"**Convex Preferences**": **a new definition***

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Abstract

We suggest a new way to think about convexity of preferences that does not require an algebraic structure on the set of alternatives. In the back of the decision maker's mind is a set of orderings interpreted as evaluation criteria. A preference relation is *convex* if it satisfies the following property: if for each evaluation criterion there is an element that is both inferior to b by the criterion and superior to a by the preference relation, then b is preferred to a. The definition expands the standard definition of convex preferences when applied to Euclidean spaces when the criteria are all algebraic linear orderings. For finite settings and under other general conditions, any strict convex preference relation can be represented by a maxmin utility function, $\min_k\{u_k(x)\}$, where each u_k is a utility representation of one of the criteria.

Key words: Convex preferences, Abstract convexity, Maxmin utility.

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1. A new definition of convex preferences

What are convex preferences and what basic intuitions do they carry? Often, convexity of preferences is thought of as decreasing marginal utility. Another common approach requires that if a is preferred to b then any other alternative on the line segment between a and b is also preferred to b. Both intuitions rely on the existence of an algebraic structure attached to the space of alternatives. We present a new definition of convex preferences which has an attractive verbal and intuitive meaning and can be applied to any abstract space X, even those without algebraic structure.

In this new approach, the agent has in mind a set of *primitive orderings* $\Lambda = \{\geq_k\}$ where each $\geq_k$ is a complete and transitive binary relation (which may have indifferences) over X. Each ordering represents a criterion for evaluating the alternatives and the decision maker internalizes those criteria when forming his preferences. Methods for constructing preference relations are the focus of Social Choice Theory where the (social) preference relation is a function of the individuals' preferences. As noted by Arrow and Raynaud (1986), the individuals' orderings can be viewed as representing basic criteria which the decision maker has in mind when forming his individual preferences. For a discussion of the related concept of "definable preferences", see Rubinstein (1978, 1998).

To be Λ-*convex,* a preference is required to satisfy the following consistency requirement: Given any two alternatives a and b, if for each criterion there is an element which is (i) inferior by that criterion to b and (ii) preferred to a, then b must be preferred to a. According to this definition, convexity is perceived as a scheme of argumentation, which can be used either by the decision maker himself or by someone trying to persuade him that b should be preferred to a. The argument goes as follows: You should prefer b to a, since for each of your relevant evaluation criteria there is an alternative inferior to b by that criterion which you prefer to a. To illustrate, assume that job candidates for a professorship are judged by their research achievements, teaching ability and charm. To persuade a colleague that b should be hired rather than a, one needs to show that there is c, a worse researcher than b, who is preferred by the colleague to a, that there is a candidate (which may be c or a different candidate) who is a worse teacher than b and is preferred by the colleague to

a and that there is a candidate who has less charm than b and whom the colleague ranks above a.

Definition: Let X be a set and $\Lambda = \{\geq_k\}$ be a set of *primitive orderings* on X. We say that a preference relation $\succsim$ on X is Λ-*convex* if for every $a, b \in X$ the following condition holds:

If for every $\geq_k \in \Lambda$, there is a $y_k \neq b$, such that $b \geq_k y_k$ and $y_k \succsim a$, then $b \succsim a$.

Two simple examples are (Section 4 contains several other intuitive examples of Λ-convexity):

(i) If X is a set of points (whether finite or not) on a horizontal line and Λ contains exactly two orderings, the increasing and decreasing orderings, then a strict preference is Λ-convex if and only if it is singled-peaked on X. To see why, suppose that there are three alternatives a, b, c ordered on the line $a - b - c$ so that $a, c \succ b$. Denote the increasing ordering by $\geq_I$ and the decreasing ordering by $\geq_D$. Without loss of generality, suppose that $a \succsim c$.Then, for each ordering, there is an alternative which is lower than b and preferred to c ($b \geq_I a$ and $a \succsim c$, and $b \geq_D c$ and $c \succsim c$). Therefore, by Λ-convexity, $b \succsim c$, a contradiction.

(ii) Given P, a partial order on a set X, if Λ is all completions of P, then Λ-convexity is equivalent to monotonicity with respect to P. In particular, when Z is a set of objects, X is the set of subsets of Z and P is the inclusion relation, the preference relation $\succsim$ is Λ-convex if and only if $A \succsim B$ whenever $A \supseteq B$.

The first claim shows that for continuous preference relations on a Euclidean space, the standard Euclidean notion of convexity is equivalent to our notion of Λ-convexity when the set Λ is taken to be the set of all algebraic linear orderings (that is, the orderings which are represented by linear functions). It is hard to think that the set of all linear orderings serves as a natural set of building blocks in the formation of a preference relation. Nevertheless, we bring this result as it demonstrates that the notion of Λ-convexity generalizes the standard convexity notion for continuous preferences and therefore Λ-convexity may be technically useful also in other interesting economic settings.

