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Repeated Two-Player Games with Ruin[1])

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Abstract: The class of repeated two-player games (with long-run average payoff criterion) is extended to accommodate initial holdings of wealth and the possibility of ruin. Equilibria of these games are studied under the assumption that each player regards his own ruin as the worst possible outcome of the game and his opponent's ruin as the best possible outcome.

1. Introduction

In recent years increasing effort has been devoted to the study of equilibria of infinitely-repeated games. Interest in this subject probably stems from the recognition that cooperative behavior is widely observed in real-life situations which do not involve binding agreements, that such behavior occurs in equilibria of noncooperative games when the indefinite future is a factor in players' preferences, and that infinite repetition of a single game is a relatively uncomplicated model in which co-operative play in equilibrium can be observed.

In modelling infinitely-repeated games one is immediately faced with the question of how a player's preferences over infinite sequences of payoffs is to be specified. One possibility is for the player to maximize some form of long-run average of his own payoff sequence, thereby implicitly assigning all of the weighting to the tail of the sequence. This is the choice in much of the literature [see, for example, *Aumann,* 1959; *Aumann/Shapley; Kohlberg,* 1975; *Rubinstein,* 1977, 1980]. (Actually, not all of these references are concerned exactly with repetition of a single game. In some the informational conditions may change as time advances, but in all cases there is significant similar structure attached to the possible games at each stage.) In all of these papers the assumed preferences of the players give no weight to the payoffs received at any particular stage of the game, and this feature has profound influence on the structure of the equilibrium (or perfect equilibrium) set.

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Another part of the literature [e.g., *Cave*] avoids the all-weight-in-the-tail objection by assuming maximization of a geometrically weighted average of the player's own payoff sequence. This "discounting" approach has many desirable features, but it still posits a significant restriction about the form of preferences.

In the present paper we study repeated games in which "impatience" in preferences is assumed to arise from the possibility of ruin. What we have in mind is that in many situations (primarily economic ones) permanent effects can occur in the "short-run" (like economic ruin, preemption, elimination of competition, etc.). These effects can be of overwhelming importance to players: and only after they are taken account of can consideration be given to, say, the long-run average of the payoff sequence. There is a small literature on games involving ruin [e.g., *Milnor/Shapley; Shubik; Shubik/Thompson*], and there has also been some attention to the related subject of repeated games with absorbing states [e.g., *Blackwell/Ferguson; Kohlberg,* 1974]; but no general picture has yet emerged concerning the general structure of the equilibria (or classes of equilibria) which can come about in such games. The goal of the present paper is to make some progress in that direction.

More specifically we shall consider here the special case of two players, starting from initial positions of wealth, repeating the same "stage game" infinitely often at discrete points of time. The wealth positions of the players are altered after each repetition by the addition (or subtraction) of the payoffs in the stage game to (from) the previous-period wealth positions. A player's ruin occurs if his wealth becomes nonpositive before his opponent's does. (We adopt the following convention in the event that both players' wealths first become nonpositive after the same stage. Assume that the payoffs from any stage game are distributed to the players uniformly in time between the successive plays of the stage game. Updating wealths continuously, if one of the players' wealth positions hits zero before his opponent's does he is the ruined player; if both hit zero simultaneously both are ruined.) Each player's least preferred payoff sequences (in the grand "ruin game") are those in which he is ruined. Each player's most preferred sequences are those in which his opponent alone is ruined. In between are all those sequences which ruin neither player, and these last sequences are ordered relatively according to the limit of the player's own sequence of average payoffs. This preference structure is admittedly quite special. What we are attempting to capture is the situation in which ruin forces a player to withdraw to some other game which is surely less favorable to him in the long-run than any sequence of payoffs in the present game and in which the ruin of his opponent leaves a player in a position for the long-run which is more favorable than any sequence of payoffs in the present game. For example, as a monopolist a player might expect earnings worth more than anything he could hope to achieve as a duopolist; and any short-run costs incurred in becoming a monopolist could be more than offset by the future benefits. (To extend the assumed preferences to games with more than two players, some choices would have to be made from among various alternatives. We shall refrain from a discussion of such options in this paper.)

The preferences described above are purely ordinal. In this paper no cardinal assumptions on preferences will be necessary, since all uncertainty is ruled out. There are no chance moves in our games, and no randomization is permitted by the players either within a stage game or across stages.

We are interested in (Nash) equilibria of the two-player ruin games described above. Although a complete characterization of such equilibria is not achieved in this paper, we do produce results which reveal a great deal of simple, interpretable geometric structure in a subset of the equilibria for these games. These equilibria are in some respects similar to the familiar equilibria of the "Folk Theorem" [see *Aumann*, 1976] in ordinary repeated games. In the equilibrium strategies of this paper (as in the "Folk Theorem") the players keep track of relatively little information about the history of play. In other respects, however, our equilibria exhibit features which seem new, and the striking new piece of information which the players monitor in these strategies is the ratio of their current wealths. The way in which this statistic is used by the players is sometimes subtle, but we suspect that it is not unrealistic.

The rest of the paper is organized as follows. Notation and definitions for the model are presented in Section 2. In Section 3 those ruin games are characterized in which one of the players can bring about the ruin of his opponent alone, no matter what the opponent does. In Section 4 the approachability idea of *Blackwell* [1956] is used to provide necessary conditions for an arbitrary positive pair of numbers to be the long-run average payoffs of some equilibrium in which neither player is ruined. In Section 5 sufficient conditions are established which are similar to but not quite the same as the necessary conditions. Still (as pointed out in Section 8) these conditions together generalize the "Folk Theorem" when it is applied to our setting. Section 7 is devoted to a constructive proof of existence of equilibrium; but this existence result requires some extra assumptions, and Section 6 is devoted to some preliminary lemmas as well as to examples which do not satisfy the assumptions and do not possess equilibria. Section 8 contains a discussion of several features of the model and results.

2. The Model

Let $\Gamma = (X, Y, u, v)$ be a two-person game in normal form. In Γ, the players are named 1 and 2; the respective action sets are X and Y, each of which is a compact subset of some topological space; and the respective payoff functions are u and v, each of which is a function from $X \times Y$ into $\mathbf{R}$ (the reals) assumed continuous in the product topology. Let $M_0 = (K_0, L_0)$ be an element of $\mathbf{R}^2_{++}$ (the subset of $\mathbf{R}^2$ composed of strictly positive pairs). We are concerned here with repetitions of Γ, payoffs from which are added to initial stocks of wealth M_0.

A *ruin game* is specified by a pair (Γ, M_0) as above and is composed as follows. A strategy for Player 1 in (Γ, M_0) is a sequence $f = (f_1, f_2, \ldots)$ where f_1 is simply an element of X and, for $n \geqslant 2, f_n$ is a function from $(X \times Y)^{n-1}$ into X (no randomization is permitted). F is the set of all such strategies for Player 1. The strategy set G for Player 2 is defined similarly. If (f, g) is any strategy pair in (Γ, M_0), $\sigma_n\ (f, g)$ denotes the pair of actions played at the n-th repetition of Γ, i.e., $\sigma_1\ (f, g) = (f_1, g_1)$, and, inductively,

$$\sigma_n(f,g) = (f_n(\sigma_1(f,g), \ldots, \sigma_{n-1}(f,g)), g_n(\sigma_1(f,g), \ldots, \sigma_{n-1}(f,g)))$$

for $n = 2, 3, \ldots$ Player 1's wealth after the n-th repetition of Γ is $K_n(f,g) = K_{n-1}(f,g) + u(\sigma_n(f,g))$, for $n \geqslant 1$ ($K_0(f,g) \equiv K_0$). Similarly, $L_n(f,g)$ is Player 2's wealth after the n-th repetition of Γ; and $M_n(f,g) = (K_n(f,g), L_n(f,g))$. For the pair (f,g), the ruin time $N = \inf \{n \geqslant 1: M_n(f,g) \notin \mathbf{R}^2_{++}\}$. If $N < \infty$, let

$$\bar{\lambda} = \min \{\lambda \in (0,1]: \lambda M_N(f,g) + (1-\lambda) M_{N-1}(f,g) \notin \mathbf{R}^2_{++}\};$$

and if

$$\bar{\lambda} K_N(f,g) + (1-\bar{\lambda}) K_{N-1}(f,g) = 0,$$

we say that (f,g) *ruins* Player 1. Similarly for Player 2. (Note that both players are ruined only if $N < \infty$ and

$$\bar{\lambda} M_N(f,g) + (1-\bar{\lambda}) M_{N-1}(f,g) = 0.)$$

In words, draw the line between M_{N-1} and M_N. If it passes through the origin, both players are ruined; if it first crosses the horizontal axis, only player 2 is ruined; if it first crosses the vertical axis, only Player 1 is ruined.

