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THE SINGLE PROFILE ANALOGUES TO MULTI PROFILE THEOREMS: MATHEMATICAL LOGIC'S APPROACH*

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1. INTRODUCTION

In most theorems in Social Choice Theory the multi profile approach is adopted. The social rule is a function F which is defined for every element in a given set of preference profiles $\mathbf{M}$. The set $\mathbf{M}$ is large enough to give substance to some inter-profile axioms which require that if two profiles M and N in $\mathbf{M}$ meet certain conditions, then there are certain dependencies between $F(M)$ and $F(N)$. For example, Arrow's impossibility theorem uses the unrestricted domain axiom — that is, the social choice rule is required to assign an order relation on the set of social alternatives for every preference profile. The inter-profile axiom in that theorem is the independence of irrelevant alternatives.

Recently another approach has begun to be used — the single profile approach. The social rule refers to a single profile M_0. The profile M_0 is "rich" enough to give substance to some intra-profile axioms which require that if some social alternatives meet certain conditions (relative to M_0) then there are certain restrictions on the way that the social rule relates to those alternatives.

The most important result in the single profile approach has been proved in several papers (Kemp and Ng [1977], Parks [1976] and Pollak [1979]). It states that when the preference profile is sufficiently "rich" (well-defined in Pollak [1979]) any order which satisfies Pareto and Single Profile Neutrality conditions is identical to the preferences of one of the individuals' preferences.

Pollak has also given single profile analogues to Sen's theorem characterizing the Pareto extension rule, and to May's theorem characterizing the Majority rule. These results led Pollak to conjecture that

> "It is likely that there are single profile analogues of virtually all the results in the theory of Social Choice" (p. 86),

and Sen [1977] writes

> "As a result of these important contributions it is now clear that the standard inter profile collective choice results have exact intra profile counterparts...." (p. 1564).

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A significant step toward generalizing Parks' results was made by Roberts [1980], but he too did not give a theorem proving the existence of single profile analogues and made do with stating

> "...the method of proof demonstrates how prop. 2 and 3 can be applied to convert multi profile into single profile results."

Is there indeed a single profile analogue to every theorem in Social Choice Theory? What are the conditions on the single profile for the analogue to hold? The works previously mentioned do not provide clear answers to these questions. It seems clear why Parks, Sen and Roberts were satisfied with a descriptive statement, and did not formulate their conjecture rigorously. The conjecture is an assertion concerning the theorems provable in Social Choice Theory and not an assertion within the theory itself. A suitable framework for formulating and proving the conjecture must be a formal theory whose universe of discourse consists of theorems; Mathematical Logic provides precisely such a framework.

The main purpose of this paper is to investigate the conjecture using basic concepts from Mathematical Logic. The theorems in section 5 provide a proof of the conjecture for a class of theorems which is characterized by a specific linguistic structure. Section 5 also includes an example demonstrating that the conjecture is not true in general. Before the theorems can be formulated some preparatory work is necessary. In section 2, the concepts of a profile and a social welfare function are defined in Mathematical Logic's terms. In section 3, the Strong Neutrality axiom is investigated. It is proved that this axiom is equivalent to a certain linguistic property.

Section 4 formulates and generalizes the single profile unrestricted domain property which is the "richness" requirement of the single profile for the analogue to be true. Only then can the main theorem be stated.

Very little has been written so far on the inter-relationship between Decision and Social Choice Theories and language. A subsidiary objective of this paper is to suggest two "linguistic" axioms for a social choice rule — the axiom of definability without quantifiers. In this paper the axioms will be used mainly in a technical manner; further analysis is needed to study their properties.

2. BASIC CONCEPTS

In this section, I will briefly introduce some basic concepts of Mathematical Logic and use them to describe the notions of 'profile' and Social Welfare Function (SWF). (For a brief overview of Mathematical Logic see Crossly [1972]. For a more thorough exposition, see for example, Robinson [1963]). The first concept which I will introduce is that of *language*.

The *alphabet* of a language consists of:

1. Variables (v_1, v_2, v_3,...).
2. Connectives ($\neg$-negation, $\vee$-disjunction, $\wedge$-conjunction, $\rightarrow$-implication

and $\leftrightarrow$-equivalence).
3. Quantifiers ($\forall$-for all, and $\exists$-there exists).
4. Predicate symbols $P_1,\ldots, P_K$.

