Solution to Problem Set Six - Choice Over Budget Sets and the Dual Problem

Lecture Notes in Microeconomic Theory by Ariel Rubinstein

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1. *In a world with two commodities, consider a consumers preferences that are represented by the utility function $u(x) = \min\{x_1, x_2\}$.*

   (a) *Calculate the consumer's demand function.*

   The consumer will set $x_1 = x_2$ by the functional form of $u(x)$ and set $p_1x_1 + p_2x_2 = w$ by monotonicity. Consequently, $p_1x_2 + p_2x_2 = w$, which implies that $x(p, w) = (\frac{w}{p_1+p_2}, \frac{w}{p_1+p_2})$.

   (b) *Verify that preferences satisfy convexity.*

   Let $y, z \succsim x$. Then $y_i, z_i \geq \min\{x_1, x_2\}$ for $i = 1, 2$, and thus $\alpha y_i + (1 - \alpha)z_i \geq \min\{x_1, x_2\}$ for any $\alpha \in (0, 1)$, $i = 1, 2$. Consequently, $\alpha y + (1 - \alpha)z \succsim x$.

   (c) *Calculate the indirect utility function $v(p, w)$.*

   $v(p, w) = u\big(x(p, w)\big) = \frac{w}{p_1+p_2}$.

   (d) *Verify Roy's Identity.*

   $$x_i(p, w) = -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\frac{-w}{(p_1+p_2)^2}}{\frac{1}{p_1+p_2}} = \frac{w}{p_1 + p_2}.$$

   (e) *Calculate the expenditure function $e(p, u)$ and verify the Dual Roy's Identity.*

   The agent's expenditure minimization problem is

   $$\min_{x \in \Re^2_+} px \qquad \text{s.t. } u(x) \geq u.$$

   It's optimal for the agent to choose $x_i = u$ for $i = 1, 2$, and thus $e(p, u) = u(p_1+p_2)$. To verify Roy's Dual Identity, note that $h_i(p, u) = \frac{\partial e(p,u)}{\partial p_i} = u$.

2. *Imagine that you are reading a paper in which the author uses the indirect utility function* $v(p_1, p_2, w) = \frac{w}{p_1} + \frac{w}{p_2}$. *You suspect that the authors conclusions in the paper are the outcome of the "fact" that the function* v *is inconsistent with the model of the rational consumer. Take the following steps to make sure that this is not the case:*

   (a) *Use Roys Equality to derive the demand function.*

   Let $i \in \{1,2\}$ and $j \in \{1,2\}$ such that $i \neq j$. Then

$$x_i(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{\frac{-w}{p_i^2}}{\frac{p_1+p_2}{p_1 p_2}} = \frac{w p_j}{p_i(p_1+p_2)}.$$

   (b) *Show that if demand is derived from a smooth utility function, then the indifference curve at the point* (x_1, x_2) *has the slope* $-\frac{\sqrt{x_2}}{\sqrt{x_1}}$.

   Let $(p,w) \in \Re^3_{++}$, and note from (a) that $x(p,w)$ is in the interior of $B(p,w)$. If we further assume that u is quasi-concave, then

$$\frac{\frac{\partial u(x)}{\partial x_1}}{\frac{\partial u(x)}{\partial x_2}} = \frac{p_1}{p_2} = \sqrt{\frac{\frac{wp_1}{p_2(p_1+p_2)}}{\frac{wp_2}{p_1(p_1+p_2)}}} = \sqrt{\frac{x_2}{x_1}}.$$

   (c) *Construct a utility function with the property that the ratio of the partial derivatives at the bundle* (x_1, x_2) *is* $\frac{\sqrt{x_2}}{\sqrt{x_1}}$.

   Let $u(x) = \left(\sqrt{x_1} + \sqrt{x_2}\right)^2$, and note that $u(x)$ satisfies the condition for the ratio of the partials.

