*Econometrica*, Vol. 47, No. 2 (March, 1979)

# A NOTE ABOUT THE "NOWHERE DENSENESS" OF SOCIETIES HAVING AN EQUILIBRIUM UNDER MAJORITY RULE[1]

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## 1. INTRODUCTION

THE PARADOX OF VOTING brought many investigators to the search for sufficient conditions on the preference relations in a decision system with majority rule which would ensure the non-emptiness of the core (existence of equilibrium points). But works such as [**1, 2, 3, 4, 6, 8, 9**, and **10**] lead to the impression that cases of a non-empty core are rare. Plott in [**9**], assuming preferences are differentiable, and Kramer in [**6**], assuming a quasi-concave and differentiable utility function, derive characterizations which seem to prove that we can always make minor perturbations of the preferences to destroy any particular core point. Schofield in [**10**] shows that for "almost all" preference $n$-tuples (relative to a certain topology different from the one used here) there is a dense set of alternatives which are not core points. Schofield also has a severe restriction on the dimension of the space.

The only assumption in this paper is that preference relations are continuous. We place no restrictions on the dimension of the space of the set of possibilities. The topology we will be using is the Kannai topology [**5**]. This topology has been used in many different contexts in mathematical economics during the last years. For example, see [**7**]. The claim of this note is that in this topology the set of social profiles with a non-empty core is a closed set with an empty interior and therefore a nowhere dense set. The nowhere denseness of a set is a criterion of its smallness from the topological point of view.

## 2. THE MODEL AND THE TOPOLOGY

There are two constant elements in this discussion: the *set of voters* $N = \{1 \ldots n\}$ and the *set of possibilities* $A \subset R^m$. It will be assumed that $A$ is a compact, convex set with a non-empty interior. For every $i \in N$ there is a preference relation on $A$ denoted by $\lesssim_i$. We will assume that for every $i$, $\lesssim_i$ is a complete, reflexive, transitive and continuous relation. As usual we will be using the expression "$a \sim_i b$" instead of "$a \lesssim_i b$ and $b \lesssim_i a$," and "$a <_i b$" instead of "$a \lesssim_i b$ and not $b \lesssim_i a$."

An $n$-tuple $\langle \lesssim_1, \ldots, \lesssim_n \rangle$ will be called a *social profile*. The social decision relation will be denoted by $<$ and will be defined in the following way:

$$a < b \qquad \text{iff} \qquad \frac{n}{2} < |\{i \mid a <_i b\}|.$$

This is the "strong" variation of majority rule according to which the society can reject alternative $a$ in favor of alternative $b$ if and only if the number of voters preferring $b$ to $a$ is larger than half of the number of voters in the society. The concept of solution we will examine here is the core. Given a social profile $M$, $M = \langle \lesssim_1 \ldots \lesssim_n \rangle$ the core will be defined as follows:

$$C(M) = \{a \in A \mid \text{there is no } b \in A \text{ such that } a < b\}.$$

Let

$$\theta = \{\lesssim \;\big|\; \lesssim \text{ is a complete, transitive, reflexive, and continuous relation on } A\}.$$

For every $i$ we will define a topology $T_i = \langle \theta, F \rangle$ where $F$ is the Kannai topology on $\theta$; i.e., it is the minimal topology where for every $A \ni x_n \to x$, $A \ni y_n \to y$ and $\theta \ni \lesssim^n \to \lesssim^0$ such that $x <^0 y$, there exists an integer $L$ such that for every $l > L$, $x_l <^l y_l$. (Here also $x <^l y$ iff $x \lesssim^l y$ and not $y \lesssim^l x$.)

[1] This paper is a part of the author's Master's Thesis in Economics prepared under the supervision of Professor M. E. Yaari.

Let us now define the product topology on $\theta^N$, i.e., the topology with the covergence $M^k=\langle\precsim_i^k\rangle_{i=1}^n\to M=\langle\precsim_i\rangle_{i=1}^n$ iff for every $i$ $\precsim_i^k\to\precsim_i$ in $F$.

## 3. THE THEOREM

First, we will prove that the set of social profiles in $\theta^N$ having non-empty core is a closed set in the Kannai topology.

PROPOSITION 1: *If* $M^k=\langle\precsim_i^k\rangle_{i=1}^n\to M=\langle\precsim_i\rangle_{i=1}^n$ *and* $C(M^k)\neq\emptyset$, *then* $C(M)\neq\emptyset$.

PROOF: Let $c^k\in C(M^k)$ and let us assume that $c^k\to c\in A$ ($A$ is compact, so a converging sub-sequence always exists).