In order to define a particular language, its predicate symbols must be specified. In this paper, I use only two languages which are suitable for formulating many results about Social Welfare Functions. Let n be a fixed number which is interpreted as the number of individuals in the society. Let $L*$ be the language which contains $n+1$ 2-places predicate symbols $P_1,\ldots, P_{n+1}$. For $1 \leq i \leq n$ the symbol P_i represents the preference relation of individual i and P_{n+1} represents the collective preference relation. We will write P for P_{n+1}. Let L be the language containing only the n predicate symbols $P_1,\ldots, P_n$.

The alphabet of the language is used to construct well-formed formulae (wff) in accordance with the following rules:
1. If x_1 and x_2 are variables $P_j(x_1, x_2)$ is a wff. (We will sometimes write $x_1P_jx_2$ instead of $P_j(x_1, x_2)$.) Such a wff is called an atomic wff.
2. If Φ and Ψ are wffs so are $\neg\Phi$, $\Phi \wedge \Psi$, $\Phi \vee \Psi$, $\Phi \rightarrow \Psi$ and $\Phi \leftrightarrow \Psi$.
3. If Φ is a wff which does not contain the symbols $\exists x$, $\forall x$ (x a variable) then $\exists x(\Phi)$ and $\forall x(\Phi)$ are wffs.

A variable x is said to be *bound* in Φ if $\forall x$ or $\exists x$ appear in Φ. Otherwise it is said to be free. A formula is said to be a *sentence* if it contains no free variables.

Example. The formula $\forall v_1v_2v_3$ $(v_1P_1v_2 \wedge v_2P_1v_3 \rightarrow v_1P_1v_3)$ is a sentence expressing the transitivity of P_1. The formula $\neg\exists v_1(v_1P_1v_2)$ is a wff in which v_2 is a free variable.

And now to the interpretation. A *model* is a $(K+1)$-tuple, $M = \langle A, R_1,\ldots, R_K\rangle$ where A is a set and for all j, R_j is a binary relation on A, i.e., $R_j \subseteq A^2$. We use $|M| = A$ to denote the *universe* of M. If $K=n$ it will be said that M is a model of L, and if $K=n+1$ it will be said that M is a model of $L*$. Thus, where the set of social alternatives is A, a profile $\langle R_1,\ldots, R_n\rangle$ can be identified with a model of L, $\langle A, R_1,\ldots, R_n\rangle$.

Let Φ be a wff in L (in $L*$) with m free variables, and let M be a model of L (of $L*$). The statement "*Φ is satisfied in M under the substitution of* $t_1,\ldots, t_m$" is denoted by $M \models \Phi[t_1,\ldots, t_m]$ and will be defined recursively according to the following rules:
1. $M \models P_j[t_1, t_2]$ if $(t_1, t_2) \in R_j$.
2. $M \models \neg\Phi[t_1,\ldots, t_m]$ if it is not true that $M \models \Phi[t_1,\ldots, t_m]$.
3. $M \models \Phi \wedge \Psi[t_1,\ldots, t_m]$ if both $M \models \Phi[t_1,\ldots, t_m]$ and $M \models \Psi[t_1,\ldots, t_m]$ are true (and analogously for $\Phi \vee \Psi$, $\Phi \rightarrow \Psi$, $\Phi \leftrightarrow \Psi$).
4. If Φ is a wff of the form $\exists x\Psi(x_1,\ldots, x_m, x)$ then $M \models \Phi[t_1,\ldots, t_m]$ if there exists $t \in |M|$ such that $M \models \Psi[t_1,\ldots, t_m, t]$.
5. If Φ is a wff of the form $\forall x\Psi(x_1,\ldots, x_m, x)$ then $M \models \Phi[t_1,\ldots, t_m]$ if for all $t \in M$, $M \models \Psi[t_1,\ldots, t_m, t]$.

When Φ is a formula of L (of L^*), we say that Φ is *universally valid* and write $\models \Phi$ if $M \models \Phi$ for all models M of L (of L^*).

Example. Let $A=\{a, b, c\}$, let $n=3$ and let $M=\langle A, R_1, R_2, R_3\rangle$ be a model of L where

$$R_1 = \{(b, c), (c, a), (b, a)\}$$

$$R_2 = \{(c, b), (b, a), (c, a)\}$$

$$R_3 = \{(c, a), (a, b), (c, b)\}.$$

Given the wff $\Phi(v_1)=\exists v(vP_1v_1 \wedge vP_2v_1 \wedge vP_3v_1)$ then $M \models \Phi[a]$ and $M \models \neg\Phi[c]$.

