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RANKING THE PARTICIPANTS IN A TOURNAMENT*

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Abstract. A tournament is a finite set whose elements are called players, together with a binary relation called beating which is complete and asymmetric. A ranking of the players is an order-relation on the set of players. A ranking method is a function from the set of tournaments to the set of possible rankings. The ranking method commonly known as the "points system" is characterized by a set of axioms.

1. Introduction. A *tournament* is an ordered pair $\langle N, \to \rangle$, where N is a finite set whose elements are called players, and $\to$ is a binary relation called *beating*. Throughout the paper it is assumed that $|N| \geqq 3$ and that the relation $\to$ is complete and asymmetric; i.e., either $a \to b$ or $b \to a$ but not both. If $a \to b$ we say that *a beats b*. An example is a round robin basketball tournament in which all possible pairs play one game, with ties not allowed. Obviously, a tournament need not be transitive.

How can one rank the set of players, N, in a tournament? A *ranking* of N is a complete, reflexive and transitive relation on N. A *ranking method* $\succsim$ is a function that ascribes a ranking $\succsim(T)$ to any tournament T. We write "$a \succ(T)\, b$" for "$a \succsim(T)\, b$ and not $b \succsim(T)\, a$" and "$a \sim(T)\, b$" for "$a \succsim(T)\, b$ and $b \succsim(T)\, a$".

The *points system* is the ranking method defined by

$$i \succsim(T)\, j \quad \text{if } S_i(T) \geqq S_j(T),$$

where $S_i(T)$ is the number of players that i beats in T. We write $\succsim$ for $\succsim(T)$ and S_i for $S_i(T)$ unless there is a possibility of ambiguity. We will prove that the points system is characterized by the following three axioms:

AXIOM I. *Let T be a tournament, σ a permutation on N, and i and j players. Denote by σT the tournament which relables the players so that $\sigma i \to \sigma j$ in σT if $i \to j$ in T. Then $i \succsim(T)\, j$ iff $\sigma i \succsim(\sigma T)\, \sigma j$.*

This is an anonymity axiom, ensuring that the ranking method does not discriminate against players because of their label.

AXIOM II. *Suppose i and j are distinct players in T and $i \succsim(T)\, j$. Let T' be identical to T, except for the existence of a third player k such that $k \to i$ in T but $i \to k$ in T'. Then $i \succ(T')\, j$.*

This expresses the positive responsiveness of the ranking method with respect to the beating relation.

AXIOM III. *Let i, j, k and l be four distinct players. Suppose T and T' are identical, except that $k \to l$ in T but $l \to k$ in T'; then $i \succsim(T)\, j$ iff $i \succsim(T')\, j$.*

This states that the relative ranking of two players is independent of those matches in which neither is involved.

We now prove that the only ranking method which satisfies these axioms is the points system.

2. The main result.

THEOREM. *The points system is the only ranking method that satisfies Axioms I, II, and III.*

It is obvious that the points system satisfies the three axioms. To prove that it is the only ranking method satisfying them, we utilize the following lemma.

LEMMA. *If a ranking method $\succsim$ satisfies I and III and if i and j are two players in T such that $S_j = S_i$, then $i \sim j$.*

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Proof. Let i, j be two players such that $S_i = S_j$, and $i \to j$. Define the following disjoint sets of players (see Fig. 1):

$$A = \{k \mid i \to k \text{ and } k \to j\},$$

$$B = \{k \mid k \to i \text{ and } j \to k\},$$

$$C = \{k \mid i \to k \text{ and } j \to k\},$$

and

$$D = \{k \mid k \to i \text{ and } k \to j\}.$$

Note that $|B| = |A| + 1$.

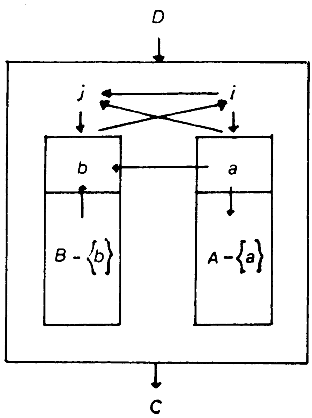

FIG. 1

If X and Y are sets of players, we write $X \to Y$ if all x in X beat all y in Y. In view of Axiom III it may be assumed that $D \to A \cup B$, $A \cup B \to C$ and $D \to C$.