In any ruin game, the players have complete, transitive preference orderings over the set of all strategy pairs which either lead to ruin of one (or both) players or generate Cesaro summable payoff sequences in the repetitions of Γ. Each player's most preferred strategy pairs are those that ruin his opponent but not himself (the player is indifferent among all these). The least preferred pairs are those that ruin himself, regardless of whether they also ruin his opponent (again indifference among all of these). In between are those pairs that ruin neither player, and those pairs are ordered by each player according to the Cesaro limit of his own payoff sequence.

Accordingly, a strategy pair (f^*, g^*) is an *equilibrium* of (Γ, M_0) if any of the following three conditions hold.

1) (f^*, g^*) ruins Player 2; and, $\forall g \in G$, (f^*, g) ruins Player 2.

2) (f^*, g^*) ruins Player 1; and, $\forall f \in F$, (f, g^*) ruins Player 1.

3) Neither player is ruined by (f^*, g^*), the limits

$$u^* = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} u_n(f^*, g^*) \text{ and } v^* = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} v_n(f^*, g^*) \text{ exist, and}$$

i) for every $f \in F$: (f, g^*) does not ruin Player 2 alone, and either (f, g^*) ruins Player 1 alone or

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (u_n(f, g^*) - u^*) \leqslant 0; \text{ and}$$

ii) for every $g \in G$: (f^*, g) does not ruin Player 1 alone, and either (f^*, g) ruins Player 2 alone or

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (v_n(f^*, g) - v^*) \leqslant 0.$$

We say that f^* is a *best strategy* for Player 1 if $\forall g \in G$, (f^*, g) ruins Player 2 *alone*. (Note that this represents a strengthening of Condition 1).) Similarly for Player 2.

Some additional notation and terminology will be helpful in the remainder of the paper. In $\mathbf{R}^2$ the locus of points lying on any straight line through the origin with nonnegative (possibly infinite) slope will often be identified with the angle between the line and the horizontal axis. In Figure 1, $l \in [0, \pi/2]$ is such a line. If a is any point in $\mathbf{R}^2$ and $l \in [0, \pi/2]$, then $a \oplus l$ denotes the line through a parallel to l (see Figure 1). Obviously, $0 \oplus l \equiv l$. For any such $a \oplus l$ let A $(a \oplus l)$ denote the closed half space above and/or to the left of $a \oplus l$ (see Figure 1), and B $(a \oplus l)$ denote the closed half-space below and/or to the right of $a \oplus l$. Similarly SA $(a \oplus l)$ (resp. SB $(a \oplus l)$) denotes the corresponding open half-space strictly above (resp. strictly below) $a \oplus l$. The pair $(u\,(x, y), v\,(x, y))$ will often be denoted $w\,(x, y)$; and $w\,(x, \cdot)$ denotes

$$\{w : w = w\,(x, y) \text{ for some } y \in Y\}.$$

Similarly for $w\,(\cdot, y)$. The set $W \subseteq \mathbf{R}^2$ is 1*-enforceable in* Γ (enforceable by Player 1) if $\exists x \in X$ such that $w\,(x, \cdot) \subseteq W$. Enforceability by Player 2 is defined similarly.

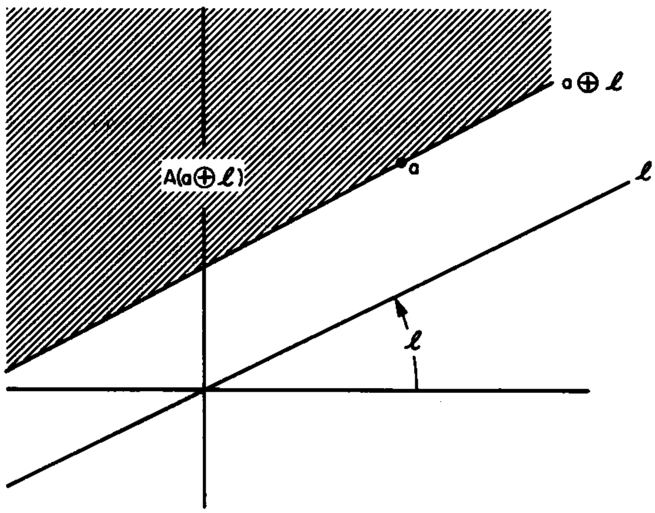

Fig. 1

3. Best Strategies

This section provides a geometric characterization of all ruin games (Γ, M_0) with the property that there exists a best strategy for Player 1. (Evidently, in such games, all equilibria involve the ruin of Player 2.) There is an obvious analogue when the players' names are reversed.

Theorem 1: Let

$$l^* = \sup\left\{ \bar{l} \in \left[0, \frac{\pi}{2}\right] : \forall\, 0 < l < \bar{l},\ SB\,(l) \text{ is 1-enforceable in } \Gamma \right\}.$$

Then Player 1 has a best strategy in (Γ, M_0) if and only if $M_0 \in SB\,(l^*)$.

Before proving Theorem 1 we shall illustrate the argument of the proof by constructing a best strategy for Player 1 in the following example.

Example 1: $X = \{\alpha, \beta, \gamma\}$, $Y = \{\phi, \psi\}$. The functions u and v are read in the customary way from the following table.

	ϕ	ψ
α	(3,4) *d*	(3,4) *d*
β	(–4,–3) *a*	(–4,–3) *a*
γ	(4,1) *c*	(–1,–4) *b*

Tab. 1

(See Figure 2.) In Example 1, it is easy to see that $l^* = \pi/2$. Therefore, according to Theorem 1 Player 1 has a best strategy for every M_0.

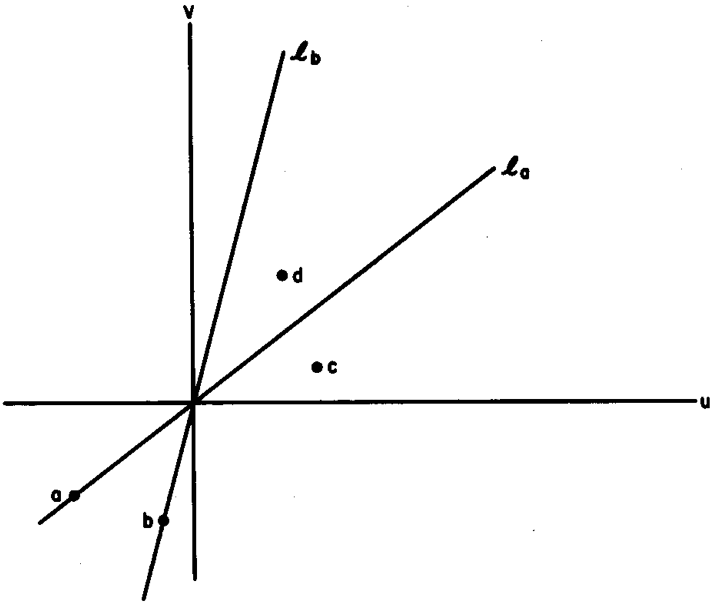

Fig. 2

Case 1: $M_0 \in SB\,(l_a)$. In this case Player 1 simply plays β. M_n moves on the line $(M_0 \oplus l_a)$ in the direction indicated in Figure 3, crossing the u-axis before the v-axis.

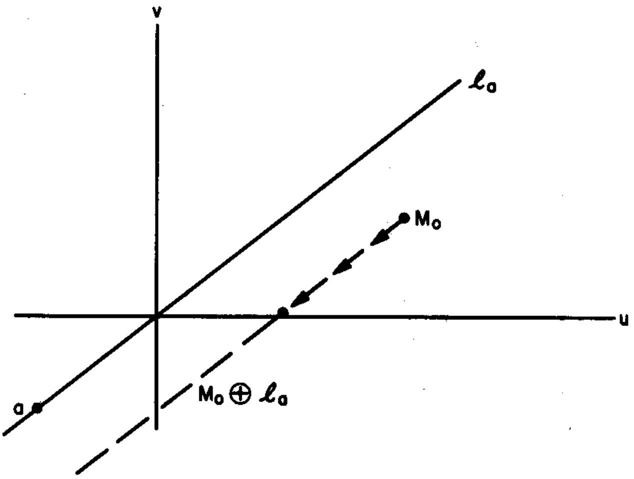

Fig. 3

Case 2: $M_0 \in SB\,(l_b) \cap A\,(l_a)$. In this case Player 1 begins by playing γ. Whatever Player 2 does, M_n is of the form $M_0 + n\,[\lambda b + (1-\lambda)\,c]$. From Figure 4 it is clear that for some N, $M_N \in SB\,(l_a)$ and $M_{N-1} \in A\,(l_a)$. If $M_N \in B\,(0)$, 2 is ruined alone; otherwise $M_N \in SB\,(l_a)$ and 1 continues as in Case 1.

