# Essay 2

# Equilibrium in Supergames


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This paper is the first part of Research Memorandum 25, The Center for Research in Game Theory and Mathematical Economy which was published in May 1977 and was based on my M.Sc. Thesis, written in 1975-6 at the Hebrew University of Jerusalem under the supervision of Professor B. Peleg. Other parts of the report were published in "Equilibrium in Supergames with the Overtaking Criterion," *J. Econ. Theory* **21** (1979) 1–9, and "Strong Perfect Equilibrium in Supergames," *International J. Game Theory* **9** (1980) 1–12. The main theorem in this paper was discovered simultaneously by R. J. Aumann and L. S. Shapley.


## 1. Introduction

There are significant differences between the situation of players undertaking to play a single game, and players who know that they will play the same game repeatedly in the future. Strategy in the first case is a single action; in the second, it is a sequence of rules, each one of which pertains to the outcomes preceding it. The preferences of the participants are determined partly by temporal considerations. The participants may adopt risky strategies, "supported" by threats of retribution in the future.

Analysis of a finite sequence of identical games shows that this model is inadequate for the analysis. If the number of games is finite and known initially, the players will treat the last game as if it were a single game. As the threats implicit in the game before last are proven to be false threats, the game before last is treated as a single game, and so on. (For a detailed analysis, see Luce and Raiffa [8].)

In order to avoid "end-points" in the model, we define a "supergame" as an infinite sequence of identical games, together with the players' evaluation relations (that is, their preference orders on utility sequences). Obviously, the assumption of an infinite planning horizon is unrealistic, but it is an approximation to the situation we wish to describe (see Aumann [1]).

The literature deals mainly with comparison of equilibrium concepts in supergames and single games. (See Aumann [1] and [2]. The results are


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derived more simply in [5]; see also [3] and [4].) Other papers emphasize the uses of the concept of supergames in economics are Friedman [6] and Kurz [7].

In this paper, as in most of the literature, it is assumed that the participants evaluate the utility flows according to the criterion of the limit of the means of the flows. The drawback in this evaluation relation is that it ignores any finite time interval.

The formal model, described in Section 2, is taken from Roth.[2]

The single game is given in strategic form. A Nash equilibrium in an $n$-player super game is an $n$-tuple of supergame strategies, such that no player may singly deviate profitably from his strategy. A steady Nash equilibrium is one which produces identical outcomes for every game played. In Section 3, the steady equilibrium points will be characterized by a "two-stage" finite game in which the time element is reduced to "present" and "future."

An equilibrium point will be called perfect if after any possible "history," the strategies planned are an equilibrium point. In other words, no player ever has a motive to change his strategy. The main theorem of this paper provides a complete characterization of steady perfect equilibrium outcomes for supergames with the limit of means evaluation relation. It is proved that the requirement of perfection does not alter the set of steady equilibrium outcomes.

## 2. The model

(i) The *single* game $G$ is a game in strategic form

$$G = \langle \{S_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n \rangle \ .$$

The set of players is $N = \{1, \ldots, n\}$. For each $i \in N$, the set of strategies of $i$ is $S_i$; $S_i$ is assumed to be non-empty and compact. $S = \prod_{i=1}^n S_i$ is the set of outcomes. An element in $S$ will be called an outcome of $G$. The preference relations of player $i$ are defined by utility function $\pi_i : S \to \Re$ (where $\Re$ is the set of the reals), which are continuous in the product topology.

Given $\sigma \in S$, a payoff vector is the $n$-tuple $\pi(\sigma) = \langle \pi_1(\sigma), \ldots, \pi_n(\sigma) \rangle$. For convenience we will denote the $(n-1)$-tuple $\langle \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n \rangle$ by $\sigma_{-i}$, and the $n$-tuple $\sigma$ by $\langle \sigma_{-i}, \sigma_i \rangle$. $\sigma$ will be called a *(Nash) equilibrium* if for all $i$ and for all $s_i \in S_i$, $\pi_i(\sigma_{-i}, s_i) \le \pi_i(\sigma)$.