   (d) *Calculate the indirect utility function derived from this utility function. Do you arrive at the original* $v(p_1, p_2, w)$? *If not, can the original indirect utility function still be derived from another utility function satisfying the property in (c)?*

   Yes, $u(x)$ corresponds to the indirect utility function:

$$u(x(p,w)) = \left(\sqrt{\frac{wp_2}{p_1(p_1+p_2)}} + \sqrt{\frac{wp_1}{p_2(p_1+p_2)}}\right)^2 =$$

$$\left(\frac{\sqrt{w}(p_1+p_2)}{\sqrt{p_1p_2(p_1+p_2)}}\right)^2 = \frac{w(p_1+p_2)^2}{p_1p_2(p_1+p_2)} = \frac{w}{p_1} + \frac{w}{p_2} = v(p,w).$$

3. *A consumer with wealth w is interested in purchasing only one unit of one of the items included in a (finite) set A. All items are indivisible. The consumer does not derive any "utility" from leftover wealth. The consumer evaluates commodity $x \in A$ by the number V_x (where the value of not purchasing any of the goods is 0). The price of commodity $x \in A$ is $p_x > 0$.*

   (a) *Formulate the consumer's problem.*

   Let d denote the consumer's action of "not purchasing anything." Set $V_d = 0$ and $p_d = 0$, ie "not purchasing anything" generates no utility and is costless. Define the set $A' = A \cup \{d\}$. The problem is:

$$\max_{x \in A'} V_x \qquad \text{s.t. } p_x \leq w.$$

   (b) *Check the properties of the indirect utility function (homogeneity of degree zero, monotonicity, continuity and quasi-convexity).*

   Define $B(p, w) = \{x \in A' \mid p_x \leq w\}$. Then

$$v(p, w) = \max_{x \in B(p,w)} V_x.$$

   Homogeneity of Degree Zero Yes. For all $\lambda > 0$, $B(p, w) = B(\lambda p, \lambda w)$, and thus

$$v(p, w) = \max_{x \in B(p,w)} V_x = \max_{x \in B(\lambda p, \lambda w)} V_x = v(\lambda p, \lambda w).$$

   Monotonicity Yes. Note that

$$p \geq p', \ w \leq w' \Rightarrow B(p.w) \subseteq B(p', w'),$$

   and thus $v(p, w) \leq v(p', w')$.

   Continuity No. Let $A = \{x\}$, $V_x = 1$ and $p = (p_d, p_x) = (0, 1)$. Note that $v\big((0, 1), 1\big) = 1$ but for every $\epsilon > 0$, $v\big((0, 1 + \epsilon), 1\big) = 0$ and $v\big((0, 1), 1 - \epsilon\big) = 0$. Thus v is not continuous in p or w.

   Quasi-Convexity Yes. Let $v(p, w) \leq v(p', w')$ and $\lambda \in [0, 1]$, and define $(p'', w'') = \lambda(p, w) + (1 - \lambda)(p', w')$. To prove quasi-convexity, we must show $v(p'', w'') \leq v(p', w')$.

   Let x^* be the best bundle in $B(p'', w'')$. Then

$$[\lambda p_{x^*} + (1 - \lambda)p'_{x^*}] = p''_{x^*} \leq w'' = [\lambda w + (1 - \lambda)w'], \text{ and thus}$$

$$p_{x^*} \leq w \qquad \text{or} \qquad p'_{x^*} \leq w'.$$

   x^* is thus affordable in either $B(p, w)$ or $B(p', w')$, and consequently $v(p'', w'') \leq v(p, w)$ or $v(p'', w'') \leq v(p', w')$, and thus in either case $v(p'', w'') \leq v(p', w')$.

