"**Convex Preferences**": **A New Definition**[*]

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Abstract

We suggest a concept of convexity of preferences without relying on an algebraic structure. The decision maker has in mind a set of orderings interpreted as evaluation criteria. A preference relation is convex: if for each criterion there is an element that is both inferior to b by the criterion and superior to a by the preference relation, then b is preferred to a. The definition expands the standard Euclidean definition with the criteria being all algebraic linear orderings. Under general conditions, any strict convex preference relation is represented by a maxmin of utility representations of the criteria.

Key words: Convex preferences, Abstract convexity, Maxmin utility.

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1. A new definition of convex preferences

The canonical definition of convex preferences requires that if a is preferred to b, then any other alternative on the line segment between them is also preferred to b. This definition relies on the existence of an algebraic structure attached to the space of alternatives. Edelman and Jamison (1985) formulated a new notion of convexity which did not rely upon any algebraic structure. In Richter and Rubinstein (2015), we used this structure and define a new abstract equilibrium notion to derive new welfare theorems which applied to a variety of economics situations.

In this paper, we focus on preferences and present a new definition of convex preferences which has an attractive verbal and intuitive meaning. This concept generalizes the standard Euclidean notion of convex preferences and can be applied to any abstract space X, especially those without algebraic structure.

In our approach, the agent has in mind a set of *primitive orderings* $\Lambda = \{\geq_k\}$ where each $\geq_k$ is a complete and transitive binary relation (which may have indifferences) over X. Each ordering represents a criterion for evaluating the alternatives. These criteria provide a structure for the decision maker's preferences and the decision maker internalizes these criteria when forming his preferences. Methods for constructing preference relations are the focus of Social Choice Theory where the (social) preference relation is a function of the individuals' preferences. As noted by Arrow and Raynaud (1986), the individuals' orderings can be viewed as representing basic criteria which the decision maker has in mind when forming his individual preferences. For a discussion of the related concept of "definable preferences", see Rubinstein (1978, 1998).

To be Λ-*convex,* a preference is required to satisfy the following consistency requirement: Given any two alternatives a and b, if for each criterion there is an element which is (i) inferior by that criterion to b and (ii) preferred to a, then b must be preferred to a. According to this definition, convexity is perceived as a scheme of argumentation, that can be used either by the decision maker himself or by someone trying to persuade him. The argument goes as follows: You should prefer b to a, since for each of your relevant evaluation criteria there is an alternative inferior to b by that criterion which you prefer to a. To illustrate, assume that job candidates are evaluated by research, teaching and charm. To persuade a colleague that b should be hired rather than a, one needs to show him: that there is a candidate c, who is a worse researcher than b and preferred by the colleague to a; that there is a candidate (who may or may not be c) who is a worse teacher than b and is preferred by the colleague to a; and that there is a candidate who has less charm than b and whom the colleague ranks above a.

Definition: Let X be a set and $\Lambda = \{\geq_k\}$ be a set of *primitive orderings* on X. We say that a preference relation $\succsim$ (complete and transitive) on X is Λ-*convex* if for every $a, b \in X$ the following condition holds:

If for every $\geq_k \in \Lambda$, there is a $y_k \neq b$, such that $b \geq_k y_k$ and $y_k \succsim a$, then $b \succsim a$.

The set of Λ-convex orderings is always non-empty. Specifically, every primitive ordering $\geq_l \in \Lambda$ is Λ-convex: if for every $\geq_k$ there is a y_k, such that $y_k \geq_l a$ and $b \geq_k y_k$, then, in particular, for l, $y_l \geq_l a$ and $b \geq_l y_l$ and thus $b \geq_l a$.

Two simple examples of Λ-convex orderings are (Sections 2 and 3 contain several other intuitive examples):

(i) If X is a set of points (whether finite or not) on a horizontal line and Λ contains exactly two orderings, the increasing and decreasing orderings, then a strict preference is Λ-convex if and only if it is singled-peaked on X. To see this, suppose that there are three alternatives a, b, c ordered on the line $a - b - c$ so that $a, c \succ b$. Denote the increasing ordering by $\geq_I$ and the decreasing ordering by $\geq_D$. Without loss of generality, suppose that $a \succsim c$. Then, for each ordering, there is an alternative which is lower than b by the ordering and preferred to c ($b \geq_I a$ and $a \succsim c$, and $b \geq_D c$ and $c \succsim c$). Therefore, by Λ-convexity, $b \succsim c$, a contradiction.

(ii) Given P, a partial ordering on a set X, if Λ is all completions of P, then Λ-convexity is equivalent to monotonicity with respect to P. In particular, when Z is a set of objects, X is the set of subsets of Z and P is the inclusion relation, the preference relation $\succsim$ is Λ-convex if and only if $A \succsim B$ whenever $A \supseteq B$.