Case 3: $M_0 \in A\,(l_b)$. Here Player 1 begins with α, eventually forcing M_n into $SB\,(l_b)$. The strategy continues as in Case 2.

Proof of Theorem 1: Suppose $M_0 \notin SB\,(l^*)$. Then $l^* \neq \pi/2$, and $SB\,(l^*)$ is not 1-enforceable. Therefore, for every $x \in X$ there is $y\,(x) \in Y$ such that $w\,(x, y\,(x)) \in A\,(l^*)$. For any $f \in F$, choose g such that 2's move at any time is $y\,(x)$ adapted to the x selected by f. Thus $M_n \in A\,(l^*)$, $\forall\, n \geqslant 0$, and 2 cannot be ruined alone.

Suppose $M_0 \in SB\,(l^*)$. Pick $\tilde{l} < l^*$ such that $M_0 \in SB\,(\tilde{l})$. It will be sufficient to show that there exists a finite collection of subintervals $I_k, \ldots, I_1$ in $[0, \tilde{l}]$ and $x_k, \ldots, x_1 \in X$ such that if M_n is on an element of I_j and Player 1 chooses x_j from $(n+1)$ on, then $\exists\, m > n$ such that M_m lies on some line in I_h, with $j > h$ (where I_0 denotes $B\,(0)$), and 1 is not ruined between n and m. For $x \in X$,

$$C_x \equiv \{l \in [0, \tilde{l}]: w\,(x, \cdot) \subseteq SB\,(l)\}$$

is an open interval in $[0, \tilde{l}]$. From the hypothesis of Theorem 1 $\{C_x\}$ is an open cover of $[0, \tilde{l}]$. Let $\{C_{x_k}, \ldots, C_{x_1}\}$ be a minimal subcover. We may assume that

$$l_j = \sup C_{x_j} > \sup C_{x_{j-1}} = l_{j-1} \quad (l_0 \equiv 0.).$$

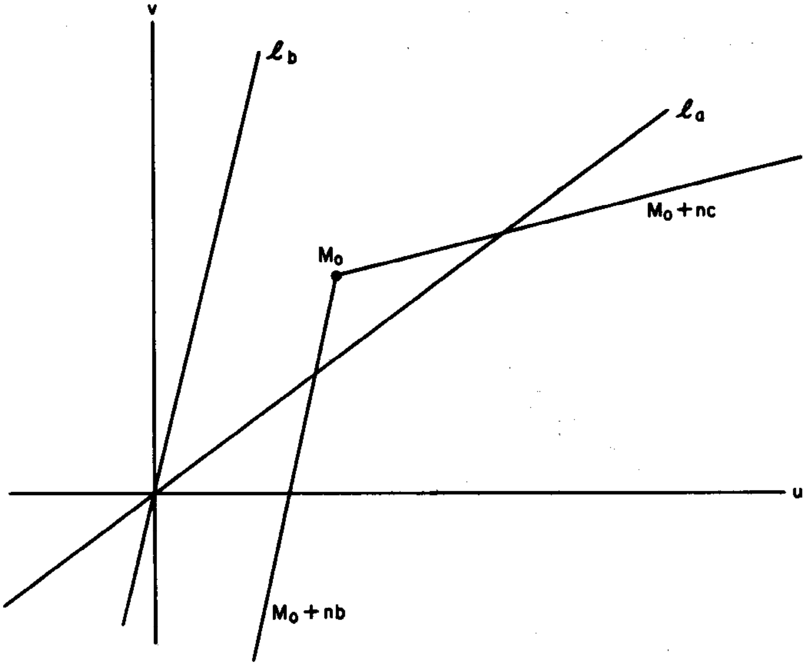
v
ℓ_b
ℓ_a
$M_0 + nc$
M_0
u
$M_0 + nb$

Fig. 4

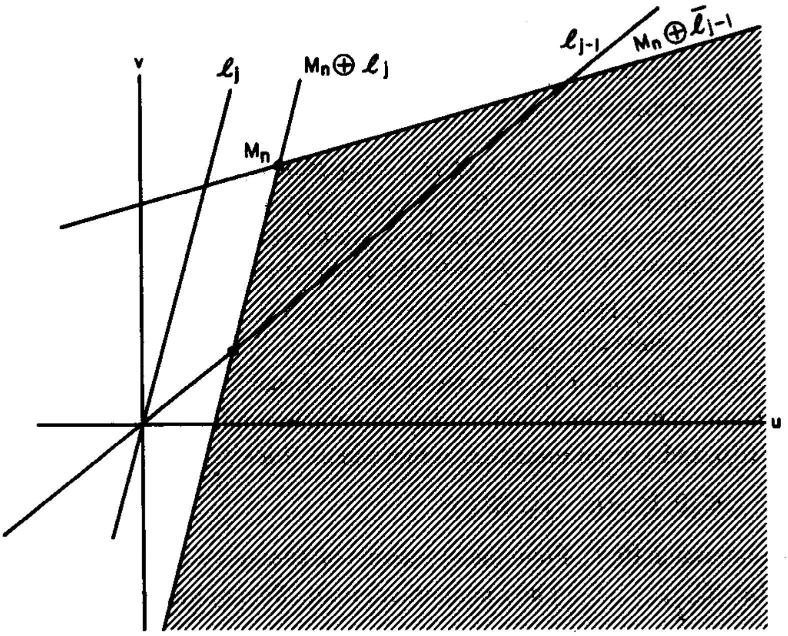
v
ℓ_j
$M_n \oplus \ell_j$
ℓ_{j-1}
$M_n \oplus \bar{\ell}_{j-1}$
M_n
u

Fig. 5

Let M_n lie on an element of $I_j \equiv C_{x_j}$. There exists $\bar{l}_{j-1} < l_{j-1}$ such that $w(x_j, \cdot) \subseteq B(l_j) \cap B(\bar{l}_{j-1})$ and $w(x_j, y)$ is bounded away from the origin uniformly for all $y \in Y$. Now $M_n \in SB(l_j)$; M_m remains in $SB(l_j)$; and therefore 1 is not ruined (see Figure 5). After a finite number of repetitions M_m must enter $SB(l_{j-1})$ and thus must be on a line in I_h, with $j > h$.

Notice that the same argument establishes.

Remark 1: If $\forall\, l \in [\underline{l}, \bar{l}]$, $SB(l)$ is 1-enforceable, then if $M_n \in (\underline{l}, \bar{l})$ Player 1 can drive M_m into $SB(\underline{l})$ without himself being ruined.

4. Necessary Conditions for Nonruining Equilibria

Section 3 was concerned with ruin games in which all equilibria involved the ruin of one player. Here we describe necessary conditions for any nonnegative pair $(u^*, v^*) \in CH$ to be the long-run average payoff at some nonruining equilibrium, where CH is the convex hull of the set of feasible payoff pairs in Γ; i.e.

$$CH = \text{conv}(\{w(x, y) : x \in X, y \in Y\}).$$

The key idea in establishing our necessary conditions is Blackwell's "approachability". First, we need to introduce notation for the average payoff in the first n repetitions of Γ. Accordingly, let

$$\bar{u}_n(f, g) = \frac{1}{n} \sum_{m=1}^{n} u_m(f, g),\ \ \bar{v}_n(f, g) = \frac{1}{n} \sum_{m=1}^{n} v_m(f, g), \text{ and } \bar{w}_n = (\bar{u}_n, \bar{v}_n).$$

Now, let $W \subseteq \mathbf{R}^2$ be a closed set. W is *1-approachable (in repetitions of* Γ*)* if $\exists\, f \in F$ such that $\forall\, \epsilon > 0$, $\exists\, N(\epsilon)$ such that for all $n > N(\epsilon)$ and for all $g \in G$, the Euclidean distance of the point $\bar{w}_n(f, g)$ from the set W is less than ϵ. (Of course, 2-approachability is defined similarly.) The first lemma is a direct application of Theorem 1 in *Blackwell* [1956] to our setting.

Lemma 1: Let $\epsilon \geqslant 0$. Let $W = B((0, -\epsilon) \oplus \bar{l}) \cap B((0, v) \oplus 0)$, and let a be the vertex of W as in Figure 6. Then W is 1-approachable if and only if $\forall\, l \in [0, \bar{l}]$, $B(a \oplus l)$ is 1-enforceable.