The proof is by induction on $|A|$. If $|A| = 0$ let k be the unique player in B. From Axiom I $j \succsim i$ iff $k \succsim j$ iff $i \succsim k$; thus $i \sim j$. Assume the result is true for $|A| = m - 1 \geqq 0$ and let $|A| = m$. Choose $a \in A$ and $b \in B$. By Axiom III we can assume that $\{a\} \to A \cup \{b\} - \{a\}$ and $B - \{b\} \to \{a, b\}$. Then $i \to a$, $a \to b$ and $b \to i$, while for all $k \notin \{i, a, b\}$ $i \to k$ if and only if $a \to k$. From the case $|A| = 0$ it follows that $a \sim i$. Now $a \to j$, and $|\{k \mid a \to k \text{ and } k \to j\}| = |A - \{a\}| = m - 1$. Furthermore $S_a = S_j$. By the induction hypothesis $a \sim j$. Thus $i \sim j$.

Proof of the Theorem. Let $\succsim$ be a ranking method satisfying Axioms I, II and III, let T be a tournament and i and j players. If $S_i > S_j \geqq 1$, let E be a subset not including j whose $S_i - S_j$ players are all beaten by i. Let T' denote the tournament obtained from T by reversing the result of i with the players in E. Then $i \sim(T')\, j$ by the lemma. Applying Axiom II $|E|$ times we have $i \succ j$.

If $S_i > S_j = 0$, let h be any third player. Let T' be the tournament obtained from T by reversing the results of the encounter between j and h. Then $S_i(T') \geqq S_j(T') \geqq 1$ so $i \succsim(T')\, j$. If $j \succsim i$ then by applying Axiom II we have $j \succ(T')\, i$, and hence $i \succ j$.

3. Remarks. (A) The three axioms are independent, as is shown by the following three examples:

(1) Let $\succsim$ be defined by

$$i \succsim j \quad \text{if } S_i > S_j \quad \text{or } S_i = S_j \text{ and } j \neq 1.$$

This ranking method is the points system modified in cases of equality of scores in favor of player 1. It satisfies II and III but not I.

(2) Let $\succsim$ be defined by

$$i \succsim j \quad \text{if } S_i \leqq S_j.$$

This ranking method satisfies I and III but not II.

(3) Let $\Phi_k = \sum_{k\to h} (S_h + 1)$, and define

$$i \succsim j \quad \text{if } \Phi_i \geqq \Phi_j.$$

It is easy to check that $\succsim$ satisfies I and II but not III.

(B) The requirement that the set of players be finite is necessary for the existence of a ranking method satisfying Axioms I, II and III, as is shown by the following example.

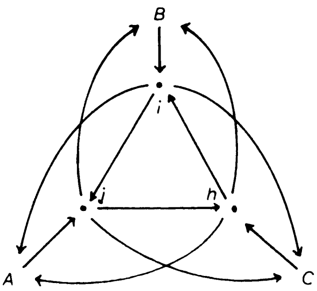

FIG. 2

Let T be the tournament described by Fig. 2. The sets A, B, C and $\{i, j, h\}$ are a partition of the set of players. A, B and C are countably infinite sets. Assume $\succsim$ is a ranking method satisfying Axioms I and III. Let σ be a one-one correspondence, $\sigma: N \to N$, such that $\sigma(i) = j$, $\sigma(j) = h$, $\sigma(h) = i$, and further $\sigma(C) = B$, $\sigma(B) = A$ and $\sigma(A) = C$. From Axiom I it follows that $i \succsim(T)\, j$ if and only if $j \succsim(\sigma T)\, h$ and if and only if $h \succsim(\sigma^2 T)\, i$. For any player k in the set $\{i, j, h\}$ and for any player l, $k \to l$ in T if and only if $k \to l$ in σT and if and only if $k \to l$ in $\sigma^2 T$. Thus from Axiom III

$$j \succsim(T)\, h \quad \text{iff } j \succsim(\sigma T)\, h$$

and

$$h \succsim(T)\, i \quad \text{iff } h \succsim(\sigma^2 T)\, i.$$

Hence $i \sim(T)\, j$.

Let τ be a one-one correspondence, $\tau: N \to N$, such that $\tau(B \cup \{h\}) = B$, $\tau(C) = C \cup \{h\}$, $\tau(A) = A$, $\tau(i) = i$ and $\tau(j) = j$. From Axiom I and $i \sim(T)\, j$ it follows that $i \sim(\tau T)\, j$. Let T' be the tournament identical to T except that $i \to h$ in T'. The results of the players i and j in τT and in T' are the same, so from Axiom III and $i \sim(\tau T)\, j$ we have $i \sim(T')\, j$. Axiom II cannot be satisfied since $i \sim(T)\, j$ and Axiom II together imply $i \succ(T')\, j$ which is contrary to $i \sim(T')\, j$.

(C) In graph theory the concept of a tournament is a complete oriented graph (see [2] and [3]). The problem of ranking the participants in a tournament arises in statistics while considering the method of paired comparisons (see [1] and [3]). The problem is also structurally similar to social choice problems (see for example [5]). The relation to social choice is brought out in [4].

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