In Euclidean space, each non-zero vector v defines an *algebraic linear ordering* by $x \geq_v y$ if $v \cdot x \geq v \cdot y$. Denote the set of all algebraic linear orderings by Ψ. The first proposition shows that for continuous preference relations on a Euclidean space, the standard Euclidean notion of convexity is equivalent to Ψ-convexity. This result demonstrates that the notion of Λ-convexity generalizes the standard convexity notion for continuous preferences. Furthermore, defining Ψ_+ (or Ψ_-) as the set of algebraic linear orderings with non-negative (non-positive) coefficients, one can similarly show that a continuous preference relation $\succsim$ is Ψ_+-convex if and only if $\succsim$ is weakly increasing and convex in the standard sense. Thus, by properly choosing the set of primitive orderings, Λ-convexity can express both convexity and monotonicity.

Proposition 1: Let X be a convex and closed subset of $\mathbb{R}^N$. The following two statements about a continuous preference relation $\succsim$ are equivalent:

(i) $\succsim$ is convex by the standard definition; and

(ii) $\succsim$ is Ψ-convex.

Proof: Assume (i). Take two different alternatives $a, b \in X$ such that for every $\geq_k$ there is a $y_k \neq b$ such that $b \geq_k y_k$ and $y_k \succsim a$. We show $b \succsim a$ by contradiction. Suppose $a \succ b$. Since $\succsim$ is continuous and convex, the set $U_{\succsim}(a) = \{z : z \succsim a\}$ is closed and convex. Thus, by the separating hyperplane theorem, there is some algebraic ordering $\geq_l$ such that b lies strictly below $U_{\succsim}(a)$. Since $b \geq_l y_l$, it follows that $y_l \notin U_{\succsim}(a)$, and therefore $a \succ y_l$, contradiction.

Assume (ii). Take two points a and b such that $b \succsim a$. Then, for any point c between a and b and any algebraic linear ordering $\geq_k$, it is the case that $c \geq_k a$ or $c \geq_k b$ and both a and b are preferred to a. Thus, by the definition of Ψ-convexity, $c \succsim a$. ■

We have just seen that Λ-convexity generalizes the standard notion of convexity and now provide a definition of Λ-strict convexity. For the case that X is a convex closed subset of $\mathbb{R}^N$ and $\Lambda = \Psi$, one can show (similarly to Proposition 1) that this definition of Λ-strict convexity is equivalent to the standard notion of strict convexity.

Definition: Let X be a set and $\Lambda = \{\geq_k\}$ be a set of orderings on X. We say that a preference relation $\succsim$ on X is Λ-*strict convex* if for every $a, b \in X$ the following condition holds:

If for every $\geq_k$, there is a $y_k \neq b$ such that $b \geq_k y_k$ and $y_k \succsim a$, then $b \succ a$.

Clearly, any Λ-strict convex preference relation is Λ-convex. While Λ-convex preferences always exist, there are degenerate cases for which there are no Λ-strict convex preferences. Specifically, when there are two alternatives a, b such that for every primitive ordering $\geq_k$, $a \sim_k b$, then Λ-convexity requires that both $a \succsim b$ and $b \succsim a$. Then, Λ-strict convexity implies that $a \succ b$ and $b \succ a$ and thus there are no Λ-strict convex preferences. (Note that in the Euclidean setting with $\Lambda = \Psi$, two points cannot be equally ranked by every primitive ordering, because for any two points there is an algebraic linear ordering that separates them.)

2. A Maxmin Representation of Convex Preferences

In this section, we explore the relationship between Λ-convex preferences and the existence of a maxmin utility representation, $\min_k U_k(y)$, where each U_k is a utility representation of $\geq_k$.

Formally, we say that a preference relation $\succsim$ over X has a Λ-*maxmin representation* if there exists a set of utility functions $\{U_k\}$ such that:

(i) for each $\geq_k$ in Λ, the function U_k is a utility representation of $\geq_k$; and

(ii) $x \succsim y \Leftrightarrow \min_k U_k(x) \geq \min_k U_k(y)$.

The existence of such a representation means that we can identify each element in the set X with a vector of numbers in $\mathbb{R}^\Lambda$ such that:

(i) for each primitive ordering, the values that are attached to the elements in X at the corresponding coordinate preserve the way in which this ordering ranks the alternatives; and

(ii) the minimum value that is attached to an alternative across the different dimensions specifies how the alternatives are ranked.

We first verify that any preference relation which has a Λ-maxmin representation is Λ-convex:

Proposition 2: If $\succsim$ has a Λ-maxmin representation, then $\succsim$ is Λ-convex.

Proof: Suppose that for every primitive ordering $\geq_k$, there is a y_k such that $b \geq_k y_k$ and $y_k \succsim a$. Then, for each $\geq_l$, $U_l(b) \geq U_l(y_l) \geq \min_k U_k(y_l) \geq \min_k U_k(a)$, and therefore $\min_k U_k(b) \geq \min_k U_k(a)$, which implies $b \succsim a$. ■

We next show that the proposition is true in the opposite direction under Λ-strict convexity. The opposite direction needs more than Λ-convexity: the total indifference preference relation is always Λ-convex, but may not have a Λ-maxmin representation: suppose $X = \{a,b,c\}$ and Λ consists of the two orderings $a >_1 b >_1 c$ and $c >_2 b >_2 a$. Then, for any Λ-maxmin representation, it must be that $U_1(b) > U_1(c)$ and $U_2(b) > U_2(a)$. Thus, $\min_k U_k(b) > \min(U_1(c), U_2(a)) \geq \min(\min_k U_k(c), \min_k U_k(a))$ and so by any maxmin utility, b must get a strictly higher value than at least one of a and c.