(c) *Calculate the indirect utility function for the case in which $A = \{a, b\}$ and $V_a > V_b > 0$.*

$$v(p, w) = \begin{cases} V_a & \text{if } p_a \leq w \\ V_b & \text{if } p_b \leq w < p_a \\ 0 & \text{if } p_a, p_b > w. \end{cases}$$

4. *Let X be a set and $\succsim$ be preferences on X. Let D be a set of choice problems and let $\succsim^*$ be the indirect preference relation defined on D. One route to elicit the choice function $c_{\succsim}$ from $\succsim^*$ is by concluding that:*

   $c_{\succsim}(A) = x^*$ when, for every $y \in A \setminus \{x^*\}$, there is a set $B(y)$ containing x^* but not y, such that $B(y) \succsim^* A$.

   Explain this definition and explain the analogy to Roys equality.

   Let $x^* \in A$ satisfy the condition described above. Then for every other $y \in A$, there is a set containing x^* and *not* y that is at least as good as A. In other words, the $\succsim$-maximal element in a set containing x^*, but not y, is at least as good as the $\succsim$-maximal element in A. Since this holds for all $y \in A$, then we can infer that x^* is the $\succsim$-maximal element in A.

   For Roy's Equality, we construct the set of price/wealth pairs such that $px(p^*, w^*) = w$. For each of these pairs, we have $B(p, w) \succsim^* B(p^*, w^*)$ since $x(p, w) \succsim x(p^*, w^*)$. Consequently, given indirect preferences, we can calculate the bundle $x(p^*, w^*)$ since the set of prices (p, w) is tangent to the indifference set through the pair (p^*, w^*).

5. *A consumer holds continuous preference relation $\succsim$ (but the optimization $\succsim$ over $B(p, w)$ does not necessarily yield a unique solution). State and prove the four properties of the induced indirect preferences $\succsim^*$ which are analogous to the four properties stated and proved for the case that $x(p, w)$ is always well defined.*

   Let $x(p, w)$ denote the *set* of optimal bundles in $B(p, w)$.

   (a) $(\lambda p, \lambda w) \sim^* (p, w)$: Since $x(p, w)$ is the set of optimal bundles in $B(p, w)$, then $u(x) = u(x')$ for all $x, x' \in x(p, w)$. Then $(\lambda p, \lambda w) \sim^* (p, w)$ because $x(\lambda p, \lambda w) = x(p, w)$.

   (b) $\succsim^*$ is non-increasing in p_k, increasing in w: Reducing the size of the budget set cannot be beneficial. Moreover, if w increases, the agent can consume more of all commodities.

   (c) If $\succsim$ is continuous, then $\succsim^*$ is continuous and there exists a continuous v representing $\succsim^*$: Let u be a continuous utility function representing $\succsim$. Define $v(p, w) = u(x^*)$, where x^* is any element of $x(p, w)$, and note that v is well-defined since all elements in $x(p,w)$ yield the same level of utility. Moreover, note that v represents $\succsim^*$.

First, let's show v is continuous in p. By contradiction, assume that there exists a convergent sequence of prices $p^n \to p$ such that $v(p^n, w)$ does not converge to $v(p,w)$. Define the sequence $\{x^n\}$, where $x^n \in x(p^n, w)$ for all n. Let's show that $\{x^n\}$ converges. Define $m = \inf\{p_k^n \mid k \in \{1, ..., K\}$ and $n \in \mathbb{N}\}$, and note $\{x^n\}$ is contained in the compact set $[0, \frac{w}{m}]^K$. Therefore, without loss of generality, we can assume that $x^n \to y$. Since, by assumption, $v(p^n, w)$ does not converge to $v(p,w)$, then it must be that $y \notin x(p,w)$. Note that $py = \lim p^n x^n \leq w$, and thus $x(p,w) \succ y$. From Question 5 in PS 5, we know that $x(p,w)$ is a closed set, and consequently there exists an $\epsilon > 0$ such that $z - \epsilon e \succ y$ for all $z \in x(p,w)$ by continuity. But then for n large enough, $p^n[z - \epsilon e] \leq w$ and $z - \epsilon e \succ x^n$ for all $z \in x(p,w)$, a contradiction to $x^n \in x(p^n, w)$. Then $x^n \to x^* \in x(p,w)$, and thus $u(x^n) \to u(x^*)$ by the continuity of u. Consequently, $v(p^n, w) \to v(p,w)$.