Claim 1: Let X be an N-dimensional Euclidean space and let Ψ be the set of all algebraic linear orderings. The following two statements about a continuous preference relation $\succsim$ are equivalent:

(i) $\succsim$ is convex by the standard definition; and

(ii) $\succsim$ is Ψ-convex.

Proof: Assume (i). Take two different alternatives $a,b \in X$ such that for every $\geq_k$ there is a $y_k \neq b$ such that $b \geq_k y_k$ and $y_k \succsim a$. We show $b \succsim a$ by contradiction. Suppose $a \succ b$. Then by the separating hyperplane theorem and the convexity of $U_{\succsim}(a) = \{z : z \succsim a\}$, there is some algebraic ordering $\geq_l$ such that b lies strictly below $U_{\succsim}(a)$. Since $b \geq_l y_l$, it follows that $y_l \notin U_{\succsim}(a)$, and therefore $a \succ y_l$, contradiction.

Assume (ii). Take two points a and b such that $b \succsim a$. Then, for any point c between a and b and any algebraic linear ordering $\geq_k$, it is the case that $c \geq_k a$ or $c \geq_k b$ and both a and b are preferred to a. Thus, by the definition of Ψ-convexity, $c \succsim a$. ■

Note that defining Ψ_+ to be the set of algebraic linear orderings with non-negative coefficients, a continuous preference relation $\succsim$ is Ψ_+-convex if and only if $\succsim$ is monotonic and convex in the standard sense.

Also, note that the set of Λ-convex orderings is always non-empty. Specifically, every primitive ordering $\geq_l \in \Lambda$ is Λ-convex. To see this, notice that if for every $\geq_k$ there is a y_k, such that $y_k \geq_l a$ and $b \geq_k y_k$, then, in particular, for l, $y_l \geq_l a$ and $b \geq_l y_l$ and thus $b \geq_l a$.

We have just seen that Λ-convexity generalizes the standard notion of convexity. We now provide a definition of Λ-strict convexity that generalizes the standard Euclidean notion of strict convexity.

Definition: Let X be a set and $\Lambda = \{\geq_k\}$ be a set of orderings on X. We say that a preference relation $\succsim$ on X is Λ-*strict convex* if for every $a,b \in X$ the following condition holds:

If for every $\geq_k$, there is a $y_k \neq b$ such that $b \geq_k y_k$ and $y_k \succsim a$, then $b \succ a$.

Clearly, any Λ-strict convex preferences is Λ-convex. While there always exist Λ-convex preferences, there are degenerate cases for which there are no Λ-strict convex preferences. Specifically, when there are two alternatives a,b such that for every primitive ordering $\geq_k$, $a \sim_k b$, then Λ-convexity requires both that $a \succsim b$ and $b \succsim a$. Then, Λ-strict convexity implies that $a \succ b$ and $b \succ a$ and thus there are no Λ-strict convex preferences. (Note that in the Euclidean setting with $\Lambda = \Psi$, two points cannot be equally ranked by every primitive ordering, because for any two points there is an algebraic linear ordering that separates them.)

2. A Maxmin Representation of Convex Preferences

In this section, we explore the relationship between Λ –convex preferences and the existence of a maxmin utility representation, $\min_k U_k(y)$, where each U_k is a utility representation of $\geq_k$.

Formally, we say that a preference relation $\succsim$ over X has a Λ *–maxmin representation* if there exists a set of utility functions $\{U_k\}$ such that:

(i) for each $\geq_k$ in Λ, the function U_k is a utility representation of $\geq_k$; and

(ii) $x \succsim y \Leftrightarrow \min_k U_k(x) \geq \min_k U_k(y)$.

The existence of such a representation means that we can identify each element in the set X with a vector of numbers in $\mathbb{R}^\Lambda$ such that:

(i) for each primitive ordering, the values that are attached to the elements in X at the corresponding coordinate preserve the way in which this ordering ranks the alternatives; and

(ii) the minimum value that is attached to an alternative across the different dimensions specifies how the alternatives are ranked.

We first verify that any preference relation which has a Λ-maxmin representation is Λ-convex:

Claim 2: If $\succsim$ has a Λ-maxmin representation, then $\succsim$ is Λ-convex.

Proof: Suppose that for every $\geq_k$, there is a y_k such that $b \geq_k y_k$ and $y_k \succsim a$. Then, for each $\geq_l$, $U_l(b) \geq U_l(y_l) \geq \min_k U_k(y_l) \geq \min_k U_k(a)$, and therefore $\min_k U_k(b) \geq \min_k U_k(a)$, which implies $b \succsim a$. ■

In order to make the claim in the opposite direction, we need to employ one more concept. Recall the familiar Euclidean property that for any indifference curve of a strict convex preference relation and any point x on that curve, there is a hyperplane tangent to the indifference curve which intersects it only at x. This motivates the following definition:

Given a preference relation $\succsim$, the set $Critical(z)$ is all $\geq_k \in \Lambda$ for which "if $y \succsim z$ and $y \neq z$, then $y >_k z$".