[2] I wish to thank Professor A. E. Roth for permitting me to use the model described in Roth [9].

If the set of strategies is finite, and it is possible to adopt mixed strategies, we can identify $S_i$ with the set of mixed strategies of $i$, and $\pi_i$ with the expected payoff.

(ii) The *supergame* $G^\infty$ is the game $\langle G \prec_1, \ldots, \prec_n \rangle$, where $G$ is a single game and the $\prec_i$ are evaluation relations on real number sequences; that is, $\prec_i$ is a binary relation on $\pi_i(S)^N$ (($A^N$ is the set of sequences of elements in $A$) where $\pi_i(S)$ is the range of $\pi_i$ on $S$. $\prec_i$ will be transitive and anti-symmetric but not necessarily a total order.

The set of outcomes at time $t$, $S(t)$, is $S$. A strategy for $i$ in $G^\infty$ is a set $\{f_i(t)\}_{t=1}^\infty$, where $f_i(1) \in S(1)$, and for $t \geq 2$, $f_i(t) : \prod_{j=1}^{t-1} S(j) \to S_i$. Thus, a supergame strategy is a choice of strategies at every stage, possibly dependent on the outcomes preceding the choice. We assume all players know all the choices made in the past by all the players.

The set of strategies of $i$ will be denoted by $F_i$. $F$ is the set of $n$-tuples of the strategies; $F = \prod_{i=1}^n F_i$.

Given $f \in F$, the outcome at time $t$ will be denoted by $\sigma(f)(t)$, and is defined inductively by

$$\sigma(f)(1) = (f_1(1), \ldots, f_n(1))$$
$$\sigma(f)(t) = (\ldots, f_i(t)(\sigma(f)(1), \ldots, \sigma(f)(t-1)), \ldots)\ .$$

We will define an evaluation relation $\bar{\prec}_i$ on $F$, induced by $\prec_i$, as follows.

$$(\forall f, g \in F)\,(f \bar{\prec}_i g \ \Leftrightarrow\ \{\pi_i(\sigma(f)(t))\}_{t=1}^\infty \prec_i \{\pi_i(\sigma(g)(t))\}_{t=1}^\infty)\ .$$

Given $f \in F$, we will denote $(f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n)$ by $f_{-i}$. $f \in F$ is a (Nash) equilibrium in the supergame $G^\infty$ if there is no $h_i \in F_i$ such that $f \bar{\prec}_i (f_{-i}, h_i)$. $f \in F$ is steady if there exists $\sigma \in S$ such that for all $t$, $\sigma(f)(t) = \sigma$. If $f \in F$ is steady, we will denote the corresponding $\sigma$ by $\hat{\sigma}(f)$.

# 3. Characterization of steady Nash equilibria

We will assume that each player is characterized by a single evaluation relation. We will merely assume that the evaluation relations are *reasonable*, in the sense that they satisfy:

**(A.1)** If for all $t$, $y_t \equiv y_0$, and $x_t \equiv x_0$, then $x_0 < y_0$ implies $x \prec y$.

**(A.2)** If $z \prec x$, and $x \leq y$ (that is, for all $t \in N$, $x_t \leq y_t$), then $z \prec y$.

**(A.3)** If there exists an $a \in A$ such that $(a, x_1, x_2, \ldots) \prec (a, y_1, y_2, \ldots)$, then $x \prec y$.

EXAMPLES:

1 The *limit of means* evaluation relation is defined by

$$x \prec y \text{ iff } \varliminf_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} x_t < \varliminf_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} y_t .$$

2 *Overtaking criterion* evaluation relation, defined by

$$x \prec y \text{ iff } \varliminf_{n\to\infty} \sum_{i=1}^{n}(y_i - x_i) > 0 .$$

3 The evaluation relation with *discount* parameter $0 < \delta < 1$, defined by

$$x \prec y \text{ iff } \sum_{t=1}^{\infty} \delta^t x_t < \sum_{t=1}^{\infty} \delta^t y_t .$$

This section has two goals. The first is to characterize the steady Nash equilibria using the equilibria of a "finite" two-stage game, $G^2$, derived from $G^\infty$. The game $G^2$ is a twofold repetition of $G$. A strategy in $G^2$ contains decisions about the "present," the first game, and the "future," the second game. The latter decision depends on the outcome of the former. The second aim is to show that the considerations included in the notion of equilibrium of a supergame can be studied by a two-stage model.