A Social Welfare Function (SWF) is a function from a set of profiles of preference relations to the set of binary relations. Thus we can consider a SWF to be a function from a set **M** of models of L with a common universe A to the set of binary relations on A. In the following definition, the relations are not required to be preferences and therefore, the notion Social Function is used:

Definition. Let **M** be a set of models of L with a common universe. A function which maps **M** into the set of binary relations on the universe is called an **M**-*Social-Function* (**M**-SF).

Using this terminology the single profile approach can be described as the analysis of $\{M_0\}-SFs$ where M_0 is a model of L. At the core of Social Choice Theory are the axioms on SWF's. Many of the axioms are in fact sentences in $L*$. For example, the Pareto condition is the sentence

$$\forall v_1 v_2[(\bigwedge_{i=1}^{n} v_1 P_i v_2)\rightarrow v_1 P v_2],$$

and the libertarianism axiom is the sentence $\bigwedge_{i=1}^{n} \exists v_1 v_2[v_1 P_i v_2 \rightarrow v_1 P v_2]$. The next definition formalizes the idea of a social function satisfying an axiom:

Definition. Let Φ be a sentence in $L*$. Let **M** be a set of models of L with a common universe A. Let F be an $\mathbf{M}-SF$. For $M=\langle A, R_1,\ldots, R_n\rangle \in \mathbf{M}$ define $M_F=\langle A, R_1,\ldots, R_n, F(M)\rangle$. ($M_F$ is a model of $L*$). F is said to satisfy Φ if for every $M \in \mathbf{M}$, $M_F \models \Phi$.

Example. Let $n=2$, and let **M** be the set of all models of L with a universe A. Let F be defined by $(x, y) \in F(M)$ if for every i $(x, y) \in R_i$ where $M=\langle A, R_1, R_2\rangle$. Let

$$\Phi = \forall v_1 v_2[P(v_1, v_2)\rightarrow(P_1(v_1, v_2) \wedge P_2(v_1, v_2))].$$

Then $\langle A, R_1, R_2, F(M)\rangle \models \Phi$, and F satisfies Φ for all M in M and F satisfies Φ.

Finally we need some more technical terms.

Denfinition. Let $x_1,\ldots, x_m$ be m variables. A *relationship* of $x_1,\ldots, x_m$ is a formula in L of the form

$$\bigwedge_{1\leq i,j\leq m} \left(\bigwedge_{h=1}^{n} \delta_{i,j,h} \cdot P_h(x_i, x_j) \right)$$

where $\delta_{i,j,h}$ is either 1 or -1, $1\cdot\Phi=\Phi$ and $(-1)\cdot\Phi=\neg\Phi$.

In other words, a relationship of $x_1,\ldots, x_m$ is a conjunction of atomic formulae and negations of atomic formulae in those variables such that for every atomic formula either it or its negation (but not both) occurs in the conjunction exactly once. We will consider as identical two relationships which are the same up to a reordering of the conjunction.

PROPOSITION 1. *(for a proof see Robinson* [1963]*). Let $\Phi(x_1,\ldots, x_m)$ be a formula without quantifiers. Then there exists a formula $\Psi(x_1,\ldots, x_m)$ which is a disjunction of relationships of $x_1,\ldots, x_m$ such that*

$$\models \forall x_1,\ldots, x_m[\Phi(x_1,\ldots, x_m) \leftrightarrow \Psi(x_1,\ldots, x_m)].$$

Ψ is called a normal form of Φ.

Example. Let

$$\Phi(x_1, x_2) = (x_1P_1x_2 \leftrightarrow \neg x_2P_1x_1) \wedge (x_1P_1x_1 \leftrightarrow x_2P_1x_2).$$

Let $\Psi_1=x_1P_1x_1$, $\Psi_2=x_2P_1x_2$, $\Psi_3=x_2P_1x_1$ and $\Psi_4=x_1P_1x_2$. A normal form of Φ is

$$(\Psi_1 \wedge \Psi_2 \wedge \neg\Psi_3 \wedge \Psi_4) \vee (\Psi_1 \wedge \Psi_2 \wedge \Psi_3 \wedge \neg\Psi_4) \vee$$
$$\vee (\neg\Psi_1 \wedge \neg\Psi_2 \wedge \neg\Psi_3 \wedge \Psi_4) \vee (\neg\Psi_1 \wedge \neg\Psi_2 \wedge \Psi_3 \wedge \neg\Psi_4).$$

