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NOTES, COMMENTS, AND LETTERS TO THE EDITOR

A Simple Model of Equilibrium in Search Procedures*

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The paper presents a simple game-theoretic model in which players decide on search procedures for a prize located in one of a set of labeled boxes. The prize is awarded to the player who finds it first. A player can decide on the number of (costly) search units he employs and on the order in which he conducts the search. It is shown that in equilibrium, the players employ an equal number of search units and conduct a completely random search. The paper demonstrates that the search procedure is intrinsically inefficient. *Journal of Economic Literature* Classification Number: D83.

1. INTRODUCTION

In this paper we analyze a simple interactive search model in which two players search for a single treasure hidden in one of a given set of labeled "boxes." Only the first player to find the treasure gets to keep it. The search is executed by elementary search units which are able to check one box per unit of time. We model the players' constraints regarding the intensity of search by treating the search units as costly. A player decides on how many search units to employ and on a strategy for the search units to follow, i.e., which boxes will be examined at each period. We will assume that the

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search strategies are selected by the players after observing each other choice regarding the number of search units.

Any model of search can be viewed as a model of decision making in which the decision maker searches for a solution to a problem (see, for example, [6]). Therefore, a game in which the procedures of search are decided strategically can be viewed as a situation in which the procedures of decision making are the result of interactive reasoning. Since the choice of decision making procedures is modeled explicitly, one can view this paper as a modest contribution to the growing literature of economic models of bounded rationality.

The assumption of a single prize runs contrary to the informational assumptions made in most of the search literature in which the values of alternatives are taken to be stochastically independent. It is, however, suitable for decision problems in which only one action is successful and where the successful action can be found only by checking the options one by one. For real life scenarios which fit the model, consider the case of two reporters who are looking for a hotel presently hosting a movie star. Each reporter wants to be the first to meet the movie star and they carry out the search by calling the hotels one by one. Or consider two problem solvers who compete to find a solution to a particular problem and can find it only by following one of a given number of equally promising routes.

We assume that the boxes have names and that the searchers can decide on the order in which to open the boxes. The more common assumptions used in the literature are that a player can call a box randomly, i.e., "bring me a box" (see, for example, [3]), or that there is a fixed ordering of the boxes such that the boxes are placed in a line and the searcher calls for them in that order, i.e., "bring me the next box" (see, for example, [2]). These two models allow an equilibrium analysis of the intensity of search. Assigning labels to the boxes permits us to consider a variety of search strategies such as opening the boxes in a pre-determined order. When more than one player searches for the prize one cannot, a priori, exclude the possibility of an equilibrium with a pair of search strategies in which each player uses different search procedures which are optimal given the other. Thus the model allows us to discuss the consequences of competition which are not connected with the intensity of search.

Clearly, the model is similar to those of the R&D race literature. The main difference lies in our interest in determining the procedure of R&D search. As an R&D model, our model resembles the trial and error of research procedure typical to the chemical and pharmaceutical industries. The assumption of only one prize has been extensively used in the theoretical R&D literature (see [5]) and seems to be empirically supported by findings that the private value of patents is extremely skewed (see [1]). The search activity in our model is carried out by search units, the number of which is determined at the outset. This assumption is in line with the empirical R&D findings which indicate that firms adjust their R&D expenditures most infrequently (see [4]).

The possibility of having a search procedure is particularly important in situations where the boxes are not a priori symmetric. If the probability of the prize being located in some subset of boxes is greater than the probability for another subset, then the ability to use box addresses affects the optimal search procedure. In a single searcher problem, obviously the optimal procedure is to first sample the boxes with the higher probability of containing the prize. Such an intuitive property, however, cannot be automatically extended to interactive search situations with more than one searcher. In fact, we show that in our setting, the equilibrium procedures do not satisfy this property.

Our main conclusion is that an interactive search situation entails several intrinsic inefficiencies:

(i) In equilibrium, the players will use an excess of costly search units even though the prize could have been discovered (although not as quickly) using only one search unit.

(ii) There is an intrinsic duplication inefficiency, i.e., the two players may frequently search the same empty box. Consequently, the two players do not reach the prize as quickly as they could have. (Although time considerations are not explicitly embedded in the model, the equilibrium is robust to the addition of some degree of impatience.)