Some additional notation and definitions are now necessary. If l is any line in $[0, \pi/2]$ and $(u, v) \geqslant 0 \in \mathbf{R}^2$ then

$$(l.v) \equiv B(l) \cap B((0, v) \oplus 0) \text{ and } (u.l) \equiv A(l) \cap A\left((u, 0) \oplus \frac{\pi}{2}\right).$$

See Figure 7.

Remark 2: If $(l.v)$ is 1-enforceable then it is 1-approachable.

Lemma 2: Let $M_0 \in l_0$. Assume that for every $l \in [l_0, \pi/2]$ the set $(l.v^*)$ is not 1-approachable in repetitions of Γ. Then for every $f \in F$, there exists $g \in G$ such that either Player 1 is ruined in (Γ, M_0) or $\liminf_{n\to\infty} \bar{v}_n (f, g) > v^*$.

Strategy g in the proof of Lemma 2 has a simple form. The idea is that $[l_0, \pi/2]$ is subdivided into a finite collection of intervals. When M_n is on a line in one of these intervals, Player 2 has a response to Player 1 which guarantees for himself strictly more than v^* on average or drives M_n into the next higher interval. Thus either 1 is ruined or 2 receives at least v^* in the long run.

Proof of Lemma 2: For every line $l \in (0, \pi/2]$, let $V(l, v^*)$ denote the vertex of the set $(l.v^*)$ and for every $\tilde{l} \in [l_0, \pi/2]$, let

$$H_{\tilde{l}} = \{l \geqslant \tilde{l} : B(V(l, v^*) \oplus \tilde{l}) \text{ is not 1-enforceable}\}.$$

(In Figure 8, for a typical $x \in X$, $w(x, \cdot) \notin B(V(l, v^*) \oplus \tilde{l})$. If this were the case for all $x \in X$, it would follow that $l \in H_{\tilde{l}}$.) The following are immediate.

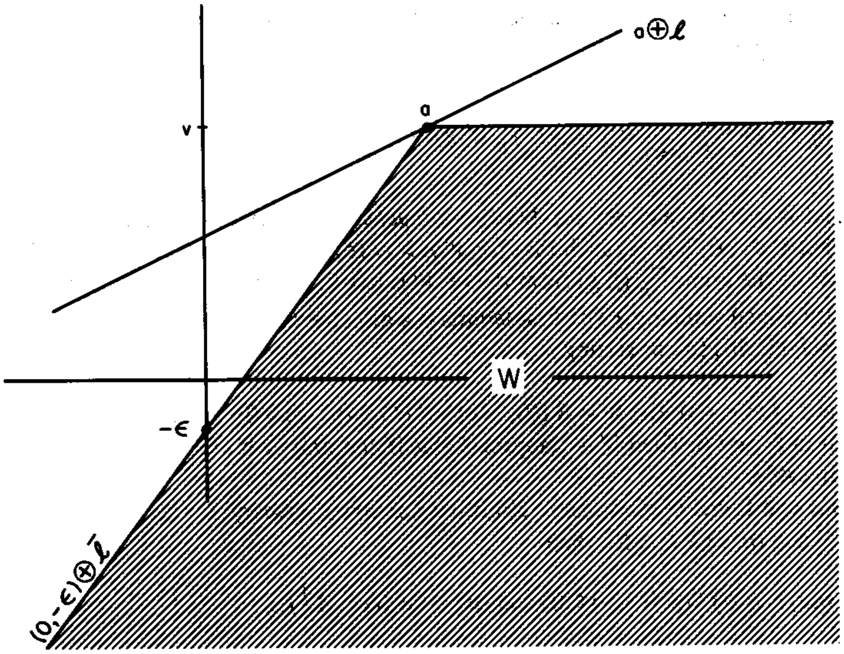

Fig. 6

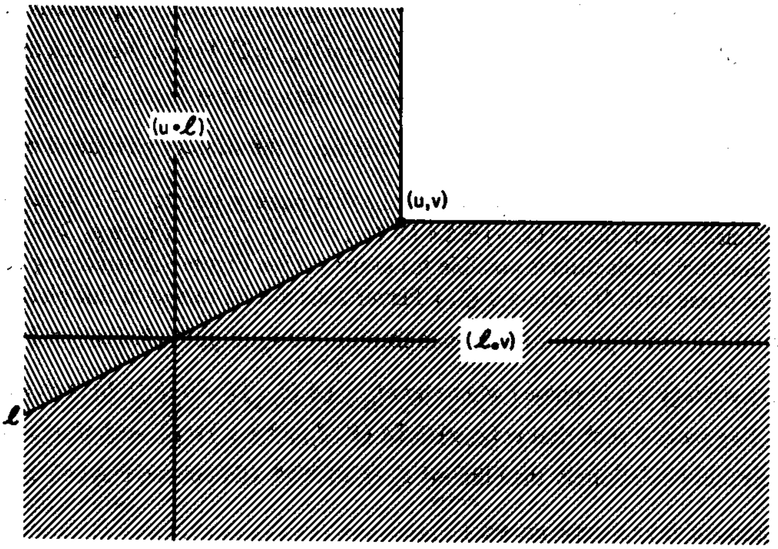

Fig. 7

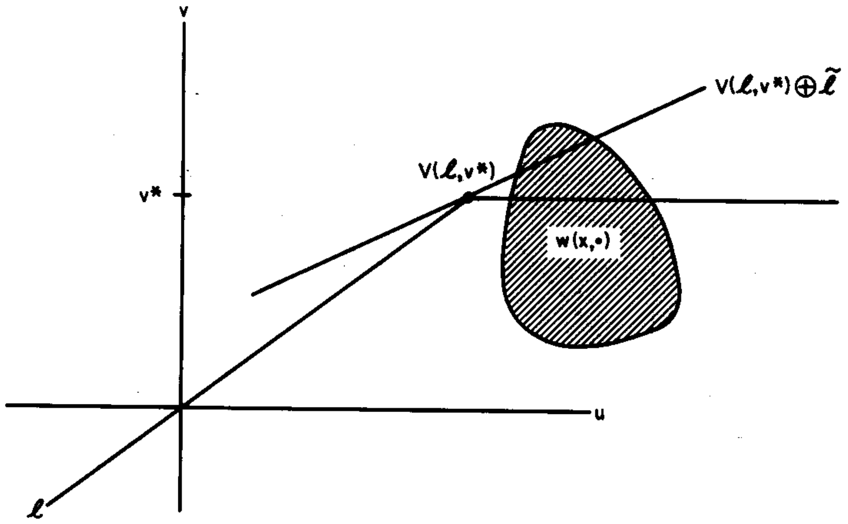

Fig. 8

1) For every $l \in [l_0, \pi/2]$, $\exists\, \tilde{l} \in [0, l]$ such that $l \in H_{\tilde{l}}$ (since l is not 1-approachable; see Lemma 1).

2) If $\pi/2 \neq l \in H_{\tilde{l}}$ then $\exists\ \epsilon > 0$ such that $[l, l + \epsilon) \subseteq H_{\tilde{l}}$.

3) If $l \in H_{\tilde{l}}$, $l > \tilde{l}$, then $[\tilde{l}, l] \subseteq H_{\tilde{l}}$.

4) If $\tilde{l} \in H_{\tilde{l}}$ then $\exists\, l < \tilde{l}$ such that $\tilde{l} \subseteq H_l$.

Now, let $I_{\tilde{l}} = (H_{\tilde{l}} \setminus \{\tilde{l}\})$. From 2) and 3), $I_{\tilde{l}}$ is an (possibly empty) open interval. From 1) and 4), $\{I_{\tilde{l}}\}_{\tilde{l} \in [0,\pi/2]}$ covers the interval $[l_0, \pi/2]$. Hence there is a minimal finite cover $\{I_{\tilde{l}_1}, \ldots, I_{\tilde{l}_k}\}$ (listed in increasing order). For $j = 1, \ldots, k-1$ choose β_j in $(I_{\tilde{l}_j} \cap I_{\tilde{l}_{j+1}})$ and $\beta_k \equiv \pi/2$. For a given strategy f the strategy g required by the Lemma will be constructed in k phases. In the first phase g selects $y \in Y$ in responce to 1's planned x such that $w(x, y) \in SA\ (V(\beta_1, v^* + \epsilon_1) \oplus \tilde{l}_1)$, where $\epsilon_1 > 0$ is small enough that such a selection is possible for every x. (See Figure 9.) Strategy g continues in this phase until M_n enters $A\ (\beta_1)$. Player 2 cannot be ruined in this phase (since $\tilde{l}_1 \leqslant l_0$); and if phase 1 never ends then $\liminf_{n\to\infty} \bar{v}_n\ (f, g) > v^*$. For $j = 2, \ldots, k$, Player 2's strategy g is in phase j whenever $M_n \in (A\ (\beta_{j-1}) \cap SB\ (\beta_j))$. In each such phase j, 2's g picks y such that $w(x, y) \in SA\ (V(\beta_j, v^* + \epsilon_j) \oplus \tilde{l}_j)$. As before if any phase continues indefinitely then $\liminf_{n\to\infty} \bar{v}_n\ (f, g) > v^*$. (Of course, Player 1 can be ruined alone in any of these phases.)