3. STRONG NEUTRALITY AND DEFINABILITY

A further concept which I wish to redefine using logic terminology is the Strong Neutrality condition. A SWF, F, is Strongly Neutral if, for every a, b, c, d (in the set of social alternatives) and for every pair of profiles $(R_1,\ldots, R_n)$ and $(R_1',\ldots, R_n')$,

$$(*) \quad \text{for all } i,\ aR_ib \text{ iff } cR_i'd \text{ and } bR_ia \text{ iff } dR_i'c$$

implies

$$(**) \quad aF(R_1,\ldots, R_n)b \ \text{ iff } \ cF(R_1',\ldots, R_n')d \ \text{ and}$$
$$bF(R_1,\ldots, R_n)a \ \text{ iff } \ dF(R_1',\ldots, R_n')c.$$

Notice that $(*)$ is the requirement that any relationship of x_1, x_2 in the language L is satisfied under the substitution of (a, b) iff it is satisfied under the substitution of (c, d). Thus we can define

Definition. An $\mathbf{M}-SF$, F, is said to satisfy the Strong Neutrality condition (SN), if for all M, $N\in\mathbf{M}$, for all $a_1,\ldots, a_m\in|M|$, $b_1,\ldots, b_m\in|N|$, and for any relationship Φ of $x_1,\ldots, x_m$ "$M\models\Phi[a_1,\ldots, a_m]$ iff $N\models\Phi[b_1,\ldots, b_m]$" implies

"$(a_1,\dots, a_m)\in F(M)$ iff $(b_1,\dots, b_m)\in F(N)$".

A crucial step in the proof of the main theorem is the establishment of the connection between SN and a linguistic concept — the definability condition.

Definition. Let $\mathbf{M}$ be a set of models of L. An $\mathbf{M}-SF$, F, is *definable* if there is a wff Φ in L with two free variables x_1, x_2 such that for all $M\in\mathbf{M}$ and for all t_1, $t_2\in M$, $M\models\Phi[t_1, t_2]$ iff $(t_1, t_2)\in F(M)$. F is called *definable without quantifiers* (DWQ) if there exists a Φ (as above) without quantifiers.

Example. Let $\mathbf{M}$ be the set of all models of L such that the relations are order relations and their universal set is the set of natural numbers. The Pareto-extension rule is definable by the following quantifier–less formula

$$\Psi(v_1, v_2) = (\bigwedge_{i=1}^{n} v_1P_iv_2)\wedge(\bigvee_{i=1}^{n} \neg v_2P_iv_1).$$

The rule which assigns priority to those elements which are not Pareto-dominated is definable by the following formula:

$$\Phi(v_1, v_2) = \neg\exists v(\Psi(v_1, v)).$$

It might be shown that this rule is not definable without quantifiers.

The following proposition proves the equivalence between DWQ and SN.

PROPOSITION 2. *Let F be an $\mathbf{M}-SF$. F satisfies SN iff F satisfies DWQ.*

PROOF. Suppose F satisfies SN. Let $M\in\mathbf{M}$ and a_1, $a_2\in|M|$. There is one and only one relationship in L which is satisfied in M under the substitution of a_1, a_2. Denote it by Φ_{M,a_1,a_2}. Define $\Phi(x_1, x_2)=\vee\,\Phi_{M,a_1,a_2}(x_1, x_2)$ where the disjunction is over all M, a_1, a_2 satisfying $(a_1, a_2)\in F(M)$. Let Δ be the set of all those Φ_{M,a_1,a_2}. Let us prove that $(b_1, b_2)\in F(N)$ iff $N\models\Phi[b_1, b_2]$. If $(b_1, b_2)\in F(N)$ then Φ_{N,b_1,b_2} is one of the conjunction in Δ. Thus $N\models\Phi[b_1, b_2]$. If $N\models\Phi[b_1, b_2]$ then there exist a_1, a_2 in a model $M\in\mathbf{M}$ such that $N\models\Phi_{M,a_1,a_2}[b_1, b_2]$ and Φ_{M,a_1,a_2} is a relationship in Δ. Thus $(a_1, a_2)\in F(M)$ and (a_1, a_2) satisfies in M the same relationship which (b_1, b_2) satisfies in N. The SN assumption implies that $(b_1, b_2)\in F(N)$.