(iii) In an asymmetric situation where there is a higher probability that the prize is in one specific box, the efficient search rule is to start by searching this box. We show that if the asymmetry is not "too large," the equilibrium is such that there is a positive probability that neither player will search the high-probability box in the first period.

Note that while issue (i) relates to the inefficient search effort or the inefficient number of search units, (ii) and (iii) deal with inefficient search procedures. The distinction is helpful in comparing our results with those of the literature on R&D. While the noncooperative over-investment in R&D races (point (i)) arises in a variety of R&D models, the inefficiencies indicated in points (ii) and (iii) have not been discussed in the literature. In all the R&D race models we are aware of, either the order of search is predetermined, i.e., the boxes are in a fixed order, or the boxes do not carry labels. Thus, in these models the duplicate searching of the same box is built into the model rather than derived as a conclusion from a more basic setting in which the duplication can be avoided.

To appreciate the intuition in our main result, consider the case in which both players employ k search units. Efficiency (in terms of minimizing the number of search operations) requires that the two players split the N boxes into blocks of N/k boxes, half of which are searched by player 1 and half by player 2. However, if $N/k > 2$, this is not an equilibrium because player 2 can deviate profitably by searching those boxes in period t which player 1 is planning to search in period $t+1$, assuring a probability of $1 - k/N > 1/2$ of winning the prize.

2. THE MODEL

Two players, 1 and 2, are searching for a prize which has a value of 1. There are N boxes $b_1, ..., b_N$, one of which contains the prize. The location of the prize is determined randomly.

We analyze a two-stage game. In the first stage each player i must choose a number, k_i, the number of search units. Each search unit is capable of opening one box at each period. For simplicity we will confine ourselves to cases where N is divisible by all k_i. This enables us to avoid unnecessary calculations without undermining the analysis.

In the second stage, each player i chooses a random search strategy P_i after learning k_j. During the search player i is not informed of which boxes have already been opened by the other player. A random search strategy is a probability distribution over the set of pure search strategies. A (pure) search strategy is a partition of the N boxes into sets $S_i^1, ..., S_i^{T_i}$ such that the number of elements in each of the cells in the partition is k_i (there is no reason to assume that the players prefer to delay the process and keep some search units idle). Consequently $T_i = N/k_i$.

Note that we have assumed that the size of player i's search apparatus, k_i, is observed by his opponent before the search starts. This assumption reflects the assumption that it is easier for a player to change his search strategy than to change the size of the resources he devotes to search. In Section 5 we will comment on the case where the search program is determined simultaneously with the size of the search apparatus.

A pair of search strategies stochastically determines the winner of the prize. Assume that the prize is in box b. If $b \in S_i^{t(i)} \cap S_j^{t(j)}$, then if $t(i) < t(j)$, player i locates the prize and if $t(i) = t(j)$, each of the players has a probability 0.5 of locating it.

For any pair of choices $(k_i, P_i)_{i=1,2}$, player i's payoff is the probability of locating the prize minus k_i times the cost of a search unit, c. Thus, we assume that there is a fixed cost of a search unit rather than a fixed cost of sampling, as is assumed in most of the search literature.

The solution concept we adopt is that of Subgame Perfect Equilibrium. Note that the second stage is a zero-sum game. This means that the second stage yields a unique expected payoff to each player. However, note that it does not necessarily yield a unique equilibrium strategy. Also, while allowing randomization in the second stage, we do not permit randomization in the choice of the number of search units.

In order to simplify the statement of the results, we assume that there is no k for which $c = 1/k$. We further assume that $1/2 > c > 1/2N$. The other cases are degenerate. If $c > 1$, the choice of $k_i = 0$ is obviously a dominating strategy. If $c > 1/2$, the only equilibrium is such that only one player employs one search unit, as the payoffs from employing a second search unit cannot exceed one-half. If $c < 1/2N$, the only equilibrium will involve both players employing N search units. It is only in the range $1/2 > c > 1/2N$ that the model has an interesting strategic content.

3. THE ANALYSIS

The analysis consists of three claims. The first claim involves calculating the value of the second stage of the game for every pair of choices (k_1, k_2):

CLAIM 1. *For any given pair* (k_1, k_2) *the value of the second stage of the game for player* i *is* $k_i/2k_j$ *if* $k_i \leqslant k_j$ *and* $1 - k_j/2k_i$ *if* $k_i \geqslant k_j$.