We will now define the derived game, $G^2$. A strategy for a player $i$ in $G^2$ is a pair, $\langle f_i(1), f_i(2)\rangle$, where $f_i(1) \in S_i$ and $f_i(2) : S(1) \to S_i$. We will denote the strategies of $i$ by $F_i^2$, and write $F^2 = \prod_{i\in N} F_i^2$.

We define a partial order $\prec_i^2$ on $\pi_i(S) \times \pi_i(S)$ as follows.

$$(b_1, b_2) = b \prec_i^2 a = (a_1, a_2) \quad \text{iff} \quad \begin{cases} 1)\ b_1 < a_1 \text{ and} \\ 2)\ (b_1, b_2, b_2, b_2, \ldots) \prec_i (a_1, a_2, a_2, a_2, \ldots), \end{cases}$$

where $\prec_i$ is the evaluation relation of a player $i$ in $G^\infty$.

The outcome of $G^2$, where a player adopts strategy $f \in F^2$ is defined by:

$$\sigma(f)(1) = (f_1(1), \ldots, f_n(1))$$
$$\sigma(f)(2) = (f_1(\sigma(f)(1)), \ldots, f_n(\sigma(f)(1))) .$$

$f \in F^2$ will be called *steady* if there exists a $\sigma \in S$ such that $\sigma(f)(1) = \sigma(f)(2) = \sigma$. Such a $\sigma$ will be denoted by $\hat{\sigma}(f)$. $f \in F^2$ will be called an *equilibrium* if there is no $i$ and $g_i \in F_i^2$ such that

$$(\pi_i(\sigma(f)(t)))_{t=1}^2 \prec_i^2 (\pi_i(\sigma(f_{-i}, g_i)(t)))_{t=1}^2 .$$

EXAMPLES:

1) If $\prec$ is the limit of means evaluation relation, the relation induced is:

$$b \prec^2 a \text{ iff } b_1 < a_1 \text{ and } b_2 < a_2 \ (a, b \in \Re^2) \ .$$

2) If $\prec$ is the overtaking evaluation relation,

$$b \prec^2 a \text{ iff } b_1 < a_1 \text{ and } b_2 \leq a_2 \ .$$

3) If $\prec$ is the evaluation relation with discount parameter $0 < \delta < 1$, then

$$b \prec^2 a \text{ iff } b_1 < a_1 \text{ and } b_1 + \frac{\delta}{1-\delta} b_2 < a_1 + \frac{\delta}{1-\delta} a_2 \ .$$

Thus, a player deviating at any one time only if he will derive certain profit in the foreseeable future, behaves according to the evaluation relation induced in $G^2$ by the limit of means relation. A player deviating only if he will not "lose" in the future, behaves according to the relation in $G^2$ induced by the overtaking criterion. If there exists $\epsilon > 0$, such that a player will deviate in the future iff the difference between his present profit and future loss exceeds $\epsilon$, (here $\frac{a_1 - b_1}{b_2 - a_2} > \epsilon$ iff $a_1 + \epsilon a_2 > b_1 + \epsilon b_2$); then, the evaluation relation in $G^2$ corresponding to this behavior is that induced by the evaluation relation with discount parameter $\delta = \frac{\epsilon}{1+\epsilon}$ .

**Remark 3.1.** For a supergame $G^\infty = \langle G, \prec_1, \ldots, \prec_n \rangle$, where for all $i$, the evaluation relations are reasonable, and where an equilibrium $\sigma$ exists in $G$, define $f \in F$ by: "For all $i$ and for all $t$, $f_i(t) \equiv \sigma_i$." Clearly, $f$ is a steady equilibrium in $G^\infty$ (even perfect; see the definition in Section 4), satisfying $\hat{\sigma}(f) = \sigma$. Thus, in a supergame where the single game has an equilibrium, we are guaranteed the existence of an equilibrium.

