

## "Convex Preferences": A New Definition \*

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### **Abstract**

We suggest a concept of convexity of preferences that does not rely on any algebraic structure. A decision maker has in mind a set of orderings interpreted as evaluation criteria. A preference relation is defined to be convex when it satisfies the following: if for each criterion there is an element that is both inferior to  $b$  by the criterion and superior to  $a$  by the preference relation, then  $b$  is preferred to  $a$ . This definition generalizes the standard Euclidean definition of convex preferences. It is shown that under general conditions, any strict convex preference relation is represented by a maxmin of utility representations of the criteria. Some economic examples are provided.

Key words: Convex preferences, Abstract convexity, Maxmin utility.

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## 1. Introduction

The canonical definition of convex preferences requires that if  $a$  is preferred to  $b$ , then any convex combination of  $a$  and  $b$  is also preferred to  $b$ . This definition relies on the existence of an algebraic structure attached to the space of alternatives. In this paper, we present a new definition of convex preferences. It has an attractive verbal and intuitive meaning, it generalizes the standard Euclidean notion of convex preferences and it can be applied also to spaces without algebraic structure.

In our approach, the agent has in mind a set of *primitive orderings*  $\Lambda = \{\geq_k\}$  where each  $\geq_k$  is a complete and transitive binary relation (which may have indifferences) over a set of alternatives  $X$ . Each ordering represents a criterion for evaluating the alternatives. The agent employs these criteria when forming his preferences. To be  $\Lambda$ -convex, a preference is required to satisfy the following consistency requirement: Given any two alternatives  $a$  and  $b$ , if for each criterion there is an element which is (i) inferior by that criterion to  $b$  and (ii) preferred to  $a$ , then  $b$  must be preferred to  $a$ .

According to this definition, convexity can be perceived as a scheme of argumentation used by either the agent himself, or someone trying to persuade him. The argument goes as follows: You should prefer  $b$  to  $a$ , since for each of your relevant evaluation criteria there is an alternative inferior to  $b$  by that criterion which you prefer to  $a$ . To illustrate, assume that job candidates are evaluated according to research, teaching and charm. To persuade a colleague that  $b$  should be hired rather than  $a$ , one needs to demonstrate that: there is a candidate  $c$ , who is a worse researcher than  $b$  and preferred by the colleague to  $a$ ; there is a candidate (who may or may not be  $c$ ) who is a worse teacher than  $b$  and is preferred by the colleague to  $a$ ; and there is a less charming candidate than  $b$  whom the colleague ranks above  $a$ .

The concept we introduce depends crucially on the set  $\Lambda$ . Obviously, the same set  $X$  endowed with different sets of primitive orderings may have different sets of convex preferences. We think about the orderings in  $\Lambda$  as the building blocks in the agent's formation of preferences (for the related concept of "definable preferences", see Rubinstein (1978, 1998)). The orderings in  $\Lambda$  may describe objective features of the alternatives (such as height, weight, or geographical position), but they may also reflect subjective criteria that the agent employs when ranking the alternatives

(such as attractiveness and charisma). In the analysis, we take these orderings to be primitives and explore the preferences which are convex with respect to them.

There are several reasons why  $\Lambda$ -convexity is an attractive concept:

(a) We find the consistency requirement compelling for its own sake. The analysis will clarify its role in shaping preference relations.

(b) For Euclidean spaces, choosing the set  $\Lambda$  to contain all algebraic linear orderings induces the standard convex preferences notion. The new definition allows an extension of the standard definition in two directions: First, it applies to setups where no algebraic structure is specified. Second, for spaces *with* algebraic structure, alternative specifications of  $\Lambda$  induce alternative convexity-like requirements.

(c) Our main analytical result states that any  $\Lambda$ -convex strict preference relation can be represented by a utility function of the form  $\min_k(u_k(x))$  where each  $u_k$  is a utility representation of some ordering in  $\Lambda$ . This is a meta-representation theorem. For any set of primitive orderings  $\Lambda$ , it delivers a representation result for the  $\Lambda$ -convex preferences, making it possible to derive both new and known maxmin-like representation results.

## 2. A new definition of convex preferences

**Definition:** Let  $X$  be a set and  $\Lambda$  be a set of primitive orderings on  $X$ .

We say that a preference relation  $\succsim$  (complete and transitive) on  $X$  is  $\Lambda$ -convex if for every  $a, b \in X$  the following condition holds:

If for every  $\succeq_k \in \Lambda$ , there is a  $y_k$ , such that  $b \succeq_k y_k$  and  $y_k \succsim a$ , then  $b \succsim a$ .