In order to proceed, we need one additional concept. Recall the familiar Euclidean property that for any indifference curve of a strict convex preference relation and any point x on that curve, there is a hyperplane tangent to the indifference curve which intersects it only at x. This motivates the following definition:

Given a preference relation $\succsim$, the set $Critical(z)$ contains each ordering $\geq_k \in \Lambda$ satisfying that "for every $y \neq z$, if $y \succsim z$, then $y >_k z$".

The set $Critical(z)$ is analogous to the subdifferential of $\succsim$ at the point z (each member of the subdifferential is characterized by a tangent of the indifference curve at z). Define $C_k = \{c \mid \geq_k \in Critical(c)\}$. Notice that the elements of C_k are strictly ordered by $\geq_k$: given any two distinct elements $x, y \in C_k$ where $x \succsim y$, we have $x >_k y$ because $\geq_k \in Critical(y)$.

A standard strictly convex preference relation has a nonempty subdifferential at every point. Analogously, we now show that if $\succsim$ is a Λ-strict convex preference relation, then $Critical(z) \neq \emptyset$ for all z.

Lemma: Let $\succsim$ be a Λ-strict convex preference relation on X. Then, $Critical(z) \neq \emptyset$ for all z and $\cup C_k = X$.

Proof: Assume that $Critical(z) = \emptyset$ for some $z \in X$. Then, for every $\geq_k$ there exist $y_k \in X - \{z\}$ such that $z \geq_k y_k$ and $y_k \succsim z$. By the definition of Λ-strict convexity $z \succ z$, a contradiction. ■

We now show that if X is finite, then any Λ-strict convex preference relation has a Λ-maxmin representation.

Proposition 3: Let X be a finite set. Any Λ-strict convex preference relation $\succsim$ on X has a Λ-maxmin representation.

Proof: Let U be a utility function representing $\succsim$. For every z, define $U_k(z) = U(z)$ for all orderings $\geq_k \in Critical(z)$. Notice that for every k the function U represents the ordering $\geq_k$ on $C_k = \{c \mid \geq_k \in Critical(c)\}$. To see this, assume that $y \neq z$ and $\geq_k \in Critical(y) \cap Critical(z)$. First, it cannot be that $U(y) = U(z)$ because then $y >_k z$ (since $\geq_k \in Critical(z)$) and similarly $z >_k y$ (since $\geq_k \in Critical(y)$), a contradiction. If $U(y) > U(z)$, then $y \succ z$ and by the definition of $Critical(z)$ we have $y >_k z$.

In order to expand the definitions of U_k (for each k) to the entire set X, count the elements of C_k by $c_1 >_k \ldots >_k c_L$ and define the partition of X as $D_0 = \{x \mid x >_k c_1\}$, $D_i = \{x \mid c_i \geq_k x >_k c_{i+1}\}$ and $D_L = \{x \mid c_L \geq_k x\}$. For every $z \neq c_i$, if $c_i \geq_k z$, then $c_i \succ z$ since c_i is critical. Therefore, for all $z \in D_i \backslash \{c_i\}$, $c_i \succ z$. Define $U_k(z) = U(c_i)$ for all $z \sim_k c_i$. Notice that in this case $U_k(z) = U(c_i) > U(z)$ since $c_i \succ z$. For all other elements in D_i define U_k to represent $\geq_k$ with values taken from the open interval $(\max\{U(z) : c_i \succ z\}, U(c_i))$, which guarantees that $U_k(z) > U(z)$ for such z. Therefore, U_k represents $\geq_k$.

Thus, for all $x \in C_k$, $U_k(x) = U(x)$ and for all $x \notin C_k$, $U_k(x) > U(x)$. Since, by the Lemma, $X = \cup C_k$, it follows that $\min_k U_k(x) = U(x)$ for all x. Recall that U represents $\succsim$, and thus, $\succsim$ has a Λ-maxmin representation.■

In order to demonstrate the concept of maxmin representation, consider the following example:

Example 1: The set X consists of two disjoint sets of $Females$ and $Males$. A relation S "objectively" ranks all members of X. The set Λ consists of two primitive orderings: $\geq_f$, which ranks all females above all males and any two members of the same gender by S, and the analogous ordering $\geq_m$. The Λ-convex orderings are exactly those that are monotonic in S when comparing any two individuals of the same gender.

As to the maxmin representation, notice that $C_f = M$ and $C_m = F$. Given a Λ-convex preference on X, let U be a utility representation with values from the interval $[0, 1]$. Define $U_f(x) = U(x)$ for $x \in M$ and $U_f(x) = U(x) + 1$ for $x \in F$ and define U_m analogously. Then, $U(x) = \min\{U_f(x), U_m(x)\}$ where U_f and U_m represent $\geq_f$ and $\geq_m$ respectively.

Example 2:.Preferences over Menus: Let Z be a finite set of alternatives and X be the set of all non-empty subsets of Z. Given a utility function over alternatives

$u : Z \to \mathbb{R}$, a preference relation $\geq_u$ is induced over the menus in X by $A \geq_u B$ if $\max_{z \in A} u(z) \geq \max_{z \in B} u(z)$. In words, each menu is evaluated by its u-best alternative. Let Λ consist of all such induced orderings over X.