If $c\notin C(M)$, then there is a $b\in A$ such that $c<b$, i.e., there is a majority of voters $i_1\dots i_l$, $(n/2<l)$ for whom $c<_{i_j}b$ for all $1\leqslant j\leqslant l$. But $c^k\to c$ and $\precsim_{i_j}^k\to\precsim_{i_j}$ for all $1\leqslant j\leqslant l$ and therefore for large enough $K$, for all $k>K$ and for all $1\leqslant j\leqslant l$, one has $c^k<_{i_j}^k b$, in contradiction to the fact that $c^k\in C(M^k)$.

We will prove the emptiness of the interior of the set of social profiles with a non-empty core in $\theta^N$ using Proposition 2.

PROPOSITION 2: *For every* $M=\langle\precsim_i\rangle_{i=1}^n(3\leqslant n)$ *there are social profiles* $M^k\to M$ *such that* $C(M^k)=\emptyset$.

Combining Propositions 1 and 2 furnishes a proof of the following theorem.

THEOREM: *For* $3\leqslant n$, *the set of social profiles with non-empty core is a closed set with an empty interior (and therefore nowhere dense) in the Kannai topology.*

PROOF OF PROPOSITION 2: Let $\{u_i\}_{i=1}^n$ be continuous utilities representing $\langle\precsim_i\rangle_{i=1}^n$. Their existence is assured by the assumption of the model. For any $0<\varepsilon$ we shall define the *$\varepsilon$-net on $R^m$* as

$$\{X_{i=1}^m\,[l_i\varepsilon,(l_i+1)\varepsilon]\,|\,l_1\dots l_m\text{ integers}\}.$$

The *$\varepsilon$-subnet on $A$* is the set of cubes in the $\varepsilon$-net on $R^m$ which are subjects of $A$. The union of the cubes in the $\varepsilon$-subnet on $A$ will be denoted by $A_\varepsilon$.

The $u_i$ are uniformly continuous on $A$: therefore for every $\delta>0$ there is $\varepsilon_1(\delta)$ such that if $\|x-y\|_1<\varepsilon_1(\delta)$, then for all $i$ $|u_i(x)-u_i(y)|<\delta$.

Int $A\neq\emptyset$; therefore for small enough $0<\varepsilon$, $A_\varepsilon\neq\emptyset$.

The convexity guarantees that for all $x\in A$, $d_1(x,A_\varepsilon)\to 0$ as $\varepsilon\to 0$ ($d_1$ = the distance in $\|\cdot\|_1$). Because $A$ is compact for all $\varepsilon_1>0$ there is $\varepsilon_2(\varepsilon_1)$ such that for every $x\in A$ and for every $\varepsilon\leqslant\varepsilon_2(\varepsilon_1)$, $d_1(x,A_\varepsilon)<\varepsilon_1$.

Now we will define a sequence of social profiles that converges to $M$.

For all $\delta>0$ define $U_i^\delta$, a function on $A$ as follows: Let

$$\varepsilon(\delta)=\varepsilon_2(\varepsilon_1(\delta)).$$

For brevity we will write $\varepsilon$ instead of $\varepsilon(\delta)$. For each cube $c$ in the $\varepsilon$-subnet of $A$ denote the center of the cube by $t^c$ and let $M_i^c=\max_{x\in c}u_i(x)$. We will now introduce the auxiliary function $U_0$, $U_1$, $U_2$ shown in Figure 1. We define $U_i^\delta$ on the set

$$T^c=[t^c-(\varepsilon/8,0,\dots,0),\,t^c+(\varepsilon/8,0,\dots,0)]$$

by

$$U_i^\delta(x)=M_i^c+U_{i(\mathrm{mod}\,3)}(x-t^c).$$

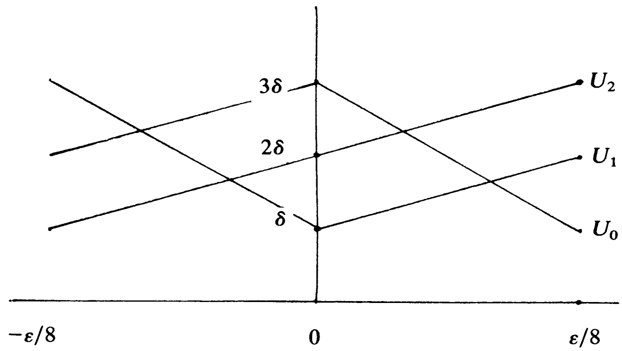


FIGURE 1

Define $S(a, \lambda) = \{x \mid \|x - a\|_1 \leq \lambda\}$. We will extend $U_i^\delta$ to the cube $S(t^c, \varepsilon/4)$ by