We say that a preference relation  $\succsim$  on  $X$  is  $\Lambda$ -strictly convex if for every  $a, b \in X$  the following stronger condition holds:

If for every  $\succeq_k \in \Lambda$ , there is a  $y_k \neq b$  such that  $b \succeq_k y_k$  and  $y_k \succsim a$ , then  $b \succ a$ .

We now present some **traits** of  $\Lambda$ -convex preferences:

(I) Every primitive ordering  $\succeq_l \in \Lambda$  is  $\Lambda$ -convex: if for every  $\succeq_k$  there is a  $y_k$ , such that  $b \succeq_k y_k$  and  $y_k \succeq_l a$ , then, in particular, for  $l$ , there is a  $y_l$  such that  $b \succeq_l y_l$  and  $y_l \succeq_l a$ , and thus  $b \succeq_l a$ .

(II) A  $\Lambda$ -convex preference relation must satisfy the weak "Pareto" property: if  $b \succeq_k a$  for every  $\succeq_k \in \Lambda$  then  $b \succsim a$  (apply the definition with  $y_k = a$  for every primitive ordering in  $\Lambda$ ). If  $\succsim$  on  $X$  is  $\Lambda$ -strictly convex then it satisfies a stronger version of Pareto: for any two distinct elements  $a$  and  $b$ , if  $b \succeq_k a$  for every  $\succeq_k \in \Lambda$

then  $b \succ a$ .

(III) If  $\Lambda$  is finite and  $\succeq$  is  $\Lambda$ -convex, then for each alternative  $a$ , there is a direction  $\succeq_l$  for which a weak decline cannot be strictly improving (for all  $y \neq a$ ,  $a \succeq_l y \Rightarrow a \succeq y$ ): it is impossible that for all  $\succeq_k$  there is  $y_k \neq a$  such that  $a \succeq_k y_k$  and  $y_k \succ a$  since let  $y_l$  be  $\succeq$ -minimal from among those  $\{y_k\}$ , then by  $\Lambda$ -convexity  $a \succeq y_l \succ a$ .

Furthermore, for any  $\Lambda$  (even infinite), if  $\succeq$  is  $\Lambda$ -strictly convex, then for each alternative  $a$ , there is a direction  $\succeq_l$  for which a weak decline is strictly disimproving (for all  $y \neq a$ ,  $a \succeq_l y \Rightarrow a \succ y$ ): it is impossible that for all  $\succeq_k$  there is  $y_k \neq a$  such that  $a \succeq_k y_k$  and  $y_k \succeq a$  since then it would follow that  $a \succ a$ .

(IV) For a finite set  $X$ , when the primitive orderings are strict, a  $\Lambda$ -convex preference relation can be built inductively as follows: Take an alternative  $x_1$  which is at the bottom of one of the primitive relations and place it at the bottom of  $\succeq$ . Then, take an alternative  $x_2$  which is at the bottom of  $X - \{x_1\}$  with respect to one of the primitive orderings and place it strongly or weakly above  $x_1$ . Continue this procedure until you exhaust all alternatives. The constructed preference is  $\Lambda$ -convex since if for every  $\succeq_k \in \Lambda$ , there is a  $y_k \neq b$ , such that  $b \succeq_k y_k$  and  $y_k \succeq a$ , then at least one such  $y_k$  must be picked by the procedure before  $b$  and thus  $b \succeq y_k$  and  $y_k \succeq a$  which implies  $b \succeq a$ .

(V) For any finite set  $X$ , where the primitive orderings are strict, every  $\Lambda$ -convex preference relation  $\succeq$  must be consistent with the procedure described in (IV). For each  $\succeq_k$ , let  $b_k$  be a  $\succeq_k$ -minimal alternative and let  $b$  be  $\succeq$ -minimal from among  $\{b_k\}$ . For every other alternative,  $a \in X \setminus \{b\}$ , for every  $\succeq_k$ , it is the case that  $a \succeq_k b_k$  and  $b_k \succeq b$ . Thus,  $\Lambda$ -convexity implies that  $a \succeq b$ . Therefore,  $b$  is  $\succeq$ -minimal. Let  $x_1 = b$  and continue with the set  $X - \{x_1\}$  to identify the sequence  $x_2, \dots, x_{|X|}$  such that  $x_i$  is  $\succeq_k$ -minimal for some  $\succeq_k$  and  $\succeq$ -minimal from among  $X - \{x_1, \dots, x_{i-1}\}$ .