Suppose F satisfies DWQ. Let Ψ be a formula with two free variables without quantifiers defining F. From Proposition 1 there exists a formula of the form $\bigvee_{j\in J}\delta_j\cdot\Phi_j(v_1, v_2)$ such that $\models\bigvee_{j\in J}\delta_j\cdot\Phi_j(v_1, v_2)\leftrightarrow\Psi(v_1, v_2)$ where $\delta_j\in\{1, -1\}$ and $\{\Phi_j\}_{j\in J}$ is a "listing" of all the relationships of v_1, v_2 in L. To prove that F satisfies SN, assume M and N are in $\mathbf{M}$, a_1, a_2 are elements in M and b_1, b_2 belong to N, and for every relationship Φ_j—

$$M\models\Phi_j[a_1, a_2] \quad\text{iff}\quad N\models\Phi_j[b_1, b_2].$$

Then

$$M\models\vee\,\delta_j\cdot\Phi_j[a_1, a_2] \quad\text{iff}\quad N\models\vee\,\delta_j\cdot\Phi_j[b_1, b_2]$$

and also

$$M \models \Psi[a_1, a_2] \quad \text{iff} \quad N \models \Psi[b_1, b_2].$$

Hence,

$$(a_1, a_2) \in F(M) \quad \text{iff} \quad (b_1, b_2) \in F(N).$$

The following proposition will be useful later:

PROPOSITION 3. *Let $\Phi*$ be a sentence in $L*$ and let* $\mathbf{M}$ *be a set of models of L. Let F be a definable* $\mathbf{M}-SF$ *and let* Ψ *be the formula in L defining F. Let* Φ *be the sentence obtained from* $\Phi*$ *by substituting* $\Psi(x_1, x_2)$ *for every occurence of* $F(x_1, x_2)$. *Then for every* $M \in \mathbf{M}$ $M \models \Phi$ *iff* $M_F \models \Phi*$.

PROOF. Follows directly from the definitions.

4. GENERALIZATION OF THE SINGLE PROFILE UNRESTRICTED DOMAIN

A further concept which requires generalization is Pollak's Unrestricted Domain over Triples. Pollak said that a profile has this property if "for every logically possible subprofile over three hypothetical alternatives x, y, z there exists a triple such that the restriction of the profile to that triple coincides with the prespecified subprofile". This property was a requirement for obtaining a "single profile analogue".

In the following $\mathbf{M}$, $\mathbf{M}_1$ and $\mathbf{M}_2$ are sets of models of L and k is a natural number. Let $R_k(\mathbf{M})$ be the set of all relationships $f(v_1,\ldots, v_k)$ for which there exist $M \in \mathbf{M}$ and $a_1,\ldots a_k \in |M|$ such that $M \models f[a_1,\ldots, a_k]$. Pollak's statement that M_0 satisfies the Unrestricted Domain over Triples is equivalent to the statement that $R_3(\{M_0\}) \supseteq R_3(\mathbf{M})$, where $\mathbf{M}$ is the set of all possible n-tuples of preference relations over a set which contains at least three elements. The following proposition relates to a sentence α for which there exists a wff without quantifiers β such that $\alpha = \forall v_1,\ldots, v_k \beta(v_1,\ldots, v_k)$. The proposition connects between "the satisfiability of α in every model in $\mathbf{M}_1$" and "the satisfiability of α in every model in $\mathbf{M}_2$", provided that $R_k(\mathbf{M}_1) \supseteq R_k(\mathbf{M}_2)$.

PROPOSITION 4. *Let* α *be a sentence of the form* $\alpha = \forall x_1,\ldots, x_k \beta(x_1,\ldots, x_k)$, *and suppose* β *is a wff without quantifiers in L. Let* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *be sets of models in L. Assume* $R_k(\mathbf{M}_1) \supseteq R_k(\mathbf{M}_2)$. *Then* $M \models \alpha$ *for every* $M \in \mathbf{M}_1$ *implies* $M \models \alpha$ *for every* $M \in \mathbf{M}_2$.