**Proposition 3.2.** *Let $G^\infty = \langle G, \prec_1, \ldots, \prec_n \rangle$ be a supergame where the $\prec_i$ are reasonable evaluation relations. If there exists $g \in F^2$, a steady equilibrium in $G^2$, the derived game, such that $\hat{\sigma}(g) = \sigma$, then there exists $f \in F$, a steady equilibrium in $G^\infty$ such that $\hat{\sigma}(f) = \sigma$.*

*Proof:* Given $s \in S$, let $r_i(s) \in S_i$ satisfy

$$\pi_i(r_i(s), s_{-i}) = \max_{t_i \in S_i} \pi_i(t_i, s_{-i}) \ .$$

Since $g$ is an equilibrium in $G^2$, there exist $\gamma^i \in S$ satisfying

$$(\pi_i(\sigma), \pi_i(\sigma)) \not\prec_i^2 (\pi_i(r_i(g(1)), g_{-i}(1)), \pi_i(r_i(\gamma^i), \gamma_{i-i})) \ .$$

We may assume that if $\pi_i(r_i(\sigma), \sigma_{-i}) \leq \pi_i(\sigma)$, then $\gamma^i = \sigma$, and we use the notation $[a] = (a, a, a \ldots)$, and $(a, [b]) = (a, b, b, b, \ldots)$.

Define $f_i \in F_i$ for $i \in N$ as follows.

$f_i(1) = \sigma_i$

$$f_i(t)(s(1), \ldots, s(t-1)) = \begin{cases} \gamma_i^j & \text{if there exists } T \leq t-1 \text{ such that:} \\ & s(1) = \cdots = s(T+1) = \sigma, \\ & s_{-j}(T) = \sigma_{-j}, \text{ and } s_j(T) \neq \sigma_j \\ \sigma_i & \text{otherwise} \end{cases}$$

Then $\hat{\sigma}(f) = \sigma$ and $f$ is a $G^\infty$ equilibrium, since if $(f_{-i}, h_i) \succ_i f$:
1) If $\pi_i(\sigma) \geq \pi_i(r_i(\sigma), \sigma_{-i})$, then $\pi_i(\sigma) \geq \pi_i(\sigma(g_{-i}, h_i)(t))$ for all $t$, and thus, according to (A.2) it will follow that $[\pi_i(\sigma)] \succ_i [\pi_i(\sigma]$ contradicting irreflexivity.
2) If $\pi_i(\sigma) < \pi_i(r_i(\sigma), \sigma_{-i})$, then let $t_0$ be the minimum satisfying $h_i(t_0)(\sigma, \sigma, \ldots, \sigma) \neq \sigma_i$. If $\{\pi_i(\sigma(f_{-i}, h_i)(t))\}_{t=1}^{\infty} \succ_i [\pi_i(\sigma)]$, then repeated applications of (A.3) yield

$$\{\pi_i(\sigma(f_{-i}, h_i)(t))\}_{t=t_0}^{\infty} \succ_i [\pi_i(\sigma)] .$$

But $\pi_i(r_i(\sigma), \sigma_{-i}) \geq \pi_i(\sigma(f_{-i}, h_i)(t_0))$ and for $t > t_0$, $\pi_i(r_i(\gamma^i), \gamma^i_{-i}) \geq \pi_i(\sigma(f_{-i}, h_i)(t))$. Applying (A.2),

$$(\pi_i(r_i(\sigma), \sigma_{-i}), [\pi_i(r_i(\gamma^i), \gamma^i_{-i})]) \succ_i [(\pi_i(\sigma)] ,$$

contradicting the choice of $\gamma^i$. ∎

**Proposition 3.3.** *Let $G^\infty = \langle G, \prec_1, \ldots, \prec_n \rangle$ where $\prec_i$ are reasonable evaluation relations. If $f \in F$ is a steady equilibrium in $G^\infty$, and $\hat{\sigma}(f) = \sigma$, then there exists a $G^2$ stationary equilibrium $g \in F^2$, such that $\hat{\sigma}(g) = \sigma$.*

*Proof:* Suppose not. Then there is an $i$ and $\tau_i \in S_i$ satisfying:

1) $\pi_i(\sigma_{-i}, \tau_i) > \pi_i(\sigma)$ .