$$U_i^\delta(x) = M_i^c + \delta + \frac{d(x, \text{boundary of } S(t^c, \varepsilon/4))}{d(x, T^c) + d(x, \text{boundary of } S(t^c, \varepsilon/4))}[U_i^\delta(y(x)) - M_i^c - \delta],$$

where $y(x)$ is the point in $T^c$ nearest to $x$. On the boundary of $c$ we identify

$$U_i^\delta(x) = U_i(x)$$

and in the region between the boundary of $c$ and $S(t^c, \varepsilon/4)$ we define $U_1^\delta$ by linear extrapolation. Outside $A_\varepsilon$ we define $U_i^\delta(x) = U_i(x)$.

The $U_i^\delta$ are continuous and they induce $\lesssim_i^\delta \in \theta$ by $x \lesssim_i^\delta y$ iff $U_i^\delta(x) \leq U_i^\delta(y)$.

We now show that $C(M^\delta) = \emptyset$.

For each $x \in A$ inside a segment $T^c$ in a cube $c$ in the $\varepsilon = \varepsilon(\delta)$ subnet on $A$ it is easy to verify that there exist $i, j \in \{1, 2, 3\}$ $(i \neq j)$ such that all the members of $\{h \mid h(\text{mod } 3) = i$ or $h(\text{mod } 3) = j\}$ prefer another point on the segment. For $3 \leq n \neq 4$ this set contains more than $n/2$ voters. (The treatment in the case $n = 4$ is similar, using other functions for perturbing the preference—see Figure 2.)

For any other point $x \in c$, $y(x)$ is preferred to $x$ by all the voters.

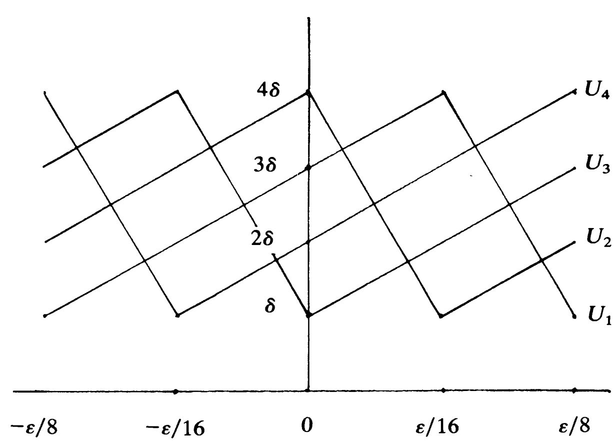


FIGURE 2

For $x \in A - A_\varepsilon$, $d_1(x, A_\varepsilon) \leq \varepsilon_1(\delta)$. Let $c$ be a cube subset of $A_\varepsilon$ which is the closest to $x$. $d(x, c) \leq \varepsilon_1(\delta)$ and, therefore, for all $i \in N$, $U_i(x) - M_i^c < \delta$ and $U_i^\delta(t^c) \geq M_i^c + \delta > U_i(x) = U_i^\delta(x)$.

It remains to be shown that $M^\delta \rightarrow_{\delta \rightarrow 0} M$. It is sufficient to show that for every $i \in N$ and for every $S_1, S_2 \subset A$ ($S_1, S_2$ closed disjoint spheres in the relative topology on $A$, satisfying that for every $x_1 \in S_1$, $x_2 \in S_2$ we have $x_2 <_i x_1$), there exist $\delta_0$ such that for every $0 < \delta \leq \delta_0$ and for all $x_1 \in S_1$ and $x_2 \in S_2$ it is true that $x_2 <_i^\delta x_1$.

Since $S_1$ and $S_2$ are closed and disjoint we can choose

$$0 < \delta_0 < \tfrac{1}{3} \min_{x_1 \in S_1} \min_{x_2 \in S_2} U_i(x_1) - U_i(x_2).$$

For every $\delta \leq \delta_0$,

$$\max_{x \in S_2} U_1^\delta(x) < \max_{x \in S_2} U_i(x) + 3\delta < \min_{x \in S_1} U_i(x) \leq \min_{x \in S_1} U_i^\delta(x)$$

and $x_2 <_i^\delta x_1$ for all $x_1 \in s_1$, $x_2 \in S_2$.

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*Manuscript received January, 1977; revision received November, 1977.*

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