Finally, take any convergent sequence $(p^n, w^n) \to (p,w)$. Then $\lim v(p^n, w^n) = \lim v(\frac{p^n}{w^n}, 1) = v(\frac{p}{w}, 1) = v(p,w)$, where the first and third equalities follow from (a) and the second equality follows from v being continuous in p. Moreover, since v is continuous, then $\succsim^*$ is continuous as well.

(d) If $(p^1, w^1) \succsim^* (p^2, w^2)$, then $(p^1, w^1) \succsim^* (p', w')$, where $p' = \lambda p^1 + (1-\lambda)p^2$ and $w' = \lambda w^1 + (1-\lambda) w^2$ for $\lambda \in [0,1]$: Let $z \in x(p', w')$. Then $zp' \leq w'$, which implies that either $p^1 z \leq w^1$ or $p^2 z \leq w^2$. Therefore, $z \in B(p^1, w^1)$ or $z \in B(p^2, w^2)$. Since z was chosen arbitrarily, then either $x(p^1, w^1) \succsim x(p', w')$ or $x(p^2, w^2) \succsim x(p', w')$, and since $x(p^1, w^1) \succsim x(p^2, w^2)$, then in either case $x(p^1, w^1) \succsim x(p', w')$. Consequently, $(p^1, w^1) \succsim^* (p', w')$.

6. *Show that if the utility function is continuous, then so is the Hicksian demand function $h(p,u)$.*

Let $(p^n, u^n) \to (p, u^0)$ be a convergent sequence of price/utility pairs. First, let's show $\{h^n\} = \{h(p^n, u^n)\}$ converges. Define $\overline{u} = \sup\{u^n\} \in (0, \infty)$,

$$m = \inf\left\{p_i^n \mid n \in \mathbb{N} \text{ and } i \in \{1, \ldots, K\}\right\} > 0,$$

$$M = \sup\left\{p_i^n \mid n \in \mathbb{N} \text{ and } i \in \{1, \ldots, K\}\right\} < \infty,$$

$p_m = (m, ..., m)$ and $p_M = (M, \ldots, M)$. Let $\overline{h} = h(p, \overline{u})$, and note that $u(\overline{h}) \geq u^n$ for all n. Then

$$p_m h^n \leq p^n h^n \leq p^n \overline{h} \leq p_M \overline{h},$$

where the first inequality follows from $p_m \leq p^n$, the second inequality follows from h^n being the cheapest bundle achieving utility u^n at prices p^n and the third inequality follows from $p^n \leq p_M$. Therefore, $h_i^n \in [0, \frac{M}{m}\overline{h}_i]$ for all n and $i = 1, ..., K$, and consequently we can assume that h^n converges to some h^*.

To complete the proof, assume by contradiction that $h^* \neq h(p, u)$. Since $u(h^n) \geq u^n$ for all n, then the continuity of u implies that $u(h^*) \geq u^0$. Then it must be that $ph^* > ph(p, u)$. Let $z \gg h(p, u)$ be such that $ph^* > pz$, and note that $u(z) > u^0$ by monotonicity. By the continuity of u, then for sufficiently large n, we have $u(z) \geq u^n$. Moreover, since $p^n \to p$ and $h^n \to h^*$, then $p^n h^n > p^n z$ for sufficiently large n, a contradiction to h^n being the cheapest bundle achieving utility u^n at prices p^n.

7. *A commodity k is a* Giffen *if the demand for the k-th good, $x_k(p, w)$, is increasing in p_k. A commodity k is* inferior *if the demand for the commodity decreases in wealth. Show that if k is Giffen in some neighborhood of (p, w), then k is inferior.*

   For this proof, I will assume that demand for x_k is strictly positive in the neighborhood of (p, w). This assumption is innocuous since if $x_k(p, w) = 0$, then for $p'_k < p_k$, we would have $x_k(p', w) = x_k(p, w) = 0$, and for $p'_k > p_k$, then the agent would still be able to afford his original bundle $x(p, w)$.