We are now able to characterize the equilibrium number of search units. The analysis indicates that in equilibrium, there is essentially no profit, a feature common to other R&D models. This result follows from our assumption that the cost of the search units is linear and from the observation noted in Claim 1 that the players' profits are linear in the number of search units.

CLAIM 2. *If* $1/2 > c > 1/2N$, *then in all equilibria of the game, the two players choose* $k_1 = k_2 = k$ *where* k *satisfies* $1/2(1+k) < c < 1/2k$.

Finally, we are able to show that when $c > 1/N$ (and thus, $k < N/2$) in equilibrium, the players randomize the order of search so that the probability that a certain box is examined at each period is equal to $1/T_i = k/N$.

CLAIM 3. *For* $k < N/2$ (*namely, when there are at least three rounds of search*) *in any equilibrium, the probability that any box is checked at period* t $(1 \leqslant t \leqslant N/k)$ *is precisely* k/N.

Proof of Claim 1. First observe that given $k_i \leqslant k_j$, the pair of strategies in which both players mix all possible pure strategies with equal probabilities is an equilibrium. In proving the formula for the probability that player i finds the prize we can avoid cumbersome calculations through the following observation. The search lasts for no more than N/k_j periods and the number of boxes which may be searched by player i is Nk_i/k_j. On condition that the prize is in one of these Nk_i/k_j boxes, the probability that player i will find it is 1/2 (conditional on it being in the set of boxes he may search during the first N/k_j periods, the probability that player j searches it in period t is equal to that of player i). Thus, the probability that i will find the prize is $(Nk_i/2k_j)/N = k_i/2k_j$. Since the second stage of the game is a zero-sum game, the value of player i when $k_i \geqslant k_j$ is $(1 - k_j/2k_i)$. Q.E.D.

Proof of Claim 2. Assume that (k_i, k_j) is an equilibrium choice in the first stage. Let i be a player such that $k_i \leqslant k_j$ for $j \neq i$. Then player i's payoff is $k_i/2k_j$, an expression that is linear in k_i. If $k_j \leqslant k$, then the marginal gain to player i from a search unit is larger than c and player i can profitably deviate by increasing k_i. If $k_j > k$, then the marginal gain to player i from another search unit is $1/2k_j \leqslant 1/2(k+1) < c$. Thus, for such an equilibrium, k_i must be 0, which implies that k_j must be 1. Since $1 = k_j > k$, this implies that k must be 0. However, in that case, player i could deviate by purchasing one search unit and achieving a payoff of $1/2 - c > 0$ in contradiction to (k_i, k_j) being an equilibrium. Thus, $k_i = k_j$.

If $k_i = k_j > k$, then a player can profitably deviate by reducing the number of search units (since $1/2k_j \leqslant 1/2(k+1) < c$). If $k_i = k_j < k$, then by increasing the number of units by one, player i increases his payoff by $1 - k_j/2(k_j+1) - 1/2 = [(k_j+2) - (k_j+1)]/[2(k_j+1)] = 1/[2(k_j+1)] \geqslant 1/2k > c$, which is thus profitable. We are left with $k_i = k_j = k$. It is easy to verify that the choice $(k_1, k_2) = (k, k)$ is indeed an equilibrium. Q.E.D.

Proof of Claim 3. A *uniform strategy* is one which assigns equal probabilities to all possible pure strategies. Note that if $k_i = k_j$, then a player can guarantee a payoff of 1/2 by using the uniform strategy.

Now, consider any equilibrium. By Claim 2, $k_i = k_j$ and by Claim 1, each player's equilibrium payoff is 1/2. Denote by $p_i(b, t)$ the probability that player i's strategy, if it operates by itself, opens box b at period t. We first show that at equilibrium, $p_i(b, t) + p_i(b, t+1) = p_i(b', t) + p_i(b', t+1)$ for any period t and boxes b and b'. To see this, assume that player i adopts a search strategy such that $p_i(b, t) + p_i(b, t+1) > p_i(b', t) + p_i(b', t+1)$ for some b, b', and t. We will show that player j has a strategy that gives him an expected payoff strictly greater than the equilibrium value of 1/2. The strategy is a "modified uniform strategy." That is, it is the mixed strategy that assigns equal probabilities to all orders of search modified so that the probability mass assigned to any pure search strategy that searches b' at period t and b at period $t+1$ is reassigned in the same order with the exception that b is searched at period t and b' at $t+1$. Since the uniform strategy is a strategy that yields the payoff 1/2 whatever the other player does, it is enough to show that this modified uniform strategy performs better, and thus achieves a payoff greater than 1/2, which is player j's equilibrium payoff.