PROOF. Suppose $M \models \alpha$ for every $M \in \mathbf{M}_1$. By Proposition 1 there is a set $\{\Phi_j\}_{j \in J}$ of relationships of $x_1,\ldots, x_k$ such that $\models \alpha(x_1,\ldots, x_k) \Leftrightarrow \bigvee_{j \in J} \Phi_j(x_1,\ldots, x_k)$. Suppose there exists $M_2 \in \mathbf{M}_2$ such that $M_2 \models \neg\alpha$. Then $\exists a_1,\ldots, a_k \in |M_2|$ such that $M_2| = -\beta[a_1,\ldots, a_k]$. Let Φ be the (unique) relationship satisfying $M_2| = \Phi[a_1,\ldots, a_k]$. Clearly $\Phi \notin \{\Phi_j\}_{j \in J}$. $R_k(\mathbf{M}_1) \supseteq R_k(\mathbf{M}_2)$, therefore, there exist $M_1 \in \mathbf{M}_1$ and $b_1,\ldots, b_k \in |M_1|$ such that $M_1 \models \Phi[b_1,\ldots, b_k]$. Then $M_1 \models \neg\beta[b_1,\ldots, b_k]$ so $M_1 \models \neg\alpha$, a contradiction.

5. THE MAIN THEOREM

We are now ready to formulate and to prove the main theorems. Let $\alpha*$ and $\gamma*$ be sentences in $L*$. Let $(T-\mathbf{M})$ denote the following statement:
$(T-\mathbf{M})$ If and $\mathbf{M}-SF$, F, satisfies SN and $\alpha*$ then F satisfies $\gamma*$.

THEOREM 1. Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be sets of models of L. Assume that $R_k(\mathbf{M}_1)=R_k(\mathbf{M}_2)$ and that there exist formulae $\beta*$ and $\delta*$ without quantifiers in $L*$ such that $\alpha*=\forall v_1\cdots v_k\beta*(v_1,\ldots, v_k)$ and $\gamma*=\forall v_1\cdots v_k\delta*(v_1,\ldots, v_k)$. Then-$(T-\mathbf{M}_1)$ is true iff $(T-\mathbf{M}_2)$ is true.

Remark. It should be emphasized that the k in the condition $R_k(\mathbf{M}_1)=R_k(\mathbf{M}_2)$ is the same k as that which indicates the number of bound variables in $\alpha*$ and $\gamma*$.

PROOF. Suppose $(T-\mathbf{M}_1)$ is true. Let F be an $\mathbf{M}_2-SF$ which satisfies SN and $\alpha*$. From proposition 2, F is DWQ. Let Φ be a formula without quantifiers in L defining F. Let α, β, γ and δ be the formulae in L obtained from $\alpha*$, $\beta*$, $\gamma*$ and $\delta*$ by substitution $\Phi(x_1, x_2)$ for every occurrence of $P(x_1, x_2)$. Thus $\alpha=\forall v_1\cdots v_k\beta(v_1,\ldots, v_k)$ and $\gamma=\forall v_1\cdots v_k\delta(v_1,\ldots, v_k)$. $R_k(\mathbf{M}_2)\supseteq R_k(\mathbf{M}_1)$ and $M\models\alpha$ for every $M\in\mathbf{M}_2$. Therefore proposition 4 implies $M\models\alpha$ for every $M\in\mathbf{M}_1$. Define an $\mathbf{M}_1-SF$, G, by $(a_1, a_2)\in G(M)$ iff $M\models\Phi[a_1, a_2]$. Clearly, G is definable and by proposition 2 G satisfies SN. From proposition 3 $M_G\models d^*$ for every $M\in M_1$. We have assumed that $(T-\mathbf{M}_1)$ is true, therefore $M_G\models\gamma*$ for every $M\in\mathbf{M}_1$ and from proposition 3 $M\models\gamma$ for every $M\in\mathbf{M}_1$. Using the fact that $R_k(\mathbf{M}_1)\supseteq R_k(\mathbf{M}_2)$ and proposition 4 we get $M\models\gamma$ for every $M\in\mathbf{M}_2$. Therefore $M_G\models\gamma*$ for every $M\in\mathbf{M}_2$ and $(T-\mathbf{M}_2)$ is true.

From the above proof it is clear that if $\mathbf{M}_1\supseteq\mathbf{M}_2$ we need proposition 4 only once and the restriction on the form of $\gamma*$ is redundant. Thus we have—

THEOREM 2. *Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be sets of models of L, $\mathbf{M}_1\supseteq\mathbf{M}_2$. Assume that $R_k(\mathbf{M}_1)=R_k(\mathbf{M}_2)$ and that there exist $\beta*$ without quantifiers in $L*$ such that*

$$\alpha* = \forall x_1,\ldots, x_k\beta*(x_1,\ldots, x_k).$$

Then-$(T-\mathbf{M}_1)$ is true implies that $(T-\mathbf{M}_2)$ is true.

Now, let us return to the question — "Is there a single profile analogue to every theorem in Social Choice Theory?" The answer depends, of course, on the meaning we give to the term "single profile analogue".