(VI) If  $\Lambda \supset \Gamma$  and  $\succeq$  is  $\Gamma$ -convex, then  $\succeq$  is  $\Lambda$ -convex. To see it, suppose that  $\succeq$  is  $\Gamma$ -convex. If for every  $\succeq_k \in \Lambda$ , there is a  $y_k$ , such that  $b \succeq_k y_k$  and  $y_k \succeq a$ , then for every  $\succeq_k \in \Gamma$ , there is a  $y_k$ , such that  $b \succeq_k y_k$  and  $y_k \succeq a$ , and by  $\Gamma$ -convexity,  $b \succeq a$ . Hence  $\succeq$  is  $\Lambda$ -convex.

(VII). If  $\succeq$  is  $\Lambda$ -convex, then the set of  $\Lambda \cup \{\succeq\}$ -convex and  $\Lambda$ -convex preferences are identical. One side of the statement follows from (VI). To see the converse, suppose that  $\succeq'$  is  $\Lambda \cup \{\succeq\}$ -convex. Assume that for every  $\succeq_k \in \Lambda$ , there is a  $y_k \neq b$ , such that  $b \succeq_k y_k$  and  $y_k \succeq' a$ . Let  $y = \min(y_k, \succeq)$ . Therefore,  $y \succeq' a$  and by the  $\Lambda$ -convexity of  $\succeq$ ,  $b \succeq y$ . Then,  $\Lambda \cup \{\succeq\}$ -convexity of  $\succeq'$  implies that  $b \succeq' a$ . Thus,

$\succsim'$  is  $\Lambda$ -convex.

(VIII) For strict preferences, there is no difference between  $\Lambda$ -convexity and  $\Lambda$ -strict convexity.

The following are three examples of  $\Lambda$ -convex orderings:

**Example 1:** Let  $X$  be a (finite or not) subset of  $\mathbb{R}$  and  $\Lambda$  contains exactly two orderings: the increasing ordering  $\geq_I$  and the decreasing ordering  $\geq_D$ .

**Observation:** A preference is  $\Lambda$ -strictly convex if and only if it is singled-peaked on  $X$ , (that is, there are no three alternatives  $x < y < z$  such that  $x, z \succsim y$ ).

**Proof:** Suppose that  $\succsim$  is singled-peaked. Assume that there are  $y_I \neq b$  and  $y_D \neq b$  such that  $b \geq_I y_I$  and  $b \geq_D y_D$  and  $y_I, y_D \succsim a$ . Then, by single-peakness of  $\succsim$  we must have  $b \succ y_I$  or  $b \succ y_D$  and thus  $b \succ a$ . Thus,  $\succsim$  is  $\Lambda$ -strictly convex.

On the other hand, if  $\succsim$  is  $\Lambda$ -strictly convex, then by trait (III), there are no three alternatives  $x < y < z$  such that  $x, z \succsim y$ . ■

**Example 2:** Let  $X$  be a convex and closed subset of  $\mathbb{R}^N$ . Each non-zero vector  $v$  defines an *algebraic linear ordering* by  $x \geq_v y$  if  $v \cdot x \geq v \cdot y$ . Denote the set of all algebraic linear orderings by  $\Psi$ . We will show that for continuous preference relations on  $X$ , the standard notion of convexity is equivalent to  $\Psi$ -convexity.

**Observation:** The following two statements about a continuous preference relation  $\succsim$  are equivalent:

- (i)  $\succsim$  is convex by the standard definition; and
- (ii)  $\succsim$  is  $\Psi$ -convex.

**Proof:** Assume (i). Take two different points  $a, b \in X$  such that for every  $\geq_k \in \Psi$  there is a  $y_k \neq b$  such that  $b \geq_k y_k$  and  $y_k \succsim a$ . We show  $b \succ a$  by contradiction. Suppose  $a \succ b$ . Since  $\succsim$  is continuous and convex, the set  $U_{\succsim}(a) = \{z : z \succsim a\}$  is closed and convex. Thus, by the separating hyperplane theorem, there is some algebraic ordering  $\geq_l$  such that  $b$  lies strictly below  $U_{\succsim}(a)$ . Since  $b \geq_l y_l$ , it follows that  $y_l \notin U_{\succsim}(a)$ , and therefore  $a \succ y_l$ , contradiction.

Assume (ii). Take two points  $a$  and  $b$  such that  $b \succ a$ . Then, for any point  $c$  between  $a$  and  $b$  and any algebraic linear ordering  $\geq_k$ , it is the case that  $c \geq_k a$  or  $c \geq_k b$  and both  $a$  and  $b$  are preferred to  $a$ . Thus, by the definition of  $\Psi$ -convexity,  $c \succ a$ . ■

The observation demonstrates that the notion of  $\Lambda$ -convexity generalizes the standard convexity notion for continuous preferences. Notice however that other familiar properties of preference relations can also be expressed in the language of  $\Lambda$ -convexity. For example, for the case that  $X$  is a convex closed subset of  $\mathbb{R}^N$ , let  $\Psi_+$  be the set of algebraic linear orderings with non-negative coefficients. Then, one can show that a continuous preference relation  $\succsim$  is  $\Psi_+$ -convex if and only if  $\succsim$  is weakly increasing and convex in the standard sense. Thus, by properly choosing the set of primitive orderings,  $\Lambda$ -convexity can express both convexity and monotonicity.