When player j searches box b' at time t, the probability of winning is $[p_i(b', t)/2 + \sum_{s>t} p_i(b', s)]/N$. Let q be the proportion of pure strategies of player j in which b' is searched at t and b at $t+1$. Player j's "loss" from the postponement of searching b' is $q[p_i(b', t) + p_i(b', t+1)]/2N$. The first term consists of the probability that player i opens the box b' at period t while player j postpones opening b' to period $t+1$. The second term is the probability that player i opens the box b' at period $t+1$ and player j, instead of opening the box at period t (before player i), opens it at period $t+1$ together with player i. Similarly, player j's "gain" from advancing the search of box b is $q[p_i(b, t) + p_i(b, t+1)]/2N$, which is larger than the loss of the postponement.

Now since for all i and t, $\sum_b p_i(b, t) = k$, we obtain $[p_i(b, t) + p_i(b, t+1)] = 2k/N$. In addition to the $N/k - 1$ equations $p_i(b, t) + p_i(b, t+1) = 2k/N$ in the case that $N/k > 2$ we have the equation $\sum_{t=1,\ldots,N/k} p_i(b, t) = 1$ and thus $p_i(b, t) = k/N$ for all b and t. Q.E.D.

Note that Claim 3 does not imply that the equilibrium is unique, that is, there may be an equilibrium in which the two players do not use "uniform strategies." For example, in the case of $k = 1$ and $N = 3$, both pairs of strategies in which each player equally randomizes between the three orders of search (b_1, b_2, b_3), (b_2, b_3, b_1), and (b_3, b_1, b_2) or (b_1, b_3, b_2), (b_2, b_1, b_3), and (b_3, b_2, b_1) are equilibria.

Note that when $k = N/2$, any pair (k_i, P_i) with $k_1 = k_2 = k$ is an equilibrium, including the pair of strategies in which the players split the N boxes equally between them.

4. DIFFERENT PROBABILITIES

So far we have considered the case in which there are equal probabilities that the prize is in each of the boxes. In this section we change this assumption and assume that the prize is in box b_1 with probability q^+ and that it is in one of the other boxes with the lower probability $q^- < q^+$ (where $q^+ + (N-1)\,q^- = 1$). The basic tradeoff in a player's considerations regarding the optimal search strategy is between assigning probability 1 to first examining the most promising box, b_1, and avoiding a deterministic plan that can be used by the other player to his advantage. We claim that if q^+ is not too large, the second effect dominates the first and no player assigns probability 1 to open b_1 in the first period.

CLAIM 4. *Consider the search game in which the probability of finding the prize in b_1 is larger than in the other boxes and assume that $c > 2/N$.*

There is $q^* > 1/N$ so that if $q^+ \in (1/N, q^*)$ then in any equilibrium there is a positive probability that b_1 will not be opened in the first period.

Proof. In Claim 2 we showed that for the case in which $q^+ = q^-$, in all equilibria the two players choose $k_1 = k_2 = k$ in the first stage (where k is defined in Claim 2). For some $q^{**} > 1/N$, it must be true that for any $q^+ \in (1/N, q^{**})$, $k_1 = k_2 = k$ in all equilibria. We will show that there is no equilibrium in which one of the players, say player 1, assigns probability 1 to opening b_1 at the first period.

The second stage of the game is a zero-sum game with value 1/2. Thus, it is sufficient to show that player 1 does not have a maxmin strategy s^* in which he opens b_1 at the first period with probability 1. Player 2, by responding to s^* with the same strategy s^*, can achieve a payoff of 1/2. Thus, it is sufficient to show that player 2 may achieve a higher payoff than he would obtain obtain by using s^* if he uses a search strategy in which he postpones opening b_1 to the last period while fully randomizing with respect to all other boxes. To see the effect of such a strategy, notice that it is equivalent to a complete randomization among all orderings in which b_1 is searched in the first period, followed by a random switch of one of the boxes that was assigned to the last period with b_1. By postponing opening b_1 to the last period, player 2 loses $q^+/2$. By moving a box from the last to the first period, player 2 gains $q^-[(k-1)/2(N-1)+1-(k-1)/(N-1)] - q^- k/2(N-1)$. The strategy s^* is worse iff $q^+/2 < [(k-1)/2(N-1)+1-(k-1)/(N-1)]\, q^- - kq^-/2(N-1)$. By substituting q^- we find this inequality to be equivalent to $q^+ < q^* = \min\{(2N-2k-1)/(N^2-2k), q^{**}\}$. The condition on c implying that $k < N/2$ guarantees that $q^* > 1/N$. Q.E.D.