We start with the assumption that $(T-\mathbf{M})$ is true, where $\mathbf{M}$ is the set of all possible models of $L*$ and where the relations are preferences on a given set A. The single profile analogue is $(T-\{M_0\})$. The condition on M_0 for the analogue to be true is $R_k(\{M_0\})=R_k(\mathbf{M})$. The above theorems proved so far confirm the hypothesis when the assertion $(T-\mathbf{M})$ satisfies a specified structure. They also show which k is required for the analogue to be true. The theorem does not hold when $(T-\mathbf{M})$ has a different structure. The following example proves that there are sentences $\alpha *$ and $\gamma *$ in $L*$ such that $(T-\mathbf{M})$ is true but given any k we can construct a model M_k such that $R_k(\{M_k\})=R_k(\mathbf{M})$ and still $(T-\{M_k\})$ is not true. Let $\alpha *$ be the formula in $L*$

$$[\exists v_1 \forall v_2 \bigwedge_{i=1}^{n} (v_2 P_i v_1 \wedge v_1 \neg v_1 P_i v_2)] \longrightarrow [\forall v_1 v_2 [v_1 P_1 v_2 \leftrightarrow v_1 P v_2]] .$$

(If there is an alternative which is Pareto-dominated by any other element, then P is identical to P_1). Let $\gamma *$ be the formula $\forall v_1 v_2 \ (v_1 P_1 v_2 \leftrightarrow v_1 P v_2)$ (1 is a dictator). Let A be a set which contains at least 3 elements and let $\mathbf{M}$ be the set of all $\langle A, \geq_1, \ldots, \geq_n \rangle$ where $\geq_i$ is a preference relation on A. Clearly $(T-\mathbf{M})$ is true. That is, for every $\mathbf{M}-SF$ satisfying SN and $\alpha *$ the $\mathbf{M}-SF$ also satisfies $\gamma *$. However for any given k, one can construct a model such that every description of $v_1, \ldots, v_k$ which is satisfied in $\mathbf{M}$ is also satisfied in M_k but in which there is no alternative which is Pareto-dominated by all the other elements. In this model, the $\mathbf{M}-SF$ which is identical to individual 2's preference satisfies $\alpha *$, and SN but does not satisfy $\gamma *$. The key to constructing such an example is of course the fact that the quantifiers $\exists$ appears in $\alpha *$. This violates the structure required of $\alpha *$ in theorems 1 and 2.

6. EXAMPLES

For the sake of simplicity all the examples are confined to the class of $(T-\mathbf{M})$'s in which $\mathbf{M}$ is the set of models of L with a common universe $A-(|A|\geq 3)$ and all the relations are connected, transitive, and asymmetric.

6.1. *Arrow's Impossibility Theorem.* Let $\alpha *$ be the sentence

$$\alpha * = \forall v_1 v_2 v_3 [((\bigwedge_{i=1}^{n} v_2 P_i v_1) \rightarrow v_2 P v_1) \wedge (v_3 P v_2 \wedge v_2 P v_1 \rightarrow v_3 P v_1) \cdots$$
$$\cdots \wedge (v_2 P v_1 \vee v_1 P v_2) \wedge (v_1 P v_1)] .$$

The sentence $\alpha *$ is in $L*$ and it expresses the Pareto condition and the requirement that the social relation be a preference relation. Let $\gamma *$ be the sentence

$$\bigvee_{i=1}^{n} \forall v_1 v_2 (v_1 P_i v_2 \leftrightarrow v_1 P v_2) .$$

The sentence $\gamma *$ is also in $L*$ and it expresses the condition that there is a dictator (but not necessarily the same dictator for every profile).

Theorem 2 provides the single profile analogue for Arrow's Impossibility Theorem. (See Pollak [1979] and Roberts [1980]): Let M_0 be a model of L satisfying $R_3(\{M_0\})=R_3(\mathbf{M})$. If F is an $\{M_0\}-SF$ which satisfies (i) SN (ii) the Pareto condition and (iii) $F(M_0)$ is a complete and transitive relation then $F(M_0)$ is one of the individuals' preferences.

6.2. *Gibbard's Oligarchy Theorem.* Gibbard's result states that any SWF satisfying Pareto, Independent of Irrelevant Alternatives and Quasi-transitivity of the social preference is oligarchic. That is, there exists a nonempty set of individuals B such that if for every $i \in B$ yP_ix then yPx and if for one of the members of B xP_iy then not yPx.