The next Lemma shows that for this choice of Λ, the property of Λ-strict convexity is equivalent to the property of strict monotonicity ($B \supset A$ implies $B \succ A$).

Lemma: Let Z be a finite set, X be the set of menus from Z and Λ be the set of all preferences over X that are induced from orderings over Z. A preference on X is Λ-strictly convex if and only if it is strictly monotonic.

Proof: Let $\succsim$ be a Λ-strictly convex preference relation. To show that $\succsim$ is strictly monotonic, consider any two menus $A \subset B \in X$. Clearly, $B \geq_k A$ for every $\geq_k \in \Lambda$ and thus $B \succ A$ by the definition of Λ-strict convexity (taking $y_k = A$).

Let $\succsim$ be a strictly monotonic preference on X and $A, B \subseteq Z$ be two menus. Suppose that $\forall \geq_k \in \Lambda$, $\exists Y_k \neq B$, such that $B \geq_k Y_k$ and $Y_k \succsim A$. Then, take $\geq_l \in \Lambda$ generated by $u(z) = 0$ for all $z \in B$ and $u(z) > 0$ for all $z \notin B$. Such $\geq_l$ bottom-ranks B and all of its subsets and ranks all other sets above it. Thus, $B \geq_l Y_l$ implies that $B \supset Y_l$ (inclusion is strict because $Y_l \neq B$) and $B \succ Y_l$ by the strict monotonicity of $\succsim$. Therefore, $B \succ Y_l \succsim A$ which implies that $B \succ A$, and thus the Λ-strict convexity condition holds. ■

Recall that Proposition 3 stated that any strict Λ-convex preference relation has a maxmin representation. Thus, given any strictly monotonic preference over menus, there exists a set U of utility functions over Z such that:

$$A \succsim B \text{ if and only if } \min_{u \in U} \max_{z \in A} u(z) \geq \min_{u \in U} \max_{z \in B} u(z).$$

One can think about such a representation as a state-dependent maxmin utility. The individual does not know his future preferences over Z, but will know them when he chooses from the menu. He evaluates each menu by the worst possible state for that menu. This conclusion was proved by Natenzon (2016) who in fact shows that any weakly monotonic preference over menus $\succsim$ can be represented in this manner. Note that this result differs from that of Kreps (1979) who provides a characterization of menu preferences that have a subjective expected utility representation which requires weak monotonicity, as well as an additional submodularity property.

3. A Maxmin Representation Theorem for compact metric spaces

In Proposition 3 we proved that when X is finite, any Λ-strictly convex preference relation has a Λ-maximin representation. Proposition 5 below is an analogous result (with additional continuity-type restrictions) for compact metric spaces. That result requires a significant amount of technical machinery and therefore we present Proposition 4 to illustrate some of the ideas in a simpler Euclidean setting. It states that any strictly convex preference relation on $\mathbb{R}^2$ according to the standard convexity has a Ψ-maxmin representation and it employs a constructive proof.

Proposition 4: Let X be a compact convex subset of $\Re^2$ and let Ψ be the set of all algebraic linear orderings on X. If $\succsim$ is a continuous Ψ-strict convex preference relation, then $\succsim$ has a Ψ-maxmin representation.

Proof: Since X is compact and the preference $\succsim$ and the primitive orderings are continuous, then for each $\geq_k$ there is an element, a_k, which is a $\succsim$-maximal element in the set of $\geq_k$ -minimal elements. By Ψ-strict convexity, a_k is unique (otherwise between two such a_k there would be a strictly $\succsim$-superior alternative). Therefore, a_k is strictly preferred to all other $\geq_k$ -minimal elements and for any $z \succsim a_k$, it must be that $z >_k a_k$ and therefore $a_k \in C_k$. Thus, for every $\geq_k$, $C_k \neq \emptyset$. Define $\overline{C_k}$ to be the set of all x such that $x \sim_k y$ for some $y \in C_k$.

Note that it cannot be that $x,y \in C_k$ and $x \sim_k y$ since WLOG $x \succsim y$ and then $y \in C_k$ leads to $x >_k y$.

We now turn to defining for each k a function U_k that represents $\geq_k$, such that $\min_k U_k(x)$ represents $\succsim$. In the construction, we use $U : X \to [0, 1]$, a continuous utility representation of $\succsim$ (whose existence is guaranteed by the continuity of $\succsim$).

First, for every $x \in \overline{C_k}$ define $U_k(x) = U(y)$ where y is the unique element in C_k for which $x \sim_k y$. To see that U_k represents $\geq_k$ on $\overline{C_k}$, let $x,y \in \overline{C_k}$. Then, there exists $\hat{x},\hat{y} \in C_k$ such that $x \sim_k \hat{x}$, $y \sim_k \hat{y}$. If $x >_k y$, then $\hat{x} >_k \hat{y}$ and since $\hat{x} \in C_k$, it must be that $\hat{x} \succ \hat{y}$, and therefore $U_k(x) = U(\hat{x}) > U(\hat{y}) = U_k(y)$. If $x \sim_k y$, then there exists $\hat{x} \in C_k$, such that $x \sim_k y \sim_k \hat{x}$ and $U_k(x) = U_k(y) = U(\hat{x})$.