The set $Critical(z)$ is analogous to the subdifferential of $\succsim$ at the point z (each member of the subdifferential is characterized by a tangent of the indifference curve at z). Define $C_k = \{c \mid \geq_k \in Critical(c)\}$.

Just as a standard strictly convex preference relation has a nonempty subdifferential at every point, we will now see that if $\succsim$ is a Λ-strict convex preference relation, then $Critical(z) \neq \emptyset$ for all z.

Lemma: Let $\succsim$ be a Λ-strict convex preference relation on X. Then, $Critical(z) \neq \emptyset$ for all z and $\cup C_k = X$.

Proof: Assume that $Critical(z) = \emptyset$ for some $z \in X$. Then, for every $\geq_k$ there exist $y_k \in X - \{z\}$ such that $z \geq_k y_k$ and $y_k \succsim z$. By the definition of Λ-strict convexity $z \succ z$, a contradiction. ■

We now show that if X is finite, then any Λ-strict convex preference relation has a Λ–maxmin representation.

Claim 3: Let X be a finite set. Any Λ-strict convex preference relation $\succsim$ on X has a Λ-maxmin representation.

Proof: Let U be a utility function representing $\succsim$.

For every z, define $U_k(z) = U(z)$ for all orderings $\geq_k \in Critical(z)$. Notice that for every k the function U represents the ordering $\geq_k$ on $C_k = \{c \mid \geq_k \in Critical(c)\}$. To see this, assume that $\geq_k \in Critical(y) \cap Critical(z)$ and $y \neq z$. First, it cannot be that $U(y) = U(z)$ because then $y >_k z$ since $\geq_k \in Critical(z)$ and $z >_k y$ since $\geq_k \in Critical(y)$, a contradiction. If $U(y) > U(z)$, then $y \succ z$ and by the definition of $Critical(z)$ we have $y >_k z$.

In order to expand the definitions of U_k (for each k) to the entire set X, count the elements of C_k by $c_1 >_k \ldots >_k c_L$ and define the partition of X as $D_0 = \{x \mid x >_k c_1\}$, $D_i = \{x \mid c_i \geq_k x >_k c_{i+1}\}$ and $D_L = \{x \mid c_L \geq_k x\}$. For every $z \neq c_i$, if $c_i \geq_k z$, then since c_i is critical $c_i \succ z$. Therefore, for all $z \in D_i \backslash \{c_i\}$, $c_i \succ z$. Define $U_k(z) = U(c_i)$ for all $z \sim_k c_i$. Notice that in this case $U_k(z) = U(c_i) > U(z)$ since $c_i \succ z$. For all other elements in D_i define U_k to represent $\geq_k$ with values taken from the open interval $(\max\{U(z) : c_i \succ z\}, U(c_i))$, which guarantees for such z that $U_k(z) > U(z)$. Therefore, U_k represents $\geq_k$.

Thus, for all $x \in C_k$, $U_k(x) = U(x)$ and for all $x \notin C_k$, $U_k(x) > U(x)$. Since $X = \cup C_k$, it follows that $\min_k U_k(x) = U(x)$ for all x. ■

It is revealing to see the analogy between Claim 3 and the following representation result for the case in which X is a compact convex set in $\Re^2$ and Λ consists of all the algebraic linear orderings. A related previous result in this spirit appears in Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011).

Claim 4: Let X be a compact convex subset of $\Re^2$ and let Ψ be the set of all algebraic linear orderings on X. If $\succsim$ is a continuous Ψ-strict convex preference relation, then $\succsim$ has a Ψ-maxmin representation.

Proof: Since X is compact and the preference $\succsim$ and the primitive orderings are continuous, then for each $\geq_k$ there is an element, a_k, which is $\succsim$-maximal element in the set of $\geq_k$ -minimal elements, and $a_k \in C_k$. Thus, for every $\geq_k$, $C_k \neq \emptyset$. Define $\overline{C}_k$ to be the set of all x such that $x \sim_k y$ for some $y \in C_k$.

Note that it cannot be that $x,y \in C_k$ and $x \sim_k y$ since WLOG $x \succsim y$ and then $y \in C_k$ leads to $x >_k y$.

We now turn to define for each k a function U_k which represents $\geq_k$, such that $\min_k U_k(x)$ represents $\succsim$. In the construction, we use $U : X \to [0,1]$, a continuous utility representation of $\succsim$ (whose existence is guaranteed by the continuity of $\succsim$).

First, for every $x \in \overline{C_k}$ define $U_k(x) = U(y)$ where y is the unique element in C_k for which $x \sim_k y$. To see that U_k represents $\geq_k$ on $\overline{C_k}$, let $x,y \in \overline{C_k}$. Then, there exists $\hat{x},\hat{y} \in C_k$ such that $x \sim_k \hat{x}$, $y \sim_k \hat{y}$. If $x >_k y$, then $\hat{x} >_k \hat{y}$ and since $\hat{x} \in C_k$, it must be that

$\hat{x} \succ \hat{y}$, and therefore $U_k(x) = U(\hat{x}) > U(\hat{y}) = U_k(y)$. If $x \sim_k y$, then there exists $\hat{x} \in C_k$, such that $x \sim_k y \sim_k \hat{x}$ and $U_k(x) = U_k(y) = U(\hat{x})$.