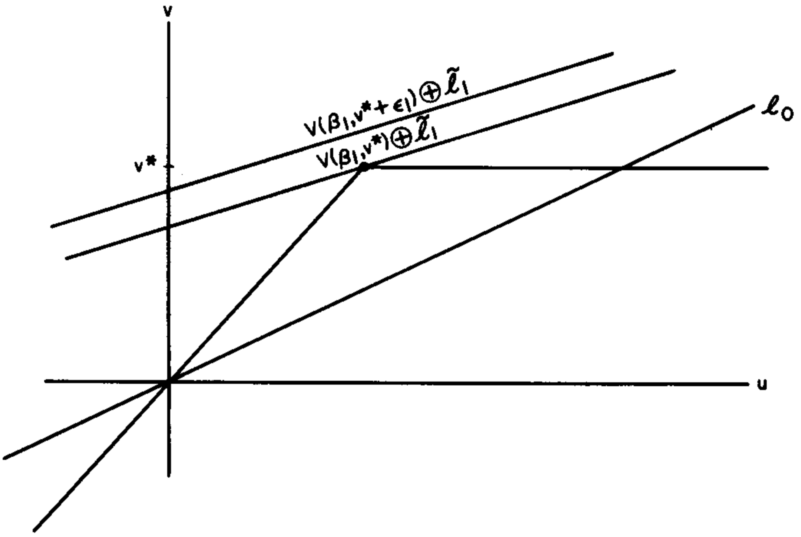

Fig. 9

Theorem 2: Suppose (Γ, M_0) has a nonruining equilibrium with long-run average payoff $w^* = (u^*, v^*)$ on the line l^*. Then

(1) if $l_0 \geqslant l^*$, $\exists\, l \geqslant l_0$ such that $(l.v^*)$ is 1-approachable; and

(2) if $l_0 < l^*$, $\exists\, l \geqslant l^*$ such that $(l.v^*)$ is 1-approachable.

Proof: (1) is immediate from Lemma 2. Assume then that $l_0 < l^*$ and for every $l \in [l^*, \pi/2]$, $(l.v^*)$ is not 1-approachable. It follows that $\exists\, \bar{l} < l^*$ such that for every $l \in [\bar{l}, l^*]$, $(l.v^*)$ is not 1-enforceable; and, indeed, $\forall\, l \in [\bar{l}, \pi/2]$, $(l.v^*)$ is not 1-enforceable. Now, at the equilibrium, $l_n \to l^*$, where $M_n \in l_n$; hence $\exists\, N$ such that $M_N \in A(\bar{l})$. But substituting M_N for M_0 in Lemma 2, we see that 2 can deviate and do better than v^* from N on.

The obvious analogues of these conditions with the players reversed are also necessary for stationary equilibria.

5. Sufficient Conditions for Nonruining Equilibria

We turn now to sufficient conditions for any strictly positive pair $(u^*, v^*) \in CH$ to be the long-run average payoff at some nonruining equilibrium. The gap between the sufficient and the necessary conditions is mainly in the notion of approachability used in this section.

For sets of the form $(l.v)$ and $(u \cdot l)$ we introduce a strengthening of approachability. $(l.v)$ is *1-strongly approachable* if $(l.v)$ is 1-approachable and $SB(l)$ is 1-enforceable. Similarly for 2-strongly approachable with $SA(l)$.

The following are useful consequences of Lemma 1 and the above definition.

Remark 3: If $(l.v)$ is 1-strongly approachable and $l < \pi/2$ then $\exists\, \bar{l} \geqslant l, N$, and f 1-approaching $(\bar{l}.v)$ such that $\forall\, g \in G$ and $\forall\, n > N$, $\bar{w}_n(f, g) \in B(\bar{l})$.

Remark 4: If $(l^*.v^*)$ is 1-approachable and $\exists\, \bar{l} > l^*$ such that $\forall\, l \in [l^*, \bar{l}]$ $SB(l)$ is 1-enforceable, then $(\bar{l}.v^*)$ is 1-strongly approachable.

Theorem 3: Let $0 \leqslant w^* = (u^*, v^*) \in CH$, where w^* is on the line l^*. Suppose M_0 is on the line l_0 and there exist lines l_1 and l_2 in $(0, \pi/2)$ with

$$l_1 > l^* > l_2,\ l_1 > l_0 > l_2,$$

$(l_1.v^*)$ 1-strongly approachable, and $(u^* \cdot l_2)$ 2-strongly approachable, then there exists a nonruining equilibrium with long-run average payoff w^* (See Figure 10.)

Proof: Let w^* be a convex combination of $\{w(x_i, y_i)\}_{i=1}^{I}$. Evidently there exists a sequence $\{z_n\}$ such that $z_n \in \{(x_i, y_i)\}_{i=1}^{I}$ for every n and $\lim (1/N) \sum_{n=1}^{N} w(z_n) = w^*$. Let $\bar{N}$ be such that for all $N > \bar{N}$

$$\frac{1}{N} \sum_{n=1}^{N} w(z_n) \in (SB(l_1) \cap SA(l_2)).$$

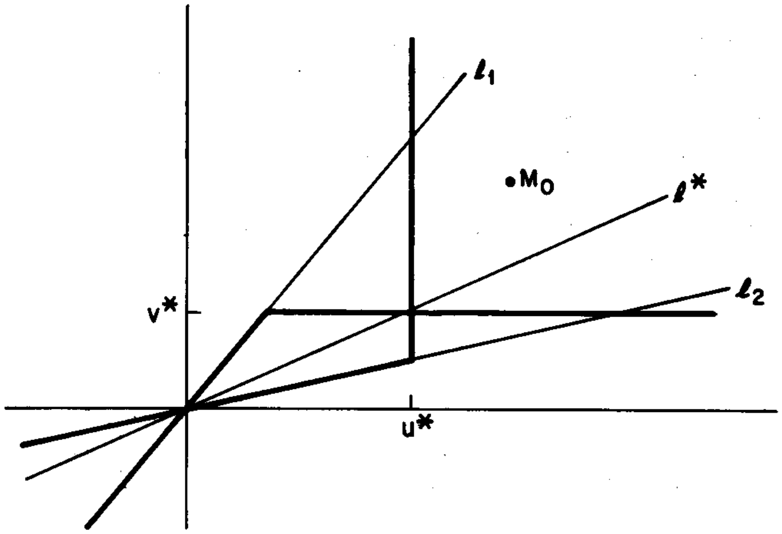

Fig. 10

According to Remark 3 there is a line $\bar{l}_1 > l_1$, a number N_1, and a strategy $f' \in F$ such that f' 1-strongly approaches $(\bar{l}_1 . v^*)$ and $\forall\, n > N_1, \bar{w}_n(f', g) \in B(\bar{l}_1)$. Similarly define g', $\underline{l}_2$ and N_2. Let

$$r = (\bar{N} + N_1 + N_2 + 1) \max_{x,y} \| w(x, y) \| .$$

We are now ready to define a nonruining equilibrium strategy pair (f^*, g^*) which realizes w^*. The strategy pair is composed of three parts.

Part 1: Player 1 begins with an action $\bar{x}$ which 1-enforces $SB(\bar{l}_1)$. Similarly Player 2 begins with $\bar{y}$ which 2-enforces $SA(\underline{l}_2)$. These actions are played until the first time n_0 at which one of the following two events occurs.

- * Neither player has deviated from $(\bar{x}, \bar{y})$ and the distance from M_{n_0} to both $\bar{l}_1$ and $\underline{l}_2$ is greater than r.
- ** Player 2 (resp. 1) has deviated from $(\bar{x}, \bar{y})$ some time in the past, and the distance from M_{n_0} to $\bar{l}_1$ (resp. $\underline{l}_2$) is greater than r.

If * occurs the strategies continue with Part 2 below. If ** occurs the strategies continue with Part 3 below.

Part 2: The players play according to the sequence $\{z_n\}$ until a deviation occurs. After a deviation they continue with Part 3.

Part 3: The players continue with (f', g').