   By contradiction, assume there exists a (p, w) such that $x_k(p, w)$ is Giffen in the neighborhood of (p, w), but $x_k(p, w)$ is not inferior. Since $x_k(p, w)$ is a Giffen, then $x_k(p + \epsilon e_k, w) > x_k(p, w)$ for some $\epsilon > 0$. For notational ease, define $x = x(p, w)$ and $x' = x(p + \epsilon e_k, w)$.

   Define a new price/wealth pair $(p + \epsilon e_k, w + x_k \epsilon)$, and define $x'' = x(p + \epsilon e_k, w + x_k \epsilon)$. Since the k-th good is not inferior, then $x''_k \geq x'_k > x_k$. Moreover, by construction

$$[p + \epsilon e_k]x = w + \epsilon x_k \Rightarrow x \in B(p + \epsilon e_k, w + \epsilon x_k),$$

   and since $x_k < x''_k$, note that

$$[p + \epsilon e_k]x'' = w + \epsilon x_k \Rightarrow px'' = w + \epsilon[x_k - x''_k] < w \Rightarrow x'' \in B(p, w).$$

   Thus $x \sim x''$ by the WA. Note, however, that there exists a $z \gg x''$ such that $pz \leq w$. But then $z \succ x'' \sim x$ by monotonicity, which contradicts x being the optimal bundle in $B(p, w)$.

8. *One way to compare budget sets is by using the relation $\succsim^*$ as defined in the text. According to this approach, the comparison between (p, w) and (p', w) is made by comparing two numbers $u(x(p, w))$ and $u(x(p', w))$, where u is a utility function defined on the space of the bundles. Following*

are two other approaches for making such comparisons using "concrete terms."
Define:

$$CV(p, p', w) = w - e(p', u) = e(p, u) - e(p', u)$$

where $u = u(x(p, w))$*. This is the answer to the question: What is the change in wealth that would be equivalent, from the perspective of* (p, w)*, to the change in price vectors from* p *to* p'*?*
Define:

$$EV(p, p', w) = e(p, u') - w = e(p, u') - e(p', u')$$

where $u' = u(p', w)$*. This is the answer to the question: What is the change in wealth that would be equivalent, from the perspective of* (p', w)*, to the change in price vectors from* p *to* p'*?*
Now, answer the following questions regarding a consumer in a two-commodity world with a utility function u*:*

(a) *For the case* $u(x_1, x_2) = x_1 + x_2$*, calculate the two "consumer surplus" measures.*

Let p, p' be two price vectors. Then $u = \frac{w}{\min\{p_1, p_2\}}$ and $u' = \frac{w}{\min\{p'_1, p'_2\}}$. From here, a bit of algebra yields

$$CV(p, p', w) = w - \frac{w \min\{p'_1, p'_2\}}{\min\{p_1, p_2\}} = w\left[\frac{\min\{p_1, p_2\} - \min\{p'_1, p'_2\}}{\min\{p_1, p_2\}}\right]$$

$$EV(p, p', w) = \frac{w \min\{p_1, p_2\}}{\min\{p'_1, p'_2\}} - w = w\left[\frac{\min\{p_1, p_2\} - \min\{p'_1, p'_2\}}{\min\{p'_1, p'_2\}}\right]$$

which are different if $\min\{p_1, p_2\} \neq \min\{p'_1, p'_2\}$.

(b) *Explain why the two measures may give different values for some other utility functions.*

When a price changes, the consumer is effected in two ways: a direct "price" effect and an indirect "wealth" effect. If the wealth effect is different at the two bundles $x(p, w)$ and $x(p', w)$, then the CV and EV will generate different values.

