Solution to Problem Set Eight - Expected Utility

Lecture Notes in Microeconomic Theory by Ariel Rubinstein

Last Update: Oct 16, 2007

1. *Consider the following preference relations that were described in the text: "The size of the support" and "comparing the most likely prize."*

   (a) *Check carefully whether they satisfy axioms I and C.*

   Size of Support: Violates I: Let $p = (1,0) \succ (0.5, 0.5) = q$ and $\alpha \in (0,1)$. Then $\alpha p \bigoplus (1-\alpha) q \sim q$.
   Violates C: Let $p = (1,0) \succ (0.5, 0.5) = q$. For any $\epsilon > 0$, $(1-\epsilon, \epsilon) \sim q$.

   Most Likely Prize: Violates I: Let z_1 be better than z_2. Then $p = (1,0) \succ (0,1) = q$. Let $\alpha \in (0, \frac{1}{2})$. Then $\alpha p \bigoplus (1-\alpha) q \sim q$.
   Violates C: Note that $(0.5.0.5) \succ (0,1)$, but for every $\epsilon > 0$, $(0.5-\epsilon, 0.5+\epsilon) \sim (0,1)$.

   (b) *These preference relations are not immune to a certain framing problem. Explain.*

   In both cases, preferences may depend on how the lotteries are presented, regardless if the lotteries are in practice idenical. For example, the lottery *win \$100 with probability 0.6 and win \$0 otherwise* would be strictly preferred to the lottery *win one \$100 bill with probability 0.3, two \$50 bills with probability 0.3 and \$0 otherwise*, even though these lotteries are equivalent.

2. *One way to construct preferences over lotteries with monetary prizes is by evaluating each lottery L on the basis of two numbers, $Ex(L)$, the expectation of L, and $var(L)$, L's variance. Such a procedure may or may not be consistent with vNM assumptions.*

   (a) *Show that $u(L) = Ex(L) - (1/4)var(L)$ induces a preference relation that is not consistent with the vNM assumptions. (For example, consider the mixtures of each of the lotteries* $[1]$ *and* $0.5[0] \bigoplus 0.5[4]$ *with the lottery* $0.5[0] \bigoplus 0.5[2]$.*)*

Note that $[1] \sim 0.5[0] \bigoplus 0.5[4]$ since

$$u([1]) = 1 - \frac{1}{4}(0) = 2 - \frac{1}{4}\left(\frac{1}{2}0^2 + \frac{1}{2}4^2 - 2^2\right) = u(0.5[0] \oplus 0.5[4]).$$

For $\alpha \in (0,1)$, I is violated because

$$\alpha[1]+(1-\alpha)\big(0.5[0]\oplus 0.5[2]\big) \succ \alpha\big(0.5[0]\oplus 0.5[4]\big)+(1-\alpha)\big(0.5[0]\oplus 0.5[2]\big).$$

To see this, note that

$$u\big(\alpha[1]\oplus(1-\alpha)\big(0.5[0]\oplus 0.5[2]\big)\big) = u\left(\alpha[1]\oplus\frac{(1-\alpha)}{2}[0]\oplus\frac{(1-\alpha)}{2}[2]\right) = \frac{3+\alpha}{4}$$

$$> \frac{3+\alpha^2}{4} = u\left(\frac{1}{2}[0]\oplus\frac{\alpha}{2}[4]\oplus\frac{(1-\alpha)}{2}[2]\right) = u\Big(\alpha\big(0.5[0]\oplus 0.5[4]\big)\oplus(1-\alpha)\big(0.5[0]\oplus 0.5[2]\big)\Big).$$

(b) *Show that $u(L) = Ex(L) - (Ex(L))^2 - var(L)$ is consistent with vNM assumptions.*

$$u(L) = Ex(L) - (Ex(L))^2 - var(L) = Ex(L) - Ex(L, z^2) = \sum_{z \in Z} L(z)(z - z^2)$$

is an expected utility function with vNM values $v(z) = z - z^2$.

3. *In this problem you will encounter the functional of Quiggin and Yaari, one of the proposed alternatives to expected utility theory. Consider a world with the prizes $z_0, z_1, ..., z_K$. A decision maker attaches a number $v(z_k)$ to each z_k such that $v(z_0) = 0 < v(z_1) < v(z_2) < ... < v(z_K)$ and evaluates each lottery L by the number $U(L) = \sum_{k=1}^{K} f(G_L(z_k))[v(z_k) - v(z_{k-1})]$, where $f : [0,1] \to [0,1]$ is a continuous increasing function and $G_L(z_k) = \sum_{j \geq k} L(z_j)$. ($L(z)$ is the probability that the lottery L yields z and G_L is the "anti-distribution" of L.)*

(a) *Verify that for $f(x) = x$, $U(L)$ is the standard expected v-utility of L.*

Recall the $v(z_0) = 0$. Then

$$\begin{aligned} U(L) &= v(z_0) + \textstyle\sum_{k=1}^{K} (\sum_{j \geq k} L(z_j))[v(z_k) - v(z_{k-1})] \\ &= L(z_K)v(z_K) + \textstyle\sum_{k=1}^{K-1} v(z_k)\big[\sum_{j \geq k} L(z_j) - \sum_{j \geq k+1} L(z_j)\big] \\ &= \textstyle\sum_{k=0}^{K} L(z_k)v(z_k). \end{aligned}$$