Figure 1 illustrates the construction of U_k. For the case where y is critical for $\geq_k$, we have $U_k(y) = U(y)$. If y is not critical for $\geq_l$ and another point $x \in C_l$ is found where $x \sim_l y$ then $U_l(y)$ is assigned $U(x)$ (of course $U_l(x)$ is also assigned $U(x)$ which is necessary because $x \sim_l y$). As demonstrated by Figure 1, for every ordering $\geq_l$ different than $\geq_k$, the point x resides on a higher indifference curve, so $U_l(y) = U_l(x) > U(y)$.

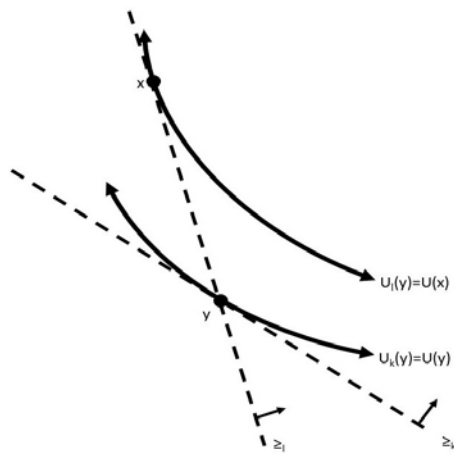

Fig.1: Assignment of $U_k(y)$ and

Note that by the convexity of $\succsim$, if $y_1 >_k x >_k y_2$ and $y_1, y_2 \in C_k$ then $x \in \overline{C}_k$. Furthermore, since a_k is $>_k$ -minimal and $a_k \in C_k$, the only x for which U_k is undefined are all strictly superior by $>_k$ to all members of C_k.

It is straightforward to expand U_k to represent $\geq_k$, by defining U_k to represent $\geq_k$ with $U_k(x) > 1$ for all such x.

It remains to be shown that for every $x \in X$, $\min_k U_k(x) = U(x)$. By the lemma, for each $x \in X$, there is some $>_k$, such that $x \in C_k$, and $U_k(x) = U(x)$. For any k such that $x \in \overline{C_k} \backslash C_k$, $x \sim_k \hat{x}$ for some $\hat{x} \in C_k$. Since $\hat{x} \in C_k$, $x \neq \hat{x}$ and $x \sim_k \hat{x}$, it must be that $U(\hat{x}) > U(x)$, and therefore $U_k(x) = U(\hat{x}) > U(x)$. Finally, for any k such that $x \notin \overline{C_k}$, $U_k(x) > 1 \geq U(x)$. ■

As mentioned above, when Λ is the set of all algebraic linear orderings, a continuous preference relation is Λ-strict convex if and only if it is strictly convex in the standard sense. Therefore, Claim 4 demonstrates that any continuous strictly convex preference relation on a compact convex subset of $\Re^2$ has a Λ-maxmin representation of the form $\min_k F_k(x \bullet e_k)$ where e_k is a vector in $\Re^2$ which points in the direction of $\geq_k$ and F_k is a strictly increasing function. In Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011), the authors establish that any continuous convex preference relation (not necessarily strict) has a representation using weakly increasing F_k. That is, they represent a larger class of preferences for R^N, but they are not Λ-maxmin representations because the F_k are $\geq_k$ -increasing but do not represent $\geq_k$.

3. A General Maxmin Representation Theorem

We now prove the existence of a Λ-maxmin representation for pairs X and Λ which satisfy the following *Betweenness* condition: for every $x,y \in X$ and ordering $\geq_l \in \Lambda$:

If $y >_l x$, then there exists $z \in X$ such that

(i) $y >_l z >_l x$ and (ii) $z \geq_k x$ or $z \geq_k y$ for all other $\geq_k$.

This condition is inspired by the Euclidean setting. In any convex subset of Euclidean space with Λ equal to any collection of linear orderings, an even stronger property holds: for any x and y and any point z on the line segment between them, z is sandwiched between x and y according to every algebraic linear ordering. Examples of non-convex sets that satisfy the Betweenness condition with $\Lambda = \Psi_+$ are depicted in Figure 2. Some of these figures, such as the square boundary, are also convex for $\Lambda = \Psi_-$, but none of them satisfy the condition when $\Lambda \supseteq \Psi_+ \cup \Psi_-$ (since different z's might be used for Ψ_+ and Ψ_-). The only closed sets in R^2 which satisfy Betweenness with $\Lambda = \Psi$ are the standard convex sets.