(c) *Explain why the two measures are identical if the individual has quasi-linear preferences in the second commodity and in a domain where the two commodities are consumed in positive quantities.*

Assume that preferences are also strictly monotonic in the second commodity (in addition to the usual assumption of continuity and monotonicity). Under this assumption, we proved in Chapter 4 that preferences can be represented by $u(x) = x_2 + \phi(x_1)$. Moreover, recall that when preferences are quasi-linear, a change in wealth causes a change in demand for *the quasi-linear commodity only*, ie demand for x_1 is independent of wealth (we proved this in Question 4, PS5).

Normalize prices such that $p_2 = p'_2 = 1$, and note

$$x_2(p, e(p,u)) + \phi\big(x_1(p, e(p,u))\big) = u = x_2(p', e(p',u)) + \phi\big(x_1(p', e(p',u))\big)$$

$$x_2(p, e(p,u')) + \phi\big(x_1(p, e(p,u'))\big) = u' = x_2(p', e(p',u')) + \phi\big(x_1(p', e(p',u'))\big).$$

Since x_1 is independent of wealth, then $x_1(p, e(p,u)) = x_1(p, e(p,u'))$ and $x_1(p', e(p',u)) = x_1(p', e(p',u'))$. By subtracting the above two equations, we have:

$$x_2(p, e(p,u)) - x_2(p, e(p,u')) = x_2(p', e(p',u)) - x_2(p', e(p',u')).$$

Since $p_2 = p'_2 = 1$, then $x_2(p,u) = e(p,u) - p_1 x_1(p,u)$ by the budget constraint, and likewise for the other three terms. Substituting into the above equation, it follows

$$[e(p,u) - p_1 x_1\big(p, e(p,u)\big)] - [e(p,u') - p_1 x_1\big(p, e(p,u')\big)] =$$

$$[e(p',u) - p'_1 x_1\big(p', e(p',u)\big)] - [e(p',u') - p'_1 x_1\big(p', e(p',u')\big)].$$

Again, since x_1 is independent of wealth, all of the x_1 terms cancel out. After rearranging the terms, it follows

$$CV(p,p',w) = e(p,u) - e(p',u) = e(p,u') - e(p',u') = EV(p,p',w).$$

(d) *Assume that the price of the second commodity is fixed and that the price vectors differ only in the price of the first commodity. What is the relation of the two measures to the "area below the demand function" (which is a standard third definition of consumer surplus)?*

Fix p_2, and let $p'_1 < p_1$, which implies that $u = v(p,w) < v(p',w) = u'$ (if, as in (c), we assume $x_1(p,w) > 0$). For the time being, assume that commodity 1 is a normal good, ie increasing in u. Then

$$h_1((\overline{p_1}, p_2), u) \le h_1((\overline{p_1}, p_2), v((\overline{p_1}, p_2), w))$$
$$\le h_1((\overline{p_1}, p_2), u') \text{ for all } \overline{p_1} \in [p'_1, p_1].$$

It follows that

$$CV(p,p',w) = \int_{p'_1}^{p_1} \frac{\partial e((\overline{p_1}, p_2), u)}{\partial \overline{p_1}} d\overline{p_1} = \int_{p'_1}^{p_1} h_1((\overline{p_1}, p_2), u) d\overline{p_1}$$

$$\le \int_{p'_1}^{p_1} h_1((\overline{p_1}, p_2), v((\overline{p_1}, p_2), w)) d\overline{p_1} \le$$

$$\int_{p'_1}^{p_1} h_1((\overline{p_1}, p_2), u') d\overline{p_1} = \int_{p'_1}^{p_1} \frac{\partial e((\overline{p_1}, p_2), u')}{\partial \overline{p_1}} d\overline{p_1} = EV(p,p',w).$$

Let A denote the area under the curve. Then $CV \le A \le EV$ when the good is normal. For inferior goods, $EV \le A \le CV$, since $h_1(p,u)$ would be *decreasing* in u. Finally, the relationship holds with equality if the good is neither normal nor inferior, as discussed in part (b).