(b) *Show that the induced preference relation satisfies the continuity axiom but may not satisfy the independence axiom.*

Preferences are continuous because U is continuous. Consider the following counterexample to I: Let $K = 2$, $f(x) = x^2$, $v(z_1) = 1$ and $v(z_2) = 4$. Consider two lotteries, $L = \frac{3}{4}[z_0] \oplus \frac{1}{4}[z_2]$ and $L' = \frac{1}{2}[z_0] \oplus \frac{1}{2}[z_1]$. Then

$$U(L) = f\left(\frac{1}{4}\right) + 3f\left(\frac{1}{4}\right) = \frac{1}{4} = f\left(\frac{1}{2}\right) = U(L').$$

Nevertheless,

$$U\left(\frac{1}{2}L \oplus \frac{1}{2}[z_1]\right) = f\left(\frac{5}{8}\right) + 3f\left(\frac{1}{8}\right) = \frac{28}{64} <$$

$$\frac{9}{16} = f\left(\frac{3}{4}\right) = U\left(\frac{1}{2}L' \oplus \frac{1}{2}[z_1]\right).$$

(c) *What are the difficulties with a functional form of the type $\sum_z f(p(z))u(z)$?*

One problem is that if the decision maker is indifferent between prizes x, y and z, then the lotteries $[x]$ and $0.5[y] \oplus 0.5[z]$ may not be indifferent under u. For example, if $f(x) = x^2$ and $v = v(x) = v(y) = v(z) > 0$, then $U([x]) = v > 0.5v = U(0.5[y] \oplus 0.5[z])$.

A second problem is that u not be monotonic with respect to first order stochastic dominance, ie the agent may be worse off after shifting some positive probability weight from a less preferred alternative to a more preferred alternative. For example, let $f(x) = x^2$, $u(z_1) = 1$ and $u(z_2) = 2$. Then $U(0.5[z_1] \oplus 0.5[z_2]) = 0.75 < 1 = U([z_1])$.

4. *A decision maker has a preference relation $\succsim$ over the space of lotteries $L(Z)$ having a set of prizes Z. On Sunday he learns that on Monday he will be told whether he has to choose between L_1 and L_2 (probability $1 > \alpha > 0$) or between L_3 and L_4 (probability $1 - \alpha$). He will make his choice at that time. Let us compare between two possible approaches the decision maker may take.*

Approach 1: He delays his decision to Monday ("why bother with the decision now when I can make up my mind tomorrow. . .").

Approach 2: He makes a contingent decision on Sunday regarding what he will do on Monday, that is, he instructs himself what to do if he faces the choice between L_1 and L_2 and what to do if he faces the choice between L_3 and L_4 ("On Monday morning I will be so busy. . .").

(a) *Formulate approach 2 as a choice between lotteries.*

Let us refer to a lottery set that the decision maker may face on Monday as a *menu*. Thus, the decision maker may will face either menu $A = \{L_1, L_2\}$ or menu $B = \{L_3, L_4\}$ on Monday, from which he will have to make a choice. In approach 2, the decision maker decides his choice on Sunday, contingent on what will happen on Monday. In other words, on Sunday he chooses an element $f(A)$ from A and an element $f(B)$ from B. When menu X is realized on Monday, he choses $f(X)$. So, the choice function f effectively yields the lottery $\alpha f(A) \oplus (1-\alpha) f(B)$. By considering the different values that $f(A)$ and $f(B)$ can take, the decision maker is effectively choosing from the set of lotteries

$$\{\alpha L_i \oplus (1-\alpha) L_j \mid i = 1, 2, j = 3, 4\}.$$

(b) *Show that if the preferences of the decision maker satisfy the independence axiom, his choice under approach 2 will always be the same as under approach 1.*

Let L_i be the lottery in menu A and L_j be the best lottery in B. If the decision maker postpones his decision to Monday (approach 1), he will choose L_i from A and L_j from B. We want to show that $f(A) = L_i$ and $f(B) = L_j$. Let L_{-i} be the suboptimal lottery in A and likewise for L_{-j}. An appeal to I several times yields

$$\alpha L_i \oplus (1-\alpha) L_j \succ \alpha L_i \oplus (1-\alpha) L_{-j} \succ \alpha L_{-i} \oplus (1-\alpha) L_{-j} \text{ and}$$

$$\alpha L_i \oplus (1-\alpha) L_j \succ \alpha L_{-i} \oplus (1-\alpha) L_j$$

and thus the decision maker will set $f(A) = L_i$ and $f(B) = L_j$.