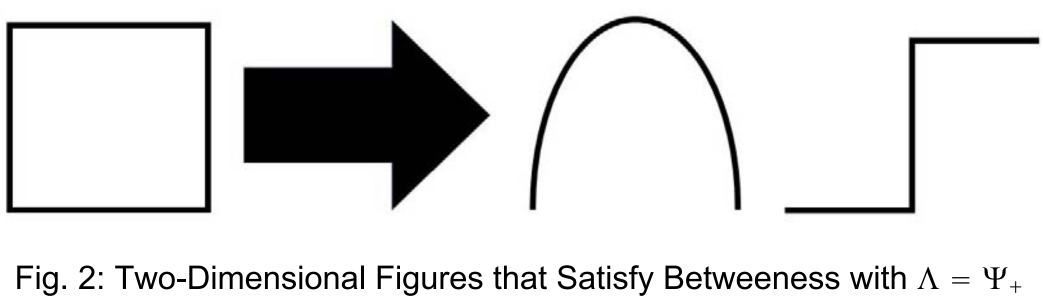

Fig. 2: Two-Dimensional Figures that Satisfy Betweeness with $\Lambda = \Psi_+$

Claim 5: Let X be a compact metric space and Λ be a set of continuous primitive orderings satisfying Betweenness. Then, any continuous Λ-strict convex preference relation $\succsim$ has a Λ-maxmin representation.

Proof: Let U and V_k be continuous functions representing $\succsim$ and $\geq_k$ respectively, each with a range of $[0,1]$. Recall that for every $\geq_k \in \Lambda$, the set C_k is defined as $C_k = \{x \mid \geq_k \in Critical(x)\}$. Notice that there cannot be $x \sim_k y$ such that $x,y \in C_k$. By $y \in C_k$, if $x \succsim y$, then $x >_k y$. Let $cl(C_k)$ denote the topological closure of C_k and define $\overline{cl(C_k)} = \{y : y \sim_k x$ for some $x \in cl(C_k)\}$.

We now define for each k a function U_k that represents $\geq_k$, such that $\min_k U_k(x)$ represents $\succsim$.

Step 1: Defining U_k on $\overline{cl(C_k)}$.

For each $x \in \overline{cl(C_k)}$, define $U_k(x) = \max\{U(y) : y \sim_k x\}$. Notice that this definition implies that for all $x \in C_k$, $U_k(x) = U(x)$. To see this, note that since $x \sim_k x$ we have $U_k(x) \geq U(x)$ and by the definition of C_k there is no y such that $U(y) > U(x)$ and $y \sim_k x$.

Step 2: If $x >_k y$ where $x \in cl(C_k)$ and y is arbitrary, then $x \succ y$.

Suppose by contradiction that $y \succsim x$. By Betweenness, there exists w such that $x >_k w >_k y$ and for all other l, $w \geq_l x$ or $w \geq_l y$.

Case (i): $x \succsim w$. Then $y \succsim x \succsim w$ and for every l, either $w \geq_l x$ or $w \geq_l y$ which implies by strict convexity that $w \succ w$, a contradiction.

Case (ii): $w \succ x$. Since $x \in cl(C_k)$, by the continuity of $\succsim$ and $\geq_k$, we have $w \succ \hat{x}$ and $\hat{x} >_k w$ for some $\hat{x} \in C_k$ sufficiently near x, thus violating $\hat{x} \in C_k$.

Step 3: U_k represents $\geq_k$ on $\overline{cl(C_k)}$.

Let $x,y \in \overline{cl(C_k)}$. By definition, if $x \sim_k y$, then $U_k(x) = U_k(y)$. Now, suppose that $x >_k y$, and take $w \in cl(C_k)$, such that $w \sim_k x$, and some $z \in X$, such that $z \sim_k y$ and $U_k(y) = U(z)$. Then $w >_k z$. Since $w \in cl(C_k)$, then by step 2, $w \succ z$. Thus, $U_k(x) = U_k(w) \geq U(w) > U(z) = U_k(y)$.

Step 4: Extension of U_k for $x \notin \overline{cl(C_k)}$.

Since $cl(C_k)$ is a closed subset of a compact set and V_k is continuous, the set of numbers $V_k(cl(C_k))$ is also closed and $V_k\left(\overline{cl(C_k)}\right) = V_k(cl(C_k))$ is therefore closed as well. Thus, the set $[0,1]\backslash V_k(cl(C_k))$ is a collection of disjoint open intervals of the form (a,b), $[0,a)$ or $(b,1]$.

Case (i) Take $x \notin \overline{cl(C_k)}$ which according to $\geq_k$ is neither strictly above nor strictly below all members of $\overline{cl(C_k)}$. Then, $V_k(x)$ lies on one such open interval $(a,b) = (V_k(\alpha), V_k(\beta))$ where $\alpha, \beta \in cl(C_k)$. Define $W(V_k(x)) = \max\{U(y) : x \sim_k y\}$. Let $\overline{W} : (a,b] \to (U_k(\alpha), U_k(\beta)]$ be the upper convex envelope of W on $[a,b]$. To see that $\overline{W}$ is strictly increasing, it must be that if $\beta >_k x >_k \alpha$, then $U_k(\beta) > W(V_k(x))$. To see it, notice that since $\beta \in cl(C_k)$ and $\beta >_k x \sim y$, then by step 2, $U(\beta) > U(y)$. Therefore $U_k(\beta) \geq U(\beta) > U(y)$ and $U(\beta) > \max\{U(y) : x \sim_k y\} = W(V_k(x))$. Define $U_k(x) = \overline{W}(V_k(x))$. The function U_k represents $\geq_k$ for any x,y, such that $b \geq V_k(x), V_k(y) \geq a$ since U_k is a strict monotonic transformation of V_k. Furthermore, $U_k(x) = \overline{W}(V_k(x)) \geq W(V_k(x)) \geq U(x)$.