Figure 1 illustrates the construction of U_k. For the case where y is critical for $\geq_k$, we have $U_k(y) = U(y)$. If y is not critical for $\geq_l$ and another point $x \in C_l$ is found where $x \sim_l y$, then $U_l(y)$ is assigned $U(x)$. As demonstrated in Figure 1, for every ordering $\geq_l$

different than $\geq_k$, the point x resides on a higher indifference curve, and so $U_l(y) = U_l(x) > U(y)$.

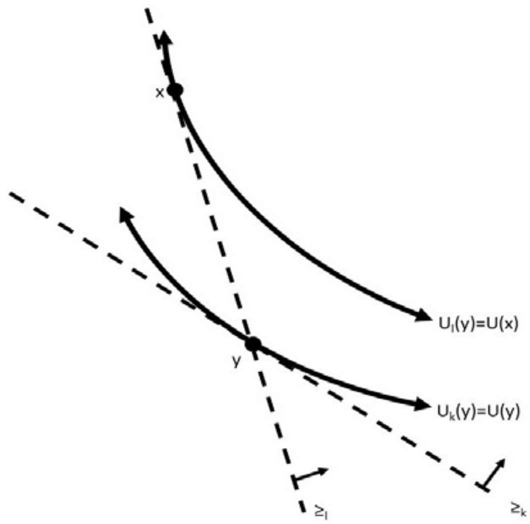

Fig.1: Assignment of $U_k(y)$ and $U_l(y)$

Note that by the convexity of $\succsim$, if $y_1 >_k x >_k y_2$ and $y_1, y_2 \in C_k$ then $x \in \overline{C}_k$. Furthermore, since a_k is $>_k$ -minimal and $a_k \in C_k$, the only x for which U_k is undefined are all strictly superior by $>_k$ to all members of C_k.

It is straightforward to expand U_k to represent $\geq_k$ by defining U_k to represent $\geq_k$ with $U_k(x) > 1$ for all such x.

It remains to be shown that for every $x \in X$, $\min_k U_k(x) = U(x)$. By the lemma, for each $x \in X$, there is some $>_k$, such that $x \in C_k$, and $U_k(x) = U(x)$. For any k such that $x \in \overline{C_k} \backslash C_k$, $x \sim_k \hat{x}$ for some $\hat{x} \in C_k$. Since $\hat{x} \in C_k$, $x \neq \hat{x}$ and $x \sim_k \hat{x}$, it must be that $U(\hat{x}) > U(x)$, and therefore $U_k(x) = U(\hat{x}) > U(x)$. Finally, for any k such that $x \notin \overline{C_k}$, $U_k(x) > 1 \geq U(x)$. ■

As mentioned above, when Λ is the set of all algebraic linear orderings, a continuous preference relation is Λ-strict convex if and only if it is strictly convex in the standard sense. Therefore, Proposition 4 demonstrates that any continuous strictly convex preference relation on a compact convex subset of $\Re^2$ has a Λ-maxmin representation of the form $\min_k F_k(x \cdot e_k)$ where e_k is a vector in $\Re^2$ that points in the direction of $\geq_k$ and F_k is a strictly increasing function. For Euclidean settings with the standard convexity, Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) establish that any continuous convex preference relation (not necessarily strict) has a representation using weakly increasing F_k. That is, they represent a larger class of preferences for R^N, but they are not Λ-maxmin representations because the F_k are $\geq_k$ -increasing but do not represent $\geq_k$. In the same setting, Connell and Rasmussen (2017) provide restrictions on the underlying preferences that guarantee the existence of a concave utility representation. Here we find that a natural restriction, namely Λ-strict convexity, delivers a Λ-maxmin utility representation in a much more abstract setting.

We now prove the existence of a Λ-maxmin representation for pairs X and Λ that satisfy the following *Betweenness* condition:

for every $x, y \in X$ and ordering $\geq_l \in \Lambda$:

If $y >_l x$, then there exists $z \in X$ such that

(i) $y >_l z >_l x$ and (ii) $z \geq_k x$ or $z \geq_k y$ for all other $\geq_k$.

This condition is inspired by the Euclidean setting. In any convex subset of Euclidean space with Λ equal to any collection of linear orderings, an even stronger property holds: for any x and y and any point z on the line segment between them, z is sandwiched between x and y according to every algebraic linear ordering. An example of a non-convex set that satisfies the Betweenness condition with $\Lambda = \Psi_+$ is the circle. The only closed sets in R^2 that satisfy Betweenness with the standard convexity are the standard convex sets.

Proposition 5: Let X be a compact metric space and Λ be a set of continuous primitive orderings satisfying Betweenness. Then, any continuous Λ-strict convex preference relation $\succsim$ has a Λ-maxmin representation.