2) For all $s \in S$, there are $t_i \in S_i$ such that

$$\pi_i(\sigma_{-i}, \tau_i), [\pi_i(t_i, s_{-i})] \succ_i [\pi_i(\sigma)] .$$

Thus, applying (A.2),

$$\left(\pi_i(\sigma_{-i}, \tau_i), \left[\min_{s_{-i}} \max_{t_i} \pi_i(t_i, s_{-i})\right]\right) \succ_i [\pi_i(\sigma)] .$$

Define $h_i$, a strategy in $G^\infty$, by

$h_i(1) = \tau_i$

$$h_i(t)(s(1)\dots s(t-1)) = r_i(f(t)(s(1)\dots s(t-1)))\ .$$

Applying (A.2), we obtain

$$\{\pi_i(\sigma(f_{-i}, h_i)(t))\}_{t=1}^{\infty} \succ_i [\pi_i(\sigma)]\ .$$

in contradiction to $f$ being an equilibrium. ∎

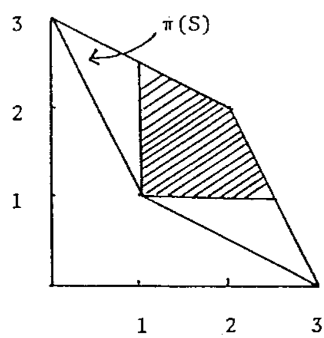
**Definition 3.4.** $s \in S$ is a *weakly forced outcome*[3] in $G$ if, for every $i$, there is an $r \in S$ such that for all $t_i \in S_i$, $\pi_i(r_{-i}, t_i) \leq \pi_i(s)$.

Thus, in a weakly forced outcome, each player's payoff is at least as large as the punishment the other players can inflict on him, that is, at least $\min_{r \in S} \max_{t_i \in S_i} \pi_i(r_{-i}, t_i)$.

**Example 3.5.** Let $S_i$ be the set of mixed strategies of $i$, $i = 1, 2$. In a matrix game with a payoff matrix

| | |
|---|---|
| 2,2 | 0,3 |
| 3,0 | 1,1 |

$\pi_i$ is the expected payoff of $i$.




[3]The term "individually rational" is by now the established terminology for this concept.

**Proposition 3.6.** *If $\prec_i$ is the limit of means evaluation relation in $G^\infty$, the steady outcomes of steady equilibria are the weakly forced outcomes.*

*Proof:* There exists $g \in F$, a steady $G^\infty$ equilibrium, such that $\hat{\sigma}(g) = \sigma$, iff there exists $f \in F^2$, a steady $G^2$ equilibrium such that $\hat{\sigma}(f) = \sigma$ and this holds iff for no $h_i \in F_i^2$ , $(\sigma, \sigma) \not\prec_i^2 (\sigma(h_i, f_{-i})(t))_{t=1}^2$, *i.e.*, iff for all $i$, either $\pi_i(r_i(\sigma), \sigma_{-i}) \leq \pi_i(\sigma)$ or

$$\pi_i(r_i(\gamma^i), \gamma^i_{-i}) = \underline{\lim}\, \tfrac{1}{n} \left[\pi_i(r_i(\sigma), \sigma_{-i}) + (n-1)\pi_i(r_i(\gamma^i), \gamma^i_{-i})\right] \leq \pi_i(\sigma)$$

and this holds iff for every $i$, there exists $\gamma^i \in S$ such that $\pi_i(r_i(\gamma^i), \gamma^i_{-i}) \leq \pi_i(\sigma)$. ∎