Case (ii) There is no $x \notin \overline{cl(C_k)}$ which according to $\geq_k$ is strictly below all members of $\overline{cl(C_k)}$. This is because a $\succsim$-maximal element of $V_k^{-1}(0)$ is necessarily in C_k.

Case (iii) Take $x \notin \overline{cl(C_k)}$ which according to $\geq_k$ is strictly above all members of $\overline{cl(C_k)}$. On such an interval $[a,1]$, we can simply define $W(1) = 2$ and then allow $\overline{W}$ to be the upper convex envelope of W on $[a,1]$. Define $U_k(x) = \overline{W}(V_k(x))$. Since $1 \geq W$, it is the case that $\overline{W}$ is strictly increasing and therefore U_k represents $\geq_k$ for x,y, such that $1 \geq V_k(x), V_k(y) \geq a$ and $U_k(x) = \overline{W}(V_k(x)) \geq W(V_k(x)) \geq U(x)$.

Step 5: $U(x) = \min_k U_k(x)$.

By construction, for all k, $U_k(x) \geq U(x)$. By the Lemma, for every x there is an ordering $\geq_k$ such that $x \in C_k$ and for this ordering $U_k(x) = U(x)$. ■

4. Examples

We now introduce a few examples in order to demonstrate the notion of Λ-convex preferences and its maxmin representation:

i) Let $X = \mathbb{R}^2$ and let Λ consists of the two primitive orderings: "right" and "up". The continuous and monotonic Λ-convex preferences are all preferences with right-angled indifference curves.

Any such preference relation has a Λ -maxmin representation of the form $U(x,y) = \min(f(x), g(y))$ where f and g are strictly increasing functions. At the corners of the indifference curves we have $f(x) = g(y)$ and the map $f \times g : (x,y) \to (f(x), g(y))$ rescales $\mathbb{R}^2$ so that all of the corners lie on the main diagonal line.

ii) Consider a set X consisting of two disjoint sets of $Females$ and $Males$. A relation S "objectively" ranks all members of X. The set Λ consists of two primitive orderings: $\geq_f$, which ranks all females above all males and any two members of the same gender by S, and the analogous ordering $\geq_m$. The Λ-convex orderings are exactly those which are monotonic in S when comparing any two individuals of the same gender.

As to the maxmin representation, notice that $C_f = M$ and $C_m = F$. Given a Λ-convex preference on X let U be a utility representation with values from the interval $[0, 1]$. Define $U_f(x) = U(x)$ for $x \in M$ and $U_f(x) = U(x) + 1$ for $x \in F$ and define U_m analogously. Then U_f, U_m represent $\geq_f$, $\geq_m$ respectively and $U(x) = \min\{U_f(x), U_m(x)\}$.

iii). Let X be the real line and Λ consist of two single-peaked orderings expressed by $D_0 = -|x|$ and $D_3 = -|x-3|$. The Λ-strictly convex preference relations are all single-peaked preferences with a peak between 0 and 3. The peak cannot be located at any $p > 3$, since $D_0(3) > D_0(p)$ and $D_3(3) > D_3(p)$, and Λ-strict convexity implies that $3 \succ p$. Similarly, the preference cannot have a peak left of 0.

To understand the Λ-maxmin representation, consider the Λ-strictly convex preference relation represented by the utility function $D_2(x) = -|x-2|$. The utility functions $U_0(x) = -|x|+2$ and $U_3(x) = -|x-3|+1$ represent the two primitive orderings and $D_2(x) = \min\{U_0(x), U_3(x)\}$ (see Figure 3).

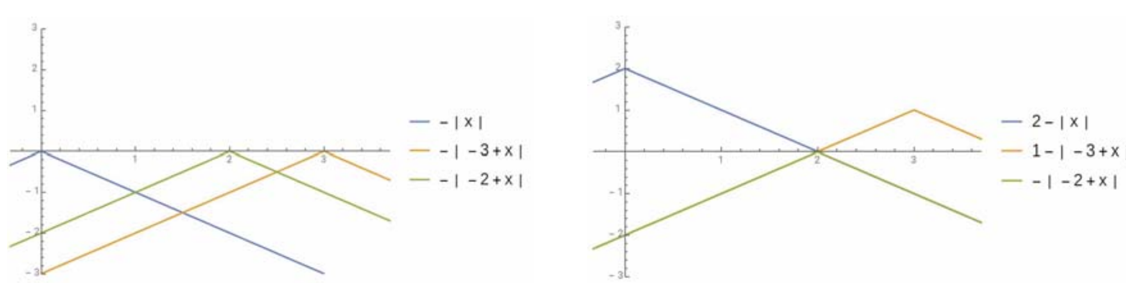

Fig. 3: Left: D_0, D_2 and D_3. Right: U_0, D_2 and U_3.

iv) Let X be any set and Λ be the set of all strict orderings over X. Then the notion of Λ-convexity is degenerate and every preference relation is strictly Λ-convex.