5. DISCUSSION

An important assumption in most of the search literature is that sampling is a random draw. In this paper, we dealt with the case in which the objects sampled (stores, projects, or candidates) have specific labels, and the search procedure may utilize these addresses so that the sample is not random. Our main result (Claim 3) indicates that in our setting, players in equilibrium use a symmetric random strategy in spite of the ability to call up specific objects.

We identify three types of inefficiencies in the model. First, the competition prevents the players from splitting the set of boxes between them and therefore they may frequently search the same box twice. This is not optimal if the search is costly or if time is valuable. Notice that we did not include time loss per se in the model; however, adding this component makes little substantial difference. If time has no value then the model exhibits a second source of inefficiency, the excessive use of search units. This second result conforms to the R&D literature which has demonstrated the inefficiency arising from excessive search intensities. The third type of inefficiency arises from the fact that in equilibrium, the order in which boxes are opened is not necessarily in accordance with the likelihood that they contain the prize.

Admittedly, the model is too simple. Initially, we had planned to analyze games in which players may use more complicated decision procedures. We failed to prove clear-cut propositions for such models and sufficed with reporting on the simple model. Nevertheless, we still believe that this area of research requires more work focusing on the equilibrium structure of decision procedures.

Let us comment on some of the special assumptions we made in this paper:

1. We have considered the case in which the number of search units of each player is chosen and observed before the players decide their own particular search programs. This assumption fits the quite realistic case where the number of search units is less flexible than the orders given to the units when and where to search.

The case in which the size of the search body and its operation plan are decided simultaneously is a simultaneous game, whose analysis is not very different from our analysis except where $N/k = 2$. The reason is that when $N/k > 2$, any search program of player i having the property $p_i(b, t) = 1/(N/k)$ for all b and t is an equilibrium strategy for any subgame with whatever k_j.

On the other hand, when $k = N/2$, the two-stage model has an equilibrium in which the players split the N boxes equally. This is not an equilibrium for the simultaneous game since player 2 can benefit by reducing the number of search units to $k-1$. This will make player 2 lose the priority in reaching the prize in one box (which he will examined together with player 1 at the second period) and thus reduce the probability to reach the prize only by $1/2N$. Given our assumption $c > 1/2N$, this is a profitable deviation.

2. Our setup is different from the standard search model as we assume that there is a fixed cost to hiring a search unit while opening a box is a free operation. A model that assumes that opening boxes is costly allows discussion of the decision maker's decision of when to stop searching. We emphasize the search structure which we have simplified by assuming that opening boxes is free. The assumption that opening boxes is costly would change some of the results: The optimal search program may involve stopping the search before opening all boxes; therefore, the second stage game would not be a zero-sum game. If the cost of opening a box is small enough, we would still receive the same qualitative results.

3. We assume that during the search, players are not informed about the boxes that have already been opened by their opponent. An alternative model would assume that after each period, each player observes which boxes have already been examined by the other player. Clearly, in such situations, players would avoid opening boxes previously opened by their opponents. We, however, adopt the assumption of no observability, which reflects situations in which players cannot follow the search conducted by opponents. For example, this assumption is a characteristic typical of mental search (such as in problem solving).

4. A main feature of our model is that boxes have addresses and there is no restriction on the order in which the search unit approaches boxes. Another possible, and interesting, search problem is the case in which players must open boxes in some pre-determined order. That is, box j may be opened only after the previous $j-1$ boxes have been opened. Such a model would fit a situation where opening the $j-1$ boxes is the necessary operation enabling the searcher to be aware of the address of box j. The equilibrium search intensity in a model under such an assumption is different from the one we found in the above analysis. Take, for example, the case in which $N=9$ and $1/8 < c < 1/6$. In our model, in equilibrium, $k_1 = k_2 = 3$ whereas in the model with pre-determined order, in equilibrium, the number of search units equals 4.

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