**Proposition 3.7.** *Let $G^\infty = \langle G, \prec_1, \ldots, \prec_n \rangle$ be a supergame with reasonable evaluation relations. A necessary condition for $f \in F$ to be a steady equilibrium is that $\hat{\sigma}(f)$ is a weakly forced outcome.*

*Proof:* By Proposition 3.3, there exists $g \in F$, a steady equilibrium in $G^2$ such that $\hat{\sigma}(g) = \hat{\sigma}(f)$. If $\hat{\sigma}(f)$ is not weakly forced, there exists $i$ such that for all $s \in S$, $\pi_i(r_i(s), s) > \pi_i(\hat{\sigma}(f))$. $S$ is compact. Thus, there exists $\epsilon > 0$ such that for all $s \in S$ $\pi_i(r_i(s), s_{-i}) > \pi_i(\hat{\sigma}(f)) + \epsilon$. Define $h_i \in F_i^2$ by:

$$h_i(1) = r_i(\hat{\sigma}(f))$$

$$h_i(2) = r_i(g(1)(\hat{\sigma}_{-i}(f), r_i(\hat{\sigma}(f)))).$$

Applying (A.1), we obtain:

$$(\pi_i(\sigma(g_{-i}, h_i)(t))_{t=1}^2 \succ_i^2 (\pi_i(\hat{\sigma}(f)), \pi_i(\hat{\sigma}(f)))) ,$$

in contradiction to $g$ being a $G^2$ equilibrium. ∎

# 4. Perfect equilibria

The definition of equilibrium as in Section 2 was shown in Section 3 to be weak. The set of equilibria is too large, and it is natural to introduce further reasonable restrictions to obtain a stronger characterization. One possible requirement is that a deviation will prove unprofitable to a player at all stages of the game, not only at the beginning; thus, under no circumstances will be induced to change his original strategy.

**Definition 4.1.** $f \in F$ is a *perfect equilibrium* if for all $r(1), \ldots, r(T) \in S$, the strategy profile $\bar{f}$ defined by

$$\bar{f}_i(t)(s(1), \ldots, s(t-1)) = f_i(t+T)(r(1), \ldots, r(T), s(1), \ldots, s(t-1))$$

is an equilibrium.

Not only is it unprofitable for one player to alter his strategy, but no player can perform manipulative maneuvers since after each "history," all players prefer not to deviate.

The following proposition characterizes the steady perfect equilibrium in a supergame with evaluation relations determined by the limit of means criterion. A similar result was discovered independently by Aumann and Shapley.

**Proposition 4.2.** *If $\sigma \in S$, there is a perfect steady equilibrium $f \in F$ such that $\hat{\sigma}(f) = \sigma$, iff $\sigma$ is a weakly forced outcome.*

*Proof:* Necessity follows from Proposition 3.6. Let $\sigma \in S$ be a weakly forced outcome. Let $\gamma^i$ be strategies satisfying

$$\max_{t_i \in S_i} \pi_i \left(\gamma^i_{-i}, t_i\right) \leq \pi_i(\sigma)$$

(($\gamma^i$ will be the strategy for punishing player $i$). Define $f_i(t+1)(s(1), \ldots, s(t))$ and $P(s(1), \ldots, s(t))$ inductively as follows. ($P(s(1), \ldots, s(t))$ is interpreted as the set of players deserving punishment after the history $(s(1), \ldots, s(t))$:

$P(\emptyset) = \emptyset$, and $f_i(1) = \sigma_i$ ,

$$P(s(1), \ldots, s(t)) = \begin{cases} \{i\} & \text{if } s_{-i}(t) = \gamma^i_{-i},\ P(s(1), \ldots, s(t-1)) = \{i\}, \\ & \quad \text{and } \frac{1}{t}\sum_{k=1}^{t} \pi(s(k)) \geq \pi(\sigma) + \frac{1}{\sqrt{t}} \\ \{i\} & \text{if } P(s(1), \ldots, s(t-1)) = \emptyset \ \&\ s_{-i}(t) = \sigma{-i} \\ & \text{but } s_i(t) \neq \sigma_i \ \&\ \frac{1}{t}\sum_{k=1}^{t} \pi_i(s(k)) \geq \pi_i(\sigma) + \frac{1}{\sqrt{t}} \\ \emptyset & \quad \text{otherwise} \end{cases}$$

$$f_i(t+1)(s(1), \ldots, s(t)) = \begin{cases} \gamma^j_i & \text{if } j \neq i \text{ and } P(s(1), \ldots, s(t)) = \{j\} \\ \sigma_i & \quad \text{otherwise} \end{cases}$$

Let $r(1), \ldots, r(T) \in S$ ; denote

$$\hat{f}_i(1) = f_i(T+1)(r(1), \ldots, r(T))$$

$$\hat{f}_i(t)(s(1), \ldots, s(t-1)) = f_i(t+T)(r(1), \ldots, r(T), s(1), \ldots, s(t-1)) .$$

We will show that $\hat{f}$ is an equilibrium. The following two lemmas will complete the proof. ▮

**Lemma 4.3.** *There exists $T_1$ such that for all $t \geq T_1$, $\sigma(\hat{f})(t) = \sigma$.*

*Proof:* If $P(r(1),\ldots,r(T)) = \emptyset$, then $\sigma(\bar{f})(t) = \sigma$ for all $t \geq 1$. If $P(r(1),\ldots,r(T)) = \{j\}$, then

$$\frac{1}{T+t}\left(\sum_{k=1}^{T}\pi_j(r(k)) + t\pi_j(\gamma^j)\right) \leq \frac{1}{T+t}\left(\sum_{k=1}^{T}\pi_j(r(k)) + t\pi_j(\sigma)\right)$$
$$\leq \frac{1}{T+t}\sum_{k=1}^{T}\pi_j(r(k)) + \pi_j(\sigma)$$
$$< \frac{1}{\sqrt{T+t}} + \pi_j(\sigma)$$

for sufficiently large $t$. ∎

**Lemma 4.4.** *Let $h_i \in F_i$. For every $t_0$, there is $t \geq t_0$ such that*

$$\frac{1}{t}\sum_{k=1}^{t}\pi_i(\sigma(\hat{f}_{-i},h_i)(k)) < \pi_i(\sigma) + \frac{1}{\sqrt{t}}\ .$$

*Proof:* It suffices to consider the case $t_0 \geq T_1$, where $T_1$ is given by Lemma 4.3; note also that $t \geq T_1$ implies $P(s(1),\ldots,s(t)) \subseteq \{i\}$. If for all $t \geq T_0$, $i$ does not deserve punishment after $\{\sigma(\hat{f}_{-i},h_i)(k)\}_{k=1}^{t}$, then $\sigma(\hat{f}_{-i},h_i) = \sigma$ for all $t_0 < k$, and

$$\frac{1}{t}\sum_{k=1}^{t}\pi_i(\sigma(\hat{f}_{-i},h_0)(k)) = \frac{1}{t}\sum_{k=1}^{t_0}\pi_i(\sigma(\hat{f}_{-i},h_i)(k)) + \frac{1}{t}(t-t_0)\pi_i(\sigma) \leq \pi_i(\sigma) + \frac{1}{\sqrt{t}}$$

for sufficiently large $t$. If there exists $t_0 \leq t_1$ such that $i$ deserves punishment after $\{\sigma(\hat{f}_{-i},h_i)(k)\}_{k=1}^{t_1}$ , then

$$\frac{1}{t}\sum_{k=1}^{t}\pi_i(\sigma(\hat{f}_{-i},h_i)(k)) = \frac{1}{t}\sum_{k=1}^{t_1}\pi_i(\sigma(\hat{f}_{-i},h_i)(k)) + \frac{1}{t}(t-t_1)\pi_i(\sigma) \leq \pi_i(\sigma) + \frac{1}{\sqrt{t}}$$

for sufficiently large $t$. ∎

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