Proof: Let U and V_k be continuous functions representing $\succsim$ and $\geq_k$, respectively, each with a range of $[0,1]$. Recall that for every $\geq_k \in \Lambda$, the set C_k is defined as $C_k = \{x \mid \geq_k \in Critical(x)\}$. Notice that there cannot be $x \sim_k y$ such that $x, y \in C_k$. By $y \in C_k$, if $x \succsim y$, then $x >_k y$. Let $cl(C_k)$ denote the topological closure of C_k and define $\overline{cl(C_k)} = \{y : y \sim_k x$ for some $x \in cl(C_k)\}$.

We now define for each k a function U_k that represents $\geq_k$, such that $\min_k U_k(x)$ represents $\succsim$.

Step 1: Defining U_k on $\overline{cl(C_k)}$.

For each $x \in \overline{cl(C_k)}$, define $U_k(x) = \max\{U(y) : y \sim_k x\}$. Notice that this definition implies that for all $x \in C_k$, $U_k(x) = U(x)$. To see this, note that since $x \sim_k x$ we have $U_k(x) \geq U(x)$ and by the definition of C_k there is no y such that $U(y) > U(x)$ and $y \sim_k x$.

Step 2: If $x >_k y$ where $x \in cl(C_k)$ and y is arbitrary, then $x \succ y$.

Suppose by contradiction that $y \succsim x$. By Betweenness, there exists w such that $x >_k w >_k y$ and for all other l, $w \geq_l x$ or $w \geq_l y$.

Case (i): $x \succsim w$. Then $y \succsim x \succsim w$ and for every l, either $w \geq_l x$ or $w \geq_l y$ which implies by strict convexity that $w \succ w$, a contradiction.

Case (ii): $w \succ x$. Since $x \in cl(C_k)$, by the continuity of $\succsim$ and $\geq_k$, we have $w \succ \hat{x}$ and $\hat{x} >_k w$ for some $\hat{x} \in C_k$ sufficiently near x, thus violating $\hat{x} \in C_k$.

Step 3: U_k represents $\geq_k$ on $\overline{cl(C_k)}$.

Let $x,y \in \overline{cl(C_k)}$. By definition, if $x \sim_k y$, then $U_k(x) = U_k(y)$. Now, suppose that $x >_k y$, and consider $w \in cl(C_k)$, such that $w \sim_k x$, and some $z \in X$, such that $z \sim_k y$ and $U_k(y) = U(z)$. Then, $w >_k z$. Since $w \in cl(C_k)$, then by step 2, $w \succ z$. Thus, $U_k(x) = U_k(w) \geq U(w) > U(z) = U_k(y)$.

Step 4: Extension of U_k for $x \notin \overline{cl(C_k)}$.

Since $cl(C_k)$ is a closed subset of a compact set and V_k is continuous, the set of numbers $V_k(cl(C_k))$ is also closed and $V_k\left(\overline{cl(C_k)}\right) = V_k(cl(C_k))$ is therefore closed as well. Thus, the set $[0,1] \backslash V_k(cl(C_k))$ is a collection I_k of disjoint open intervals of the form (a,b), $[0,b)$ or $(a,1]$.

Case (i) Take $x \notin \overline{cl(C_k)}$ which according to $\geq_k$ is neither strictly above nor strictly below all members of $\overline{cl(C_k)}$. Then, $V_k(x)$ lies on a member of I_k, call it $(a,b) = (V_k(\alpha), V_k(\beta))$ where $\alpha, \beta \in cl(C_k)$. Define $W(V_k(x)) = \max\{U(y) : x \sim_k y\}$. Let $\overline{W} : (a,b] \to (U_k(\alpha), U_k(\beta)]$ be the upper convex envelope of W on $[a,b]$. To see that $\overline{W}$ is strictly increasing, it must be that if $\beta >_k x >_k \alpha$, then $U_k(\beta) > W(V_k(x))$. To see this, notice that since $\beta \in cl(C_k)$ and $\beta >_k x \sim y$, then by step 2, $U(\beta) > U(y)$. Therefore, $U_k(\beta) \geq U(\beta) > U(y)$ and $U(\beta) > \max\{U(y) : x \sim_k y\} = W(V_k(x))$. Define

$U_k(x) = \overline{W}(V_k(x))$. The function U_k represents $\geq_k$ for any x,y, such that $b \geq V_k(x), V_k(y) \geq a$ since U_k is a strict monotonic transformation of V_k. Furthermore, $U_k(x) = \overline{W}(V_k(x)) \geq W(V_k(x)) \geq U(x)$.

Case (ii) There is no $x \notin \overline{cl(C_k)}$ which according to $\geq_k$ is strictly below all members of $\overline{cl(C_k)}$. This is because a $\succsim$-maximal element of $V_k^{-1}(0)$ is necessarily in C_k.

Case (iii) Take $x \notin \overline{cl(C_k)}$ which according to $\geq_k$ is strictly above all members of $\overline{cl(C_k)}$. For interval $(a,1] \in I_k$, we can simply define $W(1) = 2$ and then allow $\overline{W}$ to be the upper convex envelope of W on $[a,1]$. Define $U_k(x) = \overline{W}(V_k(x))$. Since $1 \geq W$, the function $\overline{W}$ is strictly increasing and therefore U_k represents $\geq_k$ for x,y, such that $1 \geq V_k(x), V_k(y) \geq a$ and $U_k(x) = \overline{W}(V_k(x)) \geq W(V_k(x)) \geq U(x)$.

