Solution to Problem Set Nine - Risk Aversion

Lecture Notes in Microeconomic Theory by Ariel Rubinstein

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1. *We say that p* second-order stochastically dominates q *and denote it by pD_2q if $p \succsim q$ for all preferences $\succsim$ satisfying the vNM assumptions, monotonicity* and *risk aversion.*

   (a) *Explain why pD_1q implies pD_2q.*
   If pD_1q, then $p \succsim q$ for all preferences satisfying the vNM assumptions and monotonicity, and thus $p \succsim q$ for all preferences satisfying the vNM assumptions, monotonicity *and* risk aversion.

   (b) *Let p and ϵ be lotteries. Define $p+\epsilon$ to be the lottery that yields the prize t with the probability $\sum_{\alpha+\beta=t} p(\alpha)\epsilon(\beta)$. Interpret $p+\epsilon$. Show that if ϵ is a lottery with expectation 0, then for all p, $pD_2(p+\epsilon)$.*
   Interpretation: $p+\epsilon$ is the combined lottery faced by an agent who simultaneously plays the independent lotteries p and ϵ.
   If ϵ has zero expectation and u is concave and increasing, then the agent is risk averse and

$$U(p+\epsilon) = \sum_{\alpha,\beta\in Z} p(\alpha)\epsilon(\beta)u(\alpha+\beta) = \sum_{\alpha\in Z} p(\alpha) \sum_{\beta\in Z} \epsilon(\beta)u(\alpha+\beta)$$

$$\leq \sum_{\alpha\in Z} p(\alpha)u\left(\sum_{\beta\in Z}\epsilon(\beta)(\alpha+\beta)\right) = \sum_{\alpha\in Z} p(\alpha)u\left(\alpha + \sum_{\beta\in Z}\epsilon(\beta)\beta\right)$$

$$= \sum_{\alpha\in Z} p(\alpha)u(\alpha) = U(p),$$

   where the inequality follows from the concavity of u and the second to last equality follows from ϵ having 0 expectation.

   (c) *Show that pD_2q iff for all $t < K$, $\sum_{k=0}^{t}[G(p,x_{k+1})-G(q,x_{k+1})][x_{k+1}-x_k] \geq 0$, where $x_0 < ... < x_K$ are all prizes in the support of either p or q and $G(p,x) = \sum_{z\geq x} p(z)$.*
   By an argument similar to PS8, 3(a), we have

$$\sum_{k=0}^{K} p(x_k)u(x_k) = \sum_{k=0}^{K-1} G(p,x_{k+1})[u(x_{k+1})-u(x_k)] + u(x_0).$$

⇒: By contradiction, assume that $\sum_{k=0}^{t}[G(p,x_{k+1})-G(q,x_{k+1})][x_{k+1}-x_k]<0$ for some $t=0,...,K-1$. Define

$$u^\epsilon(x)=\begin{cases} x & \text{if } x\leq x_{t+1} \\ \epsilon(x-x_{t+1})+x_{t+1} & \text{if } x>x_{t+1}. \end{cases}$$

If $\epsilon\in(0,1]$, then u^ϵ is concave and increasing in x and therefore represents preferences that satisfy the vNM assumptions, monotonicity and risk aversion. Moreover, using the preferences represented by u^ϵ, it follows

$$U(p)-U(q)=\sum_{k=0}^{K-1}[G(p,x_{k+1})-G(q,x_{k+1})][u^\epsilon(x_{k+1})-u^\epsilon(x_k)]$$

$$=\sum_{k=0}^{t}[G(p,x_{k+1})-G(q,x_{k+1})][x_{k+1}-x_k]+$$
$$\sum_{k=t+1}^{K-1}[G(p,x_{k+1})-G(q,x_{k+1})]\epsilon[x_{k+1}-x_k],$$

and as $\epsilon\to 0$,

$$=\sum_{k=0}^{t}[G(p,x_{k+1})-G(q,x_{k+1})][x_{k+1}-x_k]<0.$$

Therefore, for $\epsilon>0$ small enough, $q\succ p$, and thus we have NOT pD_2q, a contradiction.

⇐: Let u be nondecreasing and concave, and define $\alpha_k=\frac{u(x_{k+1})-u(x_k)}{x_{k+1}-x_k}$ for $k=0,...,K-1$ and $\alpha_K=0$. Then α_k is a nonincreasing sequence, so

$$U(p)-U(q)=\sum_{k=0}^{K-1}[G(p,x_{k+1})-G(q,x_{k+1})][u(x_{k+1})-u(x_k)]$$

$$=\sum_{k=0}^{K-1}[G(p,x_{k+1})-G(q,x_{k+1})][x_{k+1}-x_k]\alpha_k$$

$$=\sum_{k=0}^{K-1}[G(p,x_{k+1})-G(q,x_{k+1})][x_{k+1}-x_k]\sum_{t=k}^{K-1}(\alpha_t-\alpha_{t+1})$$

$$=\sum_{t=0}^{K-1}\left\{(\alpha_t-\alpha_{t+1})\sum_{k=0}^{t}[G(p,x_{k+1})-G(q,x_{k+1})][x_{k+1}-x_k]\right\}\geq 0,$$

as desired.

