists at least one $x \in X$ such that for every other $y \in X$, either $x \succ y$ or x and y are not compared by $\succ$.

Proof of Lemma: Let $|X| = N$. By contradiction, assume that for every $x \in X$, there is a $y \in X$ such that $y \succ x$. Choose any element in X, say x_1. Then there exists another element in X, say x_2, such that $x_2 \succ x_1$. Again, there exists an $x_3 \in X$ such that $x_3 \succ x_2$ and so on. Thus we can find a sequence such that $x_{N+1} \succ x_N \succ ... \succ x_1$. Since $|X| = N$, then for some $i \in \{1, ..., N-1\}$, $x_{N+1} = x_i$. But then $x_{N+1} \succ ... \succ x_{i+1}$ and $x_{i+1} \succ x_{N+1}$, a contradiction to $\succ$ being acyclic. □

Since X is finite, then there exists an element in X, say x_1, such that no other element in X is better than x_1 by the Lemma. For all other $y \in X$, let $x_1 \succ^* y$. Again, the Lemma implies that there exists an element in $X \setminus \{x_1\}$, say x_2, such that no element in $X \setminus \{x_1\}$ is better than x_2. Define $x_2 \succ^* y$ for all other $y \in X \setminus \{x_1\}$. Continue this process inductively, ie for all $n < N$, there is an x_{n+1} that is one of the "best" elements in $X \setminus \{x_1, ..., x_n\}$; let $x_{n+1} \succ^* y$ for all other $y \in X \setminus \{x_1, ..., x_n\}$.

Extension: If $a \succ b$, then a is defined as an element in the sequence $\{x_1, ..., x_N\}$ before b, and hence $a \succ^* b$.

Completeness: Implied by construction.

Asymmetry: Implied by construction.

Transitivity: Let $a \succ^* b$ and $b \succ^* c$. Then a is defined as an element in the sequence $\{x_1, ..., x_N\}$ before b, and b is defined before c. Consequently, $a \succ^* c$.