Let $\succsim$ be a strictly Λ-convex preference relation and let $U, V_k : X \to [0,1]$ represent $\succsim$ and $\geq_k$ respectively. For each $\geq_k$, let m_k be the $\geq_k$ -minimal element of X. Note that $m_k \in C_k$. Define $U_k(m_k) = U(m_k)$ and $U_k(x) = V_k(x) + 1$ otherwise. It is clear that U_k represents $\geq_k$ for all $\geq_k$ and $\min_k U_k(x) = U(x)$ for all x.

In this degenerate case and for any other Λ-convexity, any representable primitive ordering $\geq_l$ has a Λ-minmax representation with $U_l(x) = U(x)$ and $U_k(x) = V_k(x) + 1$ for all $k \neq l$.

5. Comments:

(i) **Maxmin models**: Maxmin functions have a long history, starting with Wald (1950). It is interesting to compare our maxmin representation with the familiar but different maxmin representation of Gilboa and Schmeidler (1989). Let S be a finite set of states and Z a set of outcomes. An act is a function from S into Z. Gilboa and Schmeidler (1989) prove that if a preference relation over the set of acts satisfies certain axioms, then there is a function $u : Z \to \Re$ and a set C of probability measures (priors) over S such that the preference relation is represented by $U(f) = \min_{p \in C}\{p \times [U \circ f]\}$. Thus, an act is transformed subjectively into a point $U \circ f \in \mathbb{R}^S$. A probability $p^k \in C$ can be thought of as an algebraic linear function $p^k \times x$ over $\mathbb{R}^S$. And the utility of an act is the minimal value the act it receieved according to the functions attached to the priors in C.

We study a general notion of convex preferences where the primitive orderings are not necessarily algebraic linear functions. The most similar case in our framework is when alternatives are objective vectors of $\mathbb{R}^S$ and the set of primitive orderings C is a set of functions of the type $p^k \times x$. In our representation, the decision maker has in mind an increasing function U^k which he applies to the value $p^k \times f$ and then takes $\min_k\{U^1(p^1 \times f), .., U^K(p^K \times f)\}$. Thus, when applying our approach in this setting, the set of probability measures C is taken as given in contrast to Gilboa and Schmeidler (1989). Furthermore, the subjective component in our approach is the transformation of the expected value of each $p^k \in C$, whereas in Gilboa and Schmeidler (1989) the subjectivity is in the transformation of the prizes into a utility space.

(ii) **Social Choice**: As mentioned in the Introduction, another setting in which there is a natural set of primtive orderings is social choice where each ordering in the set Λ can be interpreted as a different individual's preference. The notion of Λ-convexity then can be a desirable property of a social choice rule. Note, however, that since the input of the social choice rule is taken to be a set rather than a profile, then (i) we restrict the domain of the social choice rule to profiles where all individuals have distinct orderings and (ii) all individuals are treated equally (anonymity). Allowing repetiton within Λ may influence the social choice rule, but not the meaning of Λ-convexity.

An example of a Λ-convex social choice function which also satisfies the monotonicity condition (if an individual upgrades an alternative, then it cannot be socially downgraded) is the following:

Let $X_1 = X$ and define inductively $M_j = \{x \in X$ | there is $\geq_k \in \Lambda$ such that x is minimal in $X_j\}$ and let $X_{j+1} = X_j - M_j$. Define $class(x) = l$ if $x \in M_l$ and let $Borda(x)$ be the Borda score of x applied to the set M_l only (i.e., $Borda(x) = \sum_k |\{y \in M_l : x >_k y|$). Now rank x socially at least as high as y if $class(x) > class(y)$, or, $class(x) = class(y)$ and $Borda(x) \geq Borda(y)$. It is easy to verify that when the primitive orderings are strict, this procedure lead to a Λ-convex preference.

(iii) **Other definitions** The new definition of Λ-convex preferences can lead to other intuitive consistency requirements based on a set of primitive orderings. For example: Let Λ be a set of N orderings. Define a preference relation $\succsim$ to be Λ-*almost convex*, if for all $a, b \in X$ satisfying that for at least $N-1$ primitive orderings $\exists y_k$ such that $b \geq_k y_k$ and $y_k \succsim a$, then $b \succsim a$. While this definition is "close" to our definition of Λ-convex preferences, its implications can be quite different. For example, it may be that the only Λ-almost convex preferences are total indifference. To see such an example, let Λ be the set of N orderings over $X = \{a_1, \ldots, a_N\}$ where ordering $\geq_k$ ranks $a_k >_k a_{k+1} >_k \ldots >_k a_{k-1}$. For each i, there are $N-1$ orderings for which $a_i \geq_k a_{i+1}$ (where $a_{N+1} = a_1$). Therefore, the Λ-almost convexity of $\succsim$ implies the cycle $a_1 \succsim a_2 \succsim \ldots \succsim a_N \succsim a_1$ and thus $\succsim$ must be total indifference. In this example, every preference relation is Λ-convex by our main definition, whereas even the primitive orderings themselves are not Λ-almost convex.