Notice the following:

a) If Player 1 uses $\bar{x}$ on every play, then whatever 2 does M_n remains in $B(\bar{l}_1)$ and after a finite number of repetitions the distance from M_n to $\bar{l}_1$ must be greather than r. Similarly for Player 2 with $\bar{y}$.

b) If $M_{n_0} \in SB(\bar{l}_1)$, the distance from M_{n_0} to $\bar{l}_1$ is greater than $(N_1 + 1) \max_{x,y} \| w(x, y) \|$, and 1 uses f' after n_0, then $M_n \in SB(\bar{l}_1)$ for every n and the lim inf of Player 2's average payoff sequence cannot exceed v^*. Similarly when the players are interchanged.

If there are no deviations, then from a) Part 1 ends after a finite number of iterations and neither player is ruined during Part 1. If there are no deviations, after Part 1 terminates Part 2 continues forever and from b) neither player is ruined. From the defining property of $\{z_n\}$, if Part 2 continues forever (f^*, g^*) realizes w^*.

We will next show that (f^*, g^*) is an equilibrium. Assume first that Player 2 deviates during Part 1. From a) Part 1 must terminate without ruining Player 1; hence Part 3 is entered with 1 playing f'. From b) Player 2 cannot gain from the deviation. Assume next that Player 2 deviates during Part 2. At the time of the deviation M_n must be in $SB(\bar{l}_1)$ and the distance from M_n to $\bar{l}_1$ must be larger than $(N_1 + N_2) \max_{x,y} \| w(x, y) \|$. Applying b), the deviation cannot be profitable for Player 2. Interchanging the players completes the proof.

6. Existence: Preliminary

We are now ready to begin discussing the general question of existence of equilibrium in ruin games. Additional assumptions are evidently necessary; consider, for example, as Γ the game "matching pennies' with no randomization. For any M_0 any fixed strategy in the ruin game by either player will obviously result in his own ruin alone if the opponent responds appropriately. We are led therefore to impose the following.

Assumption 1. X and Y are convex (compact) subsets of vector spaces. The function u is concave in its first argument and convex in its second. The function v is convex in its first argument and concave in its second.

In the familiar situation of X and Y being the mixed extensions of finite sets, the restrictions of Assumption 1 are not strong ones. In our context with no randomization permitted, however, they are. On the other hand, since we hope that our results can be extended ultimately to situations involving randomization and other sources of uncertainty, we shall exploit what this assumption permits.

In fact, although familiar from classical existence results, Assumption 1 is not quite enough to guarantee existence of an equilibrium here. Before presenting a counterexample, however, it is helpful to establish an additional lemma which employs Assumption 1. For fixed Γ consider the following partition of $[0, \pi/2]$. Let

$$T_1 = \{l: SB\,(l) \text{ is 1-enforceable}\}$$

$$T_2 = \{l: SA\,(l) \text{ is 2-enforceable}\}$$

$$T_0 = \left(\left[0, \frac{\pi}{2}\right] \setminus (T_1 \cup T_2)\right).$$

Obviously T_1 and T_2 are disjoint. Furthermore, since T_1 and T_2 are evidently open in $[0, \pi/2]$, T_0 must be closed. Any of these sets may, of course, be empty.

Lemma 3: Under Assumption 1, for every $l \in T_0$, $B\,(l)$ is 1-enforceable and $A\,(l)$ is 2-enforceable.

Proof: Fix $l \in T_0$. For every $y \in Y$, let $\phi\,(y)$ be the set of $x \in X$ which maximize the distance between the parallel lines $(w\,(x, y) \oplus l)$ and l, subject to the constraint that $w\,(x, y) \in B\,(l)$. Since $l \notin T_2$, $\phi\,(y)$ is nonempty; and from Assumption 1 ϕ is a convex-valued (see Figure 11) compact-valued, upper-semicontinuous correspondence. Similarly, let $\psi\,(x)$ denote the set of maximizers of the same distance, subject to the constraint that $w\,(x, y) \in A\,(l)$. The correspondence (ϕ, ψ) mapping $X \times Y$ into itself satisfies the hypotheses of Kakutani's Theorem and therefore has a fixed point. Any fixed point of this correspondence is, however, a pair (x, y) with the property that x 1-enforces $B\,(l)$ and y 2-enforces $A\,(l)$.

Now for the counterexample.

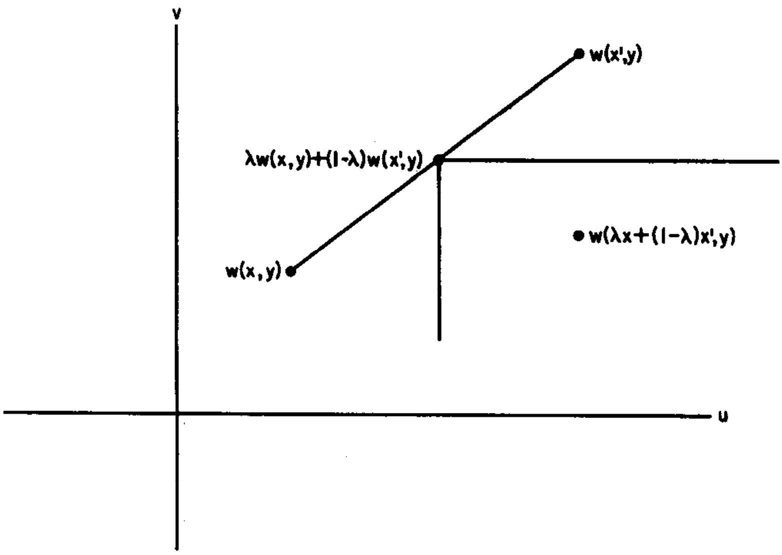

Fig. 11

Example 2: X and Y are both copies of the unit simplex in $\mathbf{R}^3$. The functions u and v are the usual linear extensions of the payoff pairs read from the following symmetric table.

	α'	β'	γ'
α	$-1,-1$	$1,1$	$1,-1$
β	$1,1$	$-1,-1$	$-2,1$
γ	$-1,1$	$1,-2$	$-1,-1$

Tab. 2

We will argue that for this Γ and for M_0 on the line $\pi/4$, (Γ, M_0) has no equilibria. First, for Γ it is a relatively simple matter to check that:

$$T_1 = \left\{l: 0 \leqslant l < \frac{\pi}{4}\right\},\ T_2 = \left\{l: \frac{\pi}{4} < l \leqslant \frac{\pi}{2}\right\}, T_0 = \left\{\frac{\pi}{4}\right\}.$$

The action α 1-enforces $B(\pi/4)$ in Γ. For any $l < \pi/4$ an appropriate convex combination of α and γ may also be seen to 1-enforce $SB(l)$. Considerations of symmetry complete the argument for the structure of T_0, T_1, and T_2. From Theorem 1, therefore, if in (Γ, M_0) M_n ever leaves $\{\pi/4\}$, one of the players has a best strategy beginning from that time. Next, suppose that (f^*, g^*) is an equilibrium of (Γ, M_0). Player 1 has a response to g^* such that he is not ruined; namely, play $(1/2, 1/2, 0)$ until the first time at which g^* calls for γ' with nonzero weight, at this time play α, and from the next repetition on initiate a strategy which ruins Player 2. It follows from symmetry that neither player is ruined at (f^*, g^*) and that M_n remains forever in $\{\pi/4\}$. Therefore at each time n Player 2 places zero weight on γ' and a weight of at least .6 on α' (otherwise Player 1 can respond to bring M_n into $SB(\pi/4)$). The symmetric argument implies that Player 1 always places no weight on γ and weight of at least .6 on α. But any such sequence ruins both players, a contradiction.

In the light of Example 2, more structure is needed to guarantee that equilibria exist. The following assumption will be seen to suffice in Section 7.

Assumption 2: For every $l \in [0, \pi/2]$: if $B(l)$ is 1-enforceable, so is $(B(l) \setminus \{0\})$; and if $A(l)$ is 2-enforceable, so is $(A(l) \setminus \{0\})$.

Without going into details, for any of the spaces of games customarily dealt with in the literature, Assumption 2 holds "generically". With Assumption 2 we can establish some more geometrical facts which will turn out to be useful in the next section.

Lemma 4: Under Assumptions 1 and 2, if $l^* \in T_0$ then there exists $\epsilon > 0$ such that either i) or ii) holds.

i) $l \in (l^*, l^* + \epsilon)$ implies $l \in T_1$
and $l \in (l^* - \epsilon, l^*)$ implies $l \in T_2$.

ii) $l \in (l^*, l^* + \epsilon)$ implies $l \in T_2$
and $l \in (l^* - \epsilon, l^*)$ implies $l \in T_1$.