Theorem 2 implies a single profile analogue for Gibbard's theorem: Let M_0 be a model of L satisfying $R_3(\{M_0\})=R_3(\mathbf{M})$. If F is an $\{M_0\}-SF$ which satisfies (i) SN (ii) the Pareto condition and (iii) $F(M_0)$ is quasi-transitive then $F(M_0)$ is oligarchic. To see this, define

$$\alpha* = \forall v_1v_2v_3[((\bigwedge_{i=1}^{n} v_2P_iv_1)\rightarrow v_2Pv_1)\wedge(v_3Pv_2\wedge v_2Pv_1\rightarrow \neg v_1Pv_3)]$$

and

$$\gamma* = \forall v_1v_2 \bigvee_{B\neq\phi}[((\bigwedge_{i\in B} v_2P_iv_1)\rightarrow v_2Pv_1)\wedge((\bigvee_{i\in B} v_1P_iv_2)\rightarrow \neg v_2Pv_1)]$$

The sentence $\alpha*$ is in $L*$ and it expresses the Pareto condition and the requirement that the social relation be quasi-transitive. The sentence $\gamma*$ expresses the assertion that there exists an oligarchy (though not necessarily the same oligarchy for every profile). As mentioned the single profile theorem is obtained directly from Theorem 2 and Gibbard's theorem.

6.3. *May's characterization of the majority rule.* May [1952] has proved that the only SWF which satisfies independence of irrelevant alternatives, neutrality, anonymity and positive responsiveness is the method of majority rule. Pollak proves a single profile analogue to May's theorem referring to profiles that satisfy the "unrestricted domain over pairs U^{*2}" in contrast to the "unrestricted domain over triples U^{*3}" which was required for getting the analogue for Arrow's impossibility theorem. Theorem 1 is useful for proving Pollak's theorem again and for understanding the differences between U^{*2} and U^{*3}. Assuming IIA and positive responsiveness anonymity can be expressed by the formula

$$\alpha* = \forall v_1v_2(\bigvee_{\Phi\in\Delta} \Phi(v_1, v_2)\leftrightarrow P(v_1, v_2))$$

where Δ is the set of all disjunctions of descriptions which satisfy—

(i) if $\bigwedge_{i=1}^{n} \delta_iP_i(v_1, v_2)$ is in the disjunction then for every permutation σ of $\{1,\ldots, n\}$ $\bigwedge_{i=1}^{n} \delta_{\sigma i}P_i(v_1, v_2)$ is in the disjunction.

(ii) if $\bigwedge_{i=1}^{n} \hat{\delta}_iP_i(v_1, v_2)$ is in disjunction and for all i $\delta_i \geq \hat{\delta}_i$ then $\bigwedge_{i=1}^{n} \delta_iP_i(v_1, v_2)$ is in the disjunction.

Define

$$\gamma* = \forall v_1v_2(\bigvee_{\Sigma\delta_i>0}\bigwedge_{i=1}^{n} \delta_iP_i(v_1, v_2)\leftrightarrow P(v_1, v_2)).$$

The sentence $\gamma*$ expresses the majority rule. Now, theorem 1 together with May's original theorem imply Pollak's theorem.

7. FURTHER IDEAS

(a) Proposition 2 reveals the DWQ to be a very strong assumption about social welfare functions. A more natural assumption in my opinion is the definability assumption.

Whereas definability without quantifiers and the Pareto condition ensure the existence of a dictator, mere definability together with the Pareto condition is not sufficient for this to be the case. The following SWF is definable in $L*$, satisfies the Pareto condition and is not dictatorial:

$$F(R^1,\dots,R^n)=\begin{cases}\text{Majority rule} & \text{if majority rule induces an order}\\ R^1 & \text{otherwise.}\end{cases}$$

The characterization of definable relations needs further research. One result about definable relation is stated in Rubinstein [1980]. If M_0 satisfies a certain richness condition then every $\{M_0\}-SF$ which is definable in L and satisfies Pareto condition is dictatorial.

(b) The theorems which have been proved here can be easily extended to other languages which consist of n predicate symbols, not necessarily two-place predicates. Such a language may be useful in analyzing $m-\mathbf{M}-SF$, that is functions which map a set of models $\mathbf{M}$ to the set of m-place relations. In a previous version of this paper (Rubinstein [1980]) the results reported have been generalized in this way. However, I believe that the current simpler version of this paper may reveal the ideas of this research better than the more general version.

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