5. *A decision maker has to choose an action from among a set A. The set of consequences is Z. For every action $a \in A$, the consequence z^* is realized with probability α, and any $z \in Z \setminus \{z^*\}$ is realized with probability $(1-\alpha) q(a, z)$.*

(a) *Assume that once he has made his choice he is told that z^* will not occur and is given a chance to change his decision. Show that if the decision maker obeys the Bayesian updating rule and follows vNM axioms, he will not change his decision.*

If the decision maker follows the vNM axioms, then there is an expected utility representation of his preferences over lotteries with vNM index v. Without knowing that z^* will not occur, he solves

$$\max_{a \in A} \left\{ \alpha v(z^*) + \sum_{z \in Z \setminus \{z^*\}} (1-\alpha) q(a, z) v(z) \right\}.$$

After learning that z^* did not occur, the decision maker updates her probabilities of $z \in Z \setminus \{z^*\}$ occurring to $q'(a,z) = \frac{1}{1-\alpha} q(a,z)$, and thus her problem reduces to

$$\max_{a \in A} \sum_{z \in Z \setminus \{z^*\}} q(a,z) v(z),$$

which generates the same choice as her original problem.

(b) *Show that this is not necessarily the case if he does not obey the Bayesian rule or is using a nonexpected utility preference relation.*

If she does not satisfy the vNM axioms (ie her utility is not linear in probabilities) or she does not use the Bayes rule (ie $p(z^*)$, $q'(a,z)$ are different from above), then in general, the maximization problems listed above are not equivalent, and thus the decision may depend on this information.

6. *Assume there is a finite number of income levels and that the distribution over income levels is defined as the proportion of individuals at each level. In other words, we can think of a distribution as a lottery over income levels, with the probability of outcomes representing the proportions at each level. We often use the phrase "one distribution is more egalitarian than another."*

(a) *Why is the von NeumannMorgenstern independence axiom inappropriate for characterizing this type of relation?*

Take two perfectly egalitarian distributions p and q, where under p everyone earns 5 and under q everyone earns 10. The distribution of $0.5p \oplus 0.5q$, where half the population earns 5 and the other half earns 10, is strictly less egalitarian.

(b) *Suggest a property that is appropriate, in your opinion, as an axiom for this type of relation. Give two examples of preference relations that satisfy your property and express the desired relation in a logical fashion.*

Property: Let p and q be two distributions according to some variable x. If $E_p(x) = E_q(x)$, then $p \succsim q$ iff the variance of p is smaller than the variance of q.

Example 1: $p \succsim q$ iff $Var(p) \leq Var(q)$.

Example 2: $p \succsim q$ iff $Var(p) - E_p(x) \leq Var(q) - E_q(x)$.

7. *A decision maker faces a trade-off between longevity and quality of life. His preference relation ranks lotteries on the set of all certain outcomes of the form (q, t), defined as "a life of quality q and length t" (where q and t are nonnegative numbers). Assume that the preference relation satisfies von NeumannMorgenstern assumptions and that it also satisfies*

   - *Indifference between "high" and "low" quality of life when longevity is 0.*
   - *Expected longevity and quality of life are desirable.*

   (a) *Formalize the two assumptions.*

   - $(q, 0) \sim (q', 0)$ for all q, q'.
   - Let π_1 and π_2 and F_π be the CDF for π. Then for any fixed q^*, if $F_{\pi_1}(q^*, t) \leq F_{\pi_2}(q^*, t)$ for all t, and for any fixed t^*, if $F_{\pi_1}(q, t^*) \leq F_{\pi_2}(q, t^*)$ for all q, then $\pi_1 \succsim \pi_2$.

   (b) *Show that the preference relation derived from maximizing the expectation of the function $v(q)t$, where v is a strictly increasing function and $v(q) > 0$ for all q, satisfies the assumptions.*

   Clearly, if $t = 0$ then $tv(q) = 0$ for all q.

   If monotonicity is satisfied, then $E_{\pi_1}[tv(q)] \geq E_{\pi_2}[tv(q)]$. If the CDFs exhibit a strict inequality at some (q, t), then $E_{\pi_1}[tv(q)] > E_{\pi_2}[tv(q)]$.

   (c) *Show that all preference relations satisfying the above assumptions can be represented by an expected utility function of the form $v(q)t$, where v is a positive and increasing function.*

   Since $\succsim$ satisfies the v-NM axioms, then we know that the decision maker maximizes expected utility for some values $w(p, t)$. According to property two, for a given $\overline{q}$, the decision maker wants to maximize expected longevity, which is represented by t. For a given $\overline{q}$, if v is such that $v > 0$, then $w(\overline{q}, t) = vt + b$ represents preferences over longevity for some number b. For two degenerate lotteries with $t = 0$, it follows that $w(q_1, 0) = w(q_2, 0)$, and thus b must be identical for all q. Thus we can normalize $b = 0$. Moreover, for two degenerate lotteries where longevity equals 1 and $q_1 > q_2$, it must be that $w(q_1, 1) > w(q_2, 1))$, and thus it must be that $v(q_1) > v(q_2)$. v, therefore, is positive and strictly increasing in q.

8. *Consider a decision maker who systematically calculates that $2 + 3 = 6$. Construct a "money pump" argument against him. Discuss the argument.*

   Tell him that if he gives you \$5.99, you will give him two checks: one for \$2 and one for \$3: since he thinks he is receiving \$6.00, he will agree. Repeat this again and again, for a profit of \$.99 in every round.