Proof: Let x^* 1-enforce $B(l^*)$ and y^* 2-enforce $A(l^*)$ (by Lemma 3). From Assumption 2 we may assume that x^* 1-enforces $(B(l^*) \setminus \{0\})$ and y^* 2-enforces $(A(l^*) \setminus \{0\})$. It follows that $w(x^*, y^*)$ is either strictly positive in both components or strictly negative in both components.

Case 1: $w(x^*, y^*) \gg 0$. Consider the intersection of the set $w(x^*, \cdot)$ with l^*. This intersection must consist only of strictly positive pairs; for otherwise for some $y \in Y$, $w(x^*, y)$ would lie in the second quadrant of $\mathbf{R}^2$, a contradiction. From the compactness of Y and the continuity of w, it should be clear that in Figure 12 x^* 1-enforces $SB(l)$, for $l \in (l^*, l^* + \epsilon)$. The lines in this interval are therefore in T_1. A symmetric argument establishes that the other half of i) holds.

Case 2: $w(x^*, y^*) \ll 0$. A similar argument establishes ii).

Lemma 5: Under Assumptions 1 and 2, $[0, \pi/2]$ is structured as follows. T_0 is a finite set. Each open interval between two neighboring elements of T_0 lies entirely in either T_1 or T_2, and these intervals alternate as to their classification. The alternation extends to the half-open (when T_0 is not empty) intervals on either end of $[0, \pi/2]$. (When T_0 is empty $[0, \pi/2]$ is entirely in either T_1 or T_2.)

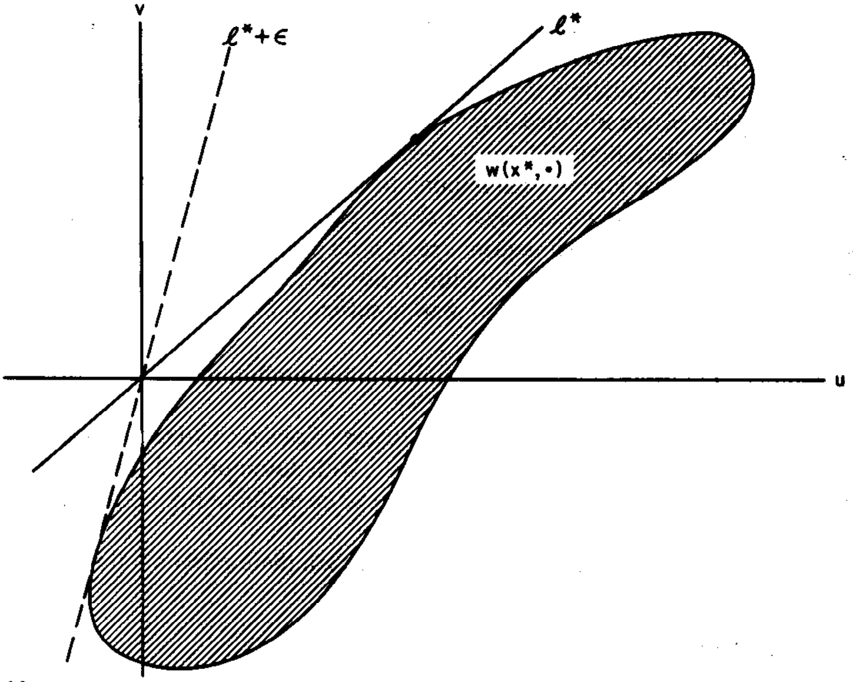

Fig. 12

Proof: All but the finiteness of T_0 follows from Lemma 4 and the fact that T_1 and T_2 are open. If T_0 were not finite, it would contain an infinite sequence of disjoint elements $\{l_j\}$ converging to some $l \in [0, \pi/2]$. But T_0 is closed, hence $l \in T_0$. From Lemma 4 there is a neighborhood of l which is contained in $(\{l\} \cup T_1 \cup T_2)$, a contradiction.

7. Existence: Construction

This section is devoted to a constructive demonstration of

Theorem 4: If (Γ, M_0) satisfies Assumptions 1 and 2, then it has an equilibrium.

The construction involves consideration of several cases. For the most part the equilibrium strategy combination has the following form: if M_0 is in the guaranteed-ruin region of Theorem 1 (or its analogue for Player 2), then (of course) the player possessing a best strategy employs it and the other player's strategy is arbitrary; otherwise, the players employ strategies as in the proof of Theorem 3. For certain starting regions certain modifications in this construction must be made.

Because of the repetitious nature of the construction, we shall be somewhat less formal with this argument than in the rest of the paper.

Case 1: $M_0 \in l_0 \in T_2$. $\pi/2 > l^* \equiv \min \{l \in T_0 : l > l_0\}$. $\exists\ (u^*, v^*) \in l^*$ which is (weakly) Pareto optimal in *CH* (see Figure 13).

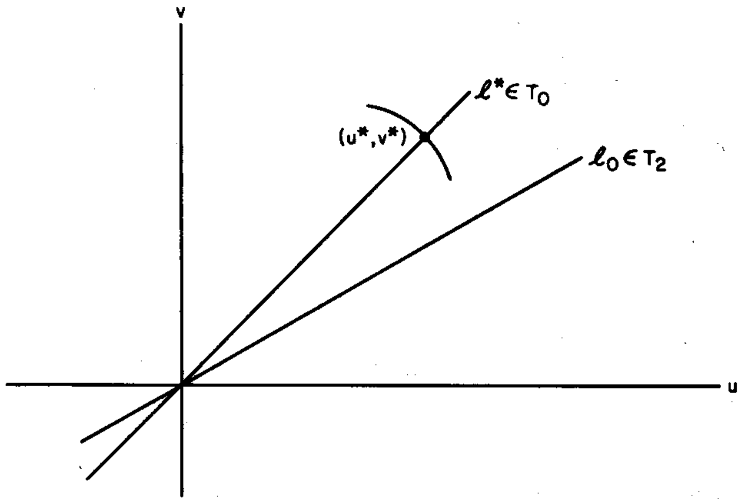

Fig. 13

First note that $(l^*.v^*)$ is 1-enforceable, since $l^* \in T_0$ and (u^*, v^*) is Pareto optimal. By Remarks 2 and 4 and Lemma 4 there exists $l_1 \in T_1$ $(l_1 > l^*)$, such that $(l_1.v^*)$ is 1-strongly approachable. Similarly there exists $l_2 < l_0$ such that $(u^* \cdot l_2)$ is 2-strongly approachable. Theorem 3 applies.

Case 2: $M_0 \in l_0 \in T_0$. $\exists\, \epsilon > 0$ such that: if $l \in (l_0 - \epsilon, l_0)$ then $l \in T_2$. $\exists\, (u^*, v^*) \in l_0$ (weakly) Pareto optimal in *CH*.

The argument is similar to that of Case 1.

Case 3: Same as Case 1 except $\pi/2 = l^* \equiv \min \{l \in T_0 : l > l_0\}$.

In this case, we define the strategies as follows. Player 1 plays throughout a constant x which 1-enforces $B\ (\pi/2)$. Player 2's strategy is in two phases. At first he plays to 2-enforce $SA\ (l_0)$. When L_n becomes sufficiently large that M_{n+1} cannot be driven below l_0, he switches to a best response to x. If Player 1 ever deviates, Player 2 picks a sequence $\{l_k\} \to \pi/2$ and plays in stages such that in the k-th stage he drives M_n into $SA\ (l_k)$ (see Remark 1). Player 2 thereby drives Player 1's average payoff to zero. Player 2 obviously has no profitable deviation.

Case 4: $M_0 \in l_0 \in T_0$. $\exists\, \epsilon > 0$ s.t.: if $l \in (l_0 - \epsilon, l_0)$ then $l \in T_1$.

a) There are no other lines in T_0.
   Here if M_n ever leaves l_0, one of the players can be ruined (Theorem 1). Our construction calls for Player 1 to 1-enforce $B\ (l_0)$ and Player 2 to 2-enforce $A\ (l_0)$ as long as M_n remains in l_0 (otherwise, the appropriate player uses his best strategy).
   By Assumption 2 the ruin of both players results, but neither player can avoid ruin by deviating.

b) $l^* \equiv \min \{l \in T_0 : l > l_0\}$. $\exists\, (u^*, v^*) \in l^*$ (weakly) Pareto optimal in *CH*.
   The equilibrium strategy pair here is similar to that of Case 1 (or, when $l^* = \pi/2$, Case 3). The only change required is at the beginning in order to move into T_2, since Player 2 can only afford to 2-enforce $A\ (l_0) \setminus \{0\}$. Player 1 therefore plays only to 1-enforce $B\ (l^*)$ at the first play while Player 2 2-enforces $A\ (l_0) \setminus \{0\}$. Deviations at the first play are treated in the obvious way. Player 1 cannot gain by attempting to keep M_n on l_0, however, since anything he plays to keep M_n on l_0 results in the ruin of both players.