Step 5: $U(x) = \min_k U_k(x)$.

By construction, for all k, $U_k(x) \geq U(x)$. By the Lemma, for every x there is an ordering $\geq_k$ such that $x \in C_k$ and for this ordering $U_k(x) = U(x)$. ■

We end the section with two examples that demonstrate the notion of Λ-convex preferences, its maxmin representation in a continuous setting and the proofs of propositions 4 and 5:

Example 3: Let $X = \mathbb{R}^2$ and let Λ consist of the two primitive orderings: "right" and "up". The continuous and monotonic Λ-convex preferences are all preferences with right-angled indifference curves.

Any such preference relation has a Λ -maxmin representation of the form $U(x,y) = \min(f(x), g(y))$ where f and g are strictly increasing functions. At the corners of the indifference curves, $f(x) = g(y)$ and the map $f \times g : (x,y) \to (f(x), g(y))$ rescales $\mathbb{R}^2$ so that all of the corners lie on the main diagonal line.

Example 4: Let X be the real line and Λ consist of two single-peaked orderings expressed by $D_0 = -|x|$ and $D_3 = -|x-3|$. The Λ-strictly convex preference relations are all single-peaked preferences with a peak between 0 and 3. The peak cannot be located at any $p > 3$, since $D_0(3) > D_0(p)$ and $D_3(3) > D_3(p)$, and Λ-strict convexity implies that $3 \succ p$. Similarly, the preference cannot have a peak left of 0.

To understand the Λ-maxmin representation, consider the Λ-strictly convex preference relation represented by the utility function $D_2(x) = -|x-2|$. The utility functions $U_0(x) = -|x|+2$ and $U_3(x) = -|x-3|+1$ represent the two primitive orderings and $D_2(x) = \min\{U_0(x), U_3(x)\}$ (see Figure 2).

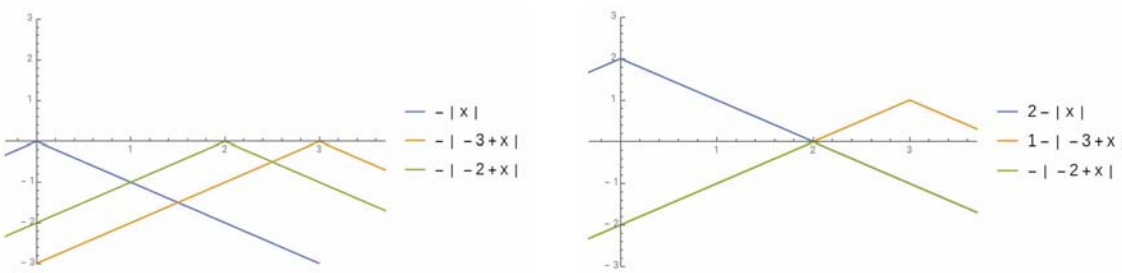

Fig. 2: Left: D_0, D_2 and D_3. Right: U_0, D_2 and U_3.

4. Comments:

(i) **Maxmin models**: Maxmin functions have a long history, originating in Wald (1950). It is interesting to compare our maxmin representation with the familiar but different maxmin representation of Gilboa and Schmeidler (1989). Let S be a finite set of states and Z a set of outcomes. An act is a function from S into Z. Gilboa and Schmeidler (1989) prove that if a preference relation over the set of acts satisfies certain axioms, then there is a function $u : Z \to \Re$ and a set C of probability measures (priors) over S such that the preference relation is represented by $U(f) = \min_{p \in C}\{p \times [U \circ f]\}$. Thus, an act is transformed subjectively into a point $U \circ f \in \mathbb{R}^S$. A probability $p^k \in C$ can be thought of as an algebraic linear function $p^k \times x$ over $\mathbb{R}^S$, and the utility of an act is the minimal value it receives according to the functions attached to the priors in C.

We study a general notion of convex preferences where the primitive orderings are not necessarily algebraic linear functions. The most similar case in our framework is when alternatives are objective vectors of $\mathbb{R}^S$ and the set of primitive orderings C is a set of functions of the type $p^k \times x$. In our representation, the decision maker has in mind an increasing function U^k which he applies to the value $p^k \times f$ and then takes $\min_k\{U^1(p^1 \times f), .., U^K(p^K \times f)\}$. Thus, when applying our approach in this setting, the set of probability measures C is taken as given, in contrast to Gilboa and Schmeidler

(1989). Furthermore, the subjective component in our approach is the transformation of the expected value of each $p^k \in C$, whereas in Gilboa and Schmeidler (1989) the subjectivity is in the transformation of the prizes into a utility space.

(ii) **Social Choice**: As mentioned in the Introduction, another setting in which there is a natural set of primitive orderings is of social choice. A social welfare function attaches a social preference relation to each profile of preferences. The notion of Λ-convexity then can be thought of as the following requirement on the social welfare function F: for every profile P the social preference $F(P)$ has to be Λ-convex where Λ consists of the preferences which appear in the profile P. Note that this property is much stronger than the Pareto condition.