2. *Consider a phenomenon called preference reversal. Let* $L_1 = 8/9[4] \oplus 1/9[0]$ *and* $L_2 = 1/9[40] \oplus 8/9[0]$.

   (a) *What is the maximal amount you are willing to pay for* L_1*? For* L_2*?*

   (b) *What lottery do you prefer?*

   (c) *Discuss the "typical" answer that ranks* L_1 *as superior to* L_2 *but attaches a lower value to* L_1 *(see Slovic, Tversky and Kahneman 1990).*

   People tend to prefer L_1, which yields a payoff of \$4 with high probability. Nevertheless, people will not pay \$4 to play L_1, while some people may be willing to pay \$4 for L_2.

3. *Consider a consumers preference over* K*-tuples of* K *uncertain assets. Denote the random return on the* k*th asset by* Z_k*. Assume that the random variables* $(Z_1, ..., Z_K)$ *are independent and take positive values with probability 1. If the consumer buys the combination of assets* $(x_1, ..., x_K)$ *and if the vector of realized returns is* $(z_1, ..., z_K)$*, then the consumers total wealth is* $\sum_k z_k x_k$*. Assume that the consumer satisfies vNM assumptions, that is, there is a function* v *(over the sum of his returns) so that he maximizes the expected value of* v*. Assume that* v *is increasing and concave. The consumer preferences over the space of the lotteries induce preferences on the space of investments. Show that the induced preferences are monotonic and convex.*

   Monotonicity: Let $x \geq x'$. Then $x \cdot z \geq x' \cdot z$ with probability one, ie $v(x \cdot z) \geq v(x' \cdot z)$ with probability one. Therefore $Ev(x \cdot z) \geq Ev(x' \cdot z)$.

   Convexity: Let x, x' be two investment levels and $\lambda \in [0, 1]$. Define $x'' = \lambda x + (1-\lambda)x'$. By the concavity of v, it follows that $v(x'', z) \geq \lambda v(x, z) + (1-\lambda)v(x', z)$ for any z. Therefore, $Ev(x'', z) \geq \lambda Ev(x, z) + (1-\lambda)Ev(x', z)$. Moreover, since v is concave in x, then it is also quasi-concave in x, ie the induced preferences on the space of investments is convex.

4. *Adam lives in the Garden of Eden and eats only apples. Time in the garden is discrete* ($t = 1, 2, ...$) *and apples are eaten only in discrete units. Adam possesses preferences over the set of streams of apple consumption. Assume that Adam*

   - *Likes to eat up to 2 apples a day and cannot bear to eat 3 apples a day.*

- *Is impatient. He will be delighted to increase his consumption at day t from 0 to 1 or from 1 to 2 apples at the expense of an apple he is promised a day later.*
- *At any period in which he does not have an apple, he prefers to get one apple immediately in exchange for two apples tomorrow.*
- *Cares only about his consumption in the first 120 years of his life.*

Show that if (poor) Adam is offered a stream of 2 apples starting in period 19 for the rest of his life (assuming he does not expect to live more than 120 years), he would be willing to exchange that offer for one apple given right away.

The original stream of consumption is $c_t = \begin{cases} 0 & \text{for } t = 1, .., 17 \\ 2 & \text{for } t = 18, .., 43800 \end{cases}$. Given Adam's preferences, it follows that

$$c_t \prec c'_t = \begin{cases} 0 & \text{for } t = 1, .., 16 \\ 1 & \text{for } t = 17 \\ 0 & \text{for } t = 18 \\ 2 & \text{for } t = 19, .., 43800 \end{cases} \quad \prec c''_t = \begin{cases} 0 & \text{for } t = 1, .., 16 \\ 1 & \text{for } t = 17, 18 \\ 0 & \text{for } t = 19 \\ 2 & \text{for } t = 20, .., 43800 \end{cases}$$

$$\prec ... \prec c'''_t = \begin{cases} 0 & \text{for } t = 1, .., 17 \\ 1 & \text{for } t = 18, .., 43799 \\ 0 & \text{for } t = 43800 \end{cases} \quad \prec c''''_t = \begin{cases} 0 & \text{for } t = 1, .., 16 \\ 2 & \text{for } t = 17, .., 21908 \\ 0 & \text{for } t = 21909, ..., 43800 \end{cases}$$

The first, second and third preferences are a result of assumption 3, and the forth preference is a result of assumption 2 (observe that the number of apples is the same in c'''_t and c''''_t). Repeat this process 16 more times and get the result that

$$c_t \prec c^*_t = \begin{cases} 2 & \text{for } t = 1 \\ 0 & \text{for } t = 2, ..., 43800 \end{cases}$$