(iv) **Connection to abstract convexity** The standard definition of convex preferences in Euclidean spaces is equivalent to the definition that for any element $a \in X$, its upper contour set $U_{\succsim}(a) = \{x : x \succsim a\}$ is a convex set. This definition requires the concept of a "convex set". In an abstract setting, where the set X is arbitrary and lacks an algebraic structure, an attractive definition of convexity is given by Edelman and Jamison (1985). It is based on a functional K on subsets of X where $K(A)$ is interpreted as the set of all elements that are "between elements in A" and is analogous to the notion of "the convex closure of A". A set A is *convex* if $K(A) = A$. For an operator to qualify under this notion of convexity, it has to satisfy a number of axioms, all of which are satisfied by the standard convexity notion:

Extensitivity: $A \subseteq K(A)$ and $K(\emptyset) = \emptyset$.

Monotonicity: $A \subseteq B$ implies $K(A) \subseteq K(B)$.

Idempotence: $K(K(A)) = K(A)$.

Anti-exchange: If A is convex, $a, b \notin A$ and $a \in K(A \cup b)$, then $b \notin K(A \cup a)$.

Richter and Rubinstein (2015) studied this notion of convexity in economic settings. In particular, we used the fact (see Edelman and Jamison (1985)) that a set of strict orderings $\Lambda = \{\geq_k\}$ *generates* an operator $K_\Lambda(A) = \{x \mid \forall k, \exists a_k \in A \ s.t.\ x \geq_k a_k\}$ which satisfies the above properties.

The primitive orderings can be viewed as a set of criteria that individuals who are using a concept of convexity have in mind. A set is convex if for any element outside the set, one of the criteria ranks it as "inferior" to all elements in the set. Thus, the separating hyperplane theorem holds for this setting.

We now verify that for any set of orderings Λ and any preference relation $\succsim$, the relation $\succsim$ is Λ-convex if and only if every upper contour of $\succsim$ is a convex set in K_Λ.

Claim 6. Let X be a set, $\Lambda = \{\geq_k\}$ be a set of orderings on X and $\succsim$ be a preference relation on X.

The following two statements are equivalent:

(a) The preference relation $\succsim$ is Λ-convex.

(b) For every a, the set $\{x \in X \mid x \succsim a\}$ is convex in K_Λ.

Proof: Assume (a). If there exists $b \in K(\{x \in X \mid x \succsim a\}) - \{x \in X \mid x \succsim a\}$, then $a \succ b$ and for each $\geq_k$ there is a $y_k \in \{x \in X \mid x \succsim a\}$ such that $b \geq_k y_k$. This implies by the Λ-convexity of $\succsim$ that $b \succsim a$, a contradiction.

Assume (b). Also assume that there is a pair $a, b \in X$ with the property that for every ordering $\geq_k$ there is a $y_k \in X$ such that $y_k \succsim a$ and $b \geq_k y_k$ but $b \not\succsim a$. Then, $b \notin \{x \in X \mid x \succsim a\} = K(\{x \in X \mid x \succsim a\})$ and therefore there is a primitive ordering $\geq_k$ such that $y >_k b$ for all $y \in \{x \in X \mid x \succsim a\}$, which contradicts the existence of y_k. ■

References

Arrow, Kenneth J. and Herve Raynaud. 1986. *Social Choice and Multicriterion Decision-Making*. MIT Press.

Cerreia-Vioglio, Simone, Fabio Maccheroni, Massimo Marinacci and Luigi Montrucchio. 2011. "Complete Monotone Quasiconcave Duality". *Mathematics of Operations Research*, 36, 321-339.

Edelman, Paul H. and Robert E. Jamison. 1985. "The Theory of Convex Geometries". *Geometriae Dedicata*, 19, 247–270.

Gilboa, Itzhak and David Schmeidler. 1989. "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 18, 141-153.

Richter, Michael and Ariel Rubinstein. 2015. "Back to Fundamentals: Convex Geometry and Economic Equilibrium". *American Economic Review*, 105, 2570-2594.

Rubinstein, Ariel. 1978. "Definable Preference relations - Three Examples". Research Memorandum 31. The Center of Research in Mathematical Economics and Game Theory. The Hebrew University, Jerusalem.

Rubinstein, Ariel. 1998. "Definable Preferences: An Example", *European Economic Review, 42*, 553-560.

Wald, Abraham. 1950. *Statistical Decision Functions*. John Wiley.