It is easy to verify that the following is an example of a social welfare function that satisfies the above condition as well as a monotonicity condition (if an individual upgrades an alternative, then it cannot be socially downgraded):

Let $X_1 = X$ and define inductively $M_j = \{x \in X$ | there is $\geq_k \in \Lambda$ such that x is minimal in $X_j\}$ and let $X_{j+1} = X_j - M_j$. Define $class(x) = l$ if $x \in M_l$ and let $Borda(x)$ be the Borda score of x applied to the set M_l only (i.e., $Borda(x) = \sum_k |\{y \in M_l : x >_k y|$). Now rank x socially at least as high as y if $class(x) > class(y)$, or, $class(x) = class(y)$ and $Borda(x) \geq Borda(y)$.

(iii) **Other definitions** Our definition of Λ-convex preferences can lead to other intuitive consistency requirements based on a set of primitive orderings. For example: Let Λ be a set of N orderings. Define a preference relation $\succsim$ to be Λ-*almost convex*, if for all $a, b \in X$ satisfying that for at least $N-1$ primitive orderings $\exists y_k$ such that $b \geq_k y_k$ and $y_k \succsim a$, then $b \succsim a$. While this definition is "close" to our definition of Λ-convex preferences, its implications can be quite different. For example, it may be that the only Λ-almost convex preferences are total indifference. For example, let Λ be the set of N orderings over $X = \{a_1, \ldots, a_N\}$ where ordering $\geq_k$ ranks $a_k >_k a_{k+1} >_k \ldots >_k a_{k-1}$. For each i, there are $N-1$ orderings for which $a_i \geq_k a_{i+1}$ (where $a_{N+1} = a_1$). Therefore, the Λ-almost convexity of $\succsim$ implies the cycle $a_1 \succsim a_2 \succsim \ldots \succsim a_N \succsim a_1$ and thus $\succsim$ must be total indifference. In this example, every preference relation is Λ-convex by our main definition, whereas even the primitive orderings themselves are not Λ-almost convex.

(iv) **Connection to abstract convexity** The standard definition of convex preferences in Euclidean spaces is equivalent to the definition that for any element $a \in X$, its upper contour set $U_{\succsim}(a) = \{x : x \succsim a\}$ is a convex set. This definition requires the concept of a "convex set". In an abstract setting, where the set X is arbitrary and lacks an algebraic structure, an attractive definition of convexity is given by Edelman and Jamison (1985). It is based on a functional K on subsets of X where $K(A)$ is interpreted as the set of all elements that are "between elements in A" and is analogous to the notion of "the convex closure of A". A set A is *convex* if $K(A) = A$. For an operator to qualify under this notion of convexity, it has to satisfy a number of axioms, all of which are satisfied by the standard convexity notion:

Extensitivity: $A \subseteq K(A)$ and $K(\emptyset) = \emptyset$.

Monotonicity: $A \subseteq B$ implies $K(A) \subseteq K(B)$.

Idempotence: $K(K(A)) = K(A)$.

Anti-exchange: If A is convex, $a,b \notin A$, $a \neq b$ and $a \in K(A \cup b)$, then $b \notin K(A \cup a)$.

Richter and Rubinstein (2015) studied this notion of convexity in economic settings. In particular, we used the fact (see Edelman and Jamison (1985)) that a set of strict orderings $\Lambda = \{\geq_k\}$ *generates* an operator $K_{\Lambda}(A) = \{x \mid \forall k, \exists a_k \in A \ s.t.\ x \geq_k a_k\}$ which satisfies the above properties.

The primitive orderings can be viewed as a set of criteria that individuals who are using a concept of convexity have in mind. A set is convex if for any element outside the set, one of the criteria ranks it as "inferior" to all elements in the set. Thus, the separating hyperplane theorem holds in this setting.

We now verify that for any set of orderings Λ and any preference relation $\succsim$, the relation $\succsim$ is Λ-convex if and only if every upper contour of $\succsim$ is a convex set in K_{Λ}.

Proposition 6. Let X be a set, $\Lambda = \{\geq_k\}$ be a set of orderings on X and $\succsim$ be a preference relation on X.

The following two statements are equivalent:

(a) The preference relation $\succsim$ is Λ-convex.

(b) For every a, the set $\{x \in X \mid x \succsim a\}$ is convex in K_{Λ}.

Proof: Assume (a). If there exists $b \in K(\{x \in X \mid x \succsim a\}) - \{x \in X \mid x \succsim a\}$, then $a \succ b$ and for each $\geq_k$ there is a $y_k \in \{x \in X \mid x \succsim a\}$ such that $b \geq_k y_k$. This implies by the Λ-convexity of $\succsim$ that $b \succsim a$, a contradiction.

Assume (b). Also assume that there is a pair $a, b \in X$ with the property that for every ordering $\geq_k$ there is a $y_k \in X$ such that $y_k \succsim a$ and $b \geq_k y_k$ but $b \not\succsim a$. Then, $b \notin \{x \in X \mid x \succsim a\} = K(\{x \in X \mid x \succsim a\})$ and therefore there is a primitive ordering $\geq_k$ such that $y >_k b$ for all $y \in \{x \in X \mid x \succsim a\}$, which contradicts the existence of y_k. ■

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