Case 5: Same as Case 1, but the largest $(u^*, v^*) \in l^*$ is not (weakly) Pareto optimal in *CH*, and all Pareto optima are on lines above l^*.

In Case 5 the difficulty with employing the strategy from Case 1 is that $(u^* \cdot l^*)$ need not be 2-enforceable, and Player 1 might have a profitable deviation from Part 2 of the strategy pair. The construction now requires that (u^*, v^*) be replaced as the target of Part 2 of the strategy pair by a new pair $(\bar{u}, \bar{v})$. The location of $(\bar{u}, \bar{v})$ depends on the location of (u', v') – the "nearest" Pareto optimum to l_0 – and is specified in the subcases below. The first parts of the equilibrium strategies are in all subcases similar to what we have already seen. Details about these parts will be omitted in the interest of brevity.

a) $(u', v') \in T_1$. (See Figure 14.)
   In this subcase $(\bar{u}, \bar{v}) = (u', v')$. There exists $l_1 \in T_1$ such that $(l_1 . v')$ is 1-strongly approachable (Remark 4) and $(u' \cdot l_0)$ is 2-strongly approachable. Theorem 3 applies.

b) $(u', v') \in T_2$. (See Figure 15.)
In this subcase l_0 must lie in an interval of T_2 below the interval containing (u', v'). The point $(\bar{u}, \bar{v})$ is set to either (u'', v'') or (u''', v''') (as in Figure 15) according to which has the larger first component. In the first event the players will use strategies to 1-enforce SB (l) for some $l \in T_1$ greater than l_0 and to 2-enforce SA (l_0). A slight modification in Theorem 3 completes the proof.

c) $(u', v') \in T_0$.
In this subcase $(\bar{u}, \bar{v}) = (u', v')$. (It is immaterial which of the two possible T_1, T_2 configurations surrounds the T_0 line containing (u', v').)

Case 6: Same as Case 5 except all Pareto optima lie on lines below l^*.

a) (u', v') in the same T_2 interval as l_0.
Set $(\bar{u}, \bar{v}) = (u', v')$.

b) (u', v') in the T_0 line which forms the lower bound of the T_2 region in subcase a).
Set $(\bar{u}, \bar{v}) = (u', v')$.

All other subcases are similar to subcases of Case 5 with the roles of the players reversed.

If we add to the above the regions in which best strategies exist and the cases which differ from the above only in the roles of the respective players, we find that all possibilities have been covered.

8. Discussion

When $w(x, y) \geqslant 0$ for all $(x, y) \in X \times Y$, ruin is impossible. In this case our equilibrium conditions are the same as those for ordinary repeated two-person games with long-run-average payoff criterion. For such games the "Folk Theorem" states that (u^*, v^*) is the long-run average payoff to an equilibrium of (Γ, M_0) if and only if: $(u^*, v^*) = (u(x, y), v(x, y))$ for some $(x, y) \in X \times Y$, and

$$u^* \geqslant \min_{y \in Y} \max_{x \in X} u(x, y) \equiv \underline{u}$$

$$v^* \geqslant \min_{x \in X} \max_{y \in Y} v(x, y) \equiv \underline{v}.$$

This theorem is implied by our Theorems 2 and 3 as follows. If $v^* \geqslant \underline{v}$ then $\exists\, x$ such that $v(x, y) \leqslant v^*$ for all $y \in Y$. Thus $((\pi/2).\, v^*)$ is 1-strongly approachable. Similarly $(u^* \cdot 0)$ is 2-strongly approachable. Hence Theorem 3 applies. Conversely, if (u^*, v^*) is the long-run average payoff at an equilibrium then (no matter where l_0 is) $\exists\, l \geqslant l^*$ such that $(l.v^*)$ is 1-approachable. Hence $\exists\, x$ such that $v^* \geqslant v(x, y)$ for all $y \in Y$, i.e. $v^* \geqslant \underline{v}$. Similarly, $u^* \geqslant \underline{u}$.

In the "Folk Theorem" the equilibrium strategies are "grim": deviations are punished (with constant punishment) forever in such a way that the deviator is made

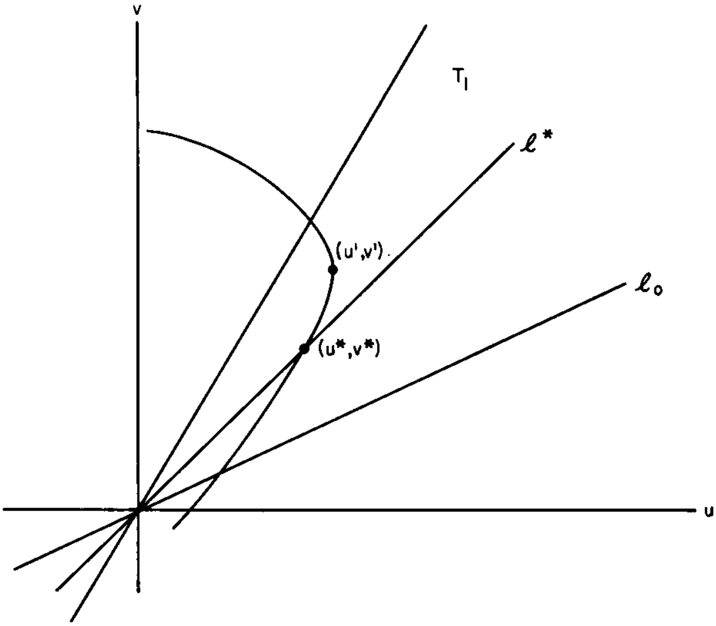

Fig. 14

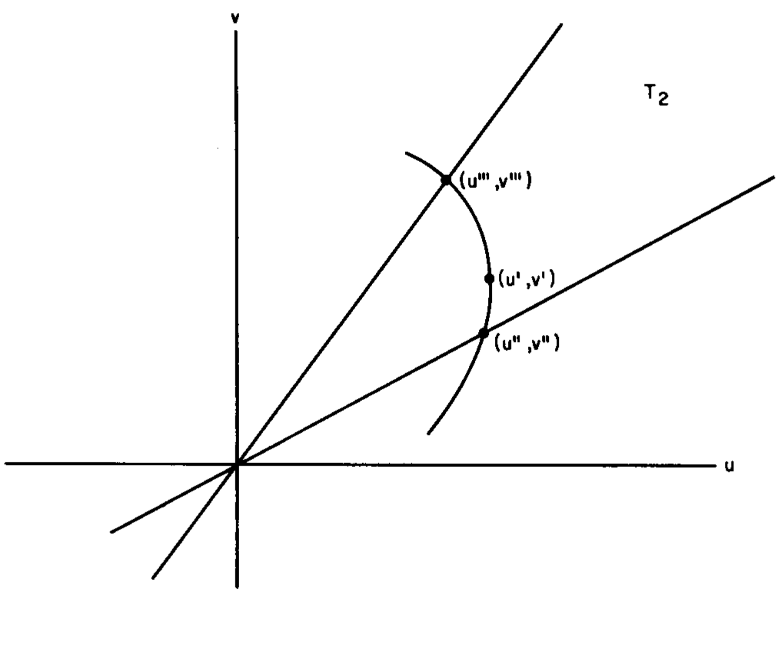

Fig. 15

worse off in the long-run than if he had not deviated. The equilibrium strategies discussed in this paper are similar even when ruin is possible. They are not always exactly as in the "Folk Theorem" in two respects. First, when one player can ruin his opponent, he does. Second, the players must protect themselves against deviations which could force them into their respective ruining regions. This makes necessary a transient phase in general for the equilibria which may, in some cases, not end.

Finally, some remarks about assumptions in our general model. First, the notion of ruin we have adopted is quite special: especially the identification of the ruined player when both wealth positions become nonpositive at the same play. An alternative would be to have both players ruined at such times. This alternative leads to somewhat messier results we think, since strategies in equilibrium must depend in even more detail on the location of M_0. A possibility is to adopt this notion of ruin, but to confine the analysis to situations where $\| M_0 \|$ is sufficiently large. This possibility leads to results quite comparable to those of this paper, although the proofs are a bit more cumbersome.

Another interesting extension of our model would be to allow randomized actions at each time. We have avoided that possibility here because it would necessitate a cardinal description of the players' preferences, and thus assumptions about trade-offs between ruining and various non-ruining outcomes. In addition, the analysis becomes significantly more difficult when a pair of strategies can lead to the ruin of either player with probability strictly between zero and one; see, for example, *Milnor/Shapley* [1957].

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