**Example 3:** Let  $X = \mathbb{R}_+^2$  (or  $X = \mathbb{R}^2$ ) and let  $\Lambda$  consist of the two primitive orderings:  $\geq_R$  ("right") and  $\geq_U$  ("up"). The following observation implies that a preference relation which is continuous,  $\Lambda$ -convex and monotonic (if  $y_1 > x_1$  and  $y_2 > x_2$ , then  $(y_1, y_2) \succ (x_1, x_2)$ ) must have indifference curves that are vertical, horizontal or right-angled only.

**Observation:** Any continuous  $\Lambda$ -convex and monotonic preference relation has a utility representation of the form  $U(x_1, x_2) = \min(f(x_1), g(x_2))$  where  $f$  and  $g$  are strictly increasing functions.

**Proof:** By monotonicity, the function  $U(x, x) = \frac{e^x}{1+e^x}$  represents  $\succsim$  along the main diagonal onto  $(1/2, 1)$ . This representation can be extended by attaching to each alternative the unique alternative on the main diagonal to which it is indifferent (its existence is guaranteed by monotonicity and continuity).

Define  $f(z_1) = \sup\{U(z_1, z_2) : z_2 \in \mathbb{R}\}$  and  $g(z_2) = \sup\{U(z_1, z_2) : z_1 \in \mathbb{R}\}$ . We first show that  $U(y_1, y_2) = \min(f(y_1), g(y_2))$ . By definition,  $U(y_1, y_2) \leq \min(f(y_1), g(y_2))$ . Suppose the inequality were strict for some  $(y_1, y_2)$ . Then, there exists  $z_2 > y_2$ ,  $z_1 > y_1$  such that  $(y_1, z_2) \succ (y_1, y_2)$  and  $(z_1, y_2) \succ (y_1, y_2)$ , violating trait (III).

The functions  $f$  and  $g$  are weakly increasing (because  $\succsim$  is monotonic). If  $f$  and  $g$  are strictly increasing everywhere, then we are done.

If not, WLOG, suppose that  $f(y_1) = f(x_1)$  where  $y_1 > x_1$ . Then, for every  $z_2$ ,  $U(y_1, z_2) = U(x_1, z_2)$  and thus  $(y_1, z_2) \sim (x_1, z_2)$ . Consequently, by monotonicity for any  $y_2 > x_2$ ,  $(y_1, y_2) \succ (x_1, x_2) \sim (y_1, x_2)$ . Therefore,  $g$  is strictly increasing everywhere and  $f(y_1) = f(x_1) \geq \sup\{g(z_2) : z_2 \in \mathbb{R}\} = 1$ . Thus,  $f(y_1) = f(x_1) = 1$ . Since  $\succsim$  is continuous and monotonic,  $\{z_1 : f(z_1) = 1\} = [m, \infty)$  for some  $m$ .

If  $m = -\infty$ , then  $f \equiv 1$  and for any  $x_1, x_2$ ,  $U(x_1, x_2) = g(x_2)$ . Therefore, all indifference curves are horizontal. Let  $h$  be a strictly increasing function such that

$h(z_1) > 1$  everywhere. Then,  $\min(h(x_1), g(x_2)) = g(x_2) = U(x_1, x_2)$  is the required representation.

If  $m > -\infty$ , then define  $h(z_1) = f(z_1) + (z_1 - m)^+$ . This function is strictly increasing since, for  $z_1 \geq m$ , the function  $(z_1 - m)^+$  is strictly increasing and  $f(z_1)$  is weakly increasing, and for  $z_1 < m$ , we have that  $(z_1 - m)^+ = 0$  and  $h(z) = f(z)$  is strictly increasing. Thus,  $h$  and  $g$  form the required representation of  $\succsim$ . ■

### 3. A Maxmin Representation of Convex Preferences

We now turn to the main analytical result, a  $\Lambda$ -maxmin utility representation of  $\Lambda$ -convex preferences. Our  $\Lambda$ -maxmin utility representation is of the form:

$$U(x) = \min\{U_k(x) \mid U_k \text{ is a utility representation of } \succeq_k \in \Lambda\}$$

The importance of this presentation is two-fold: First, it is an attractive procedure for comparing two alternatives. In the hiring example, each candidate receives a score on research, teaching, and charm, and a candidate is evaluated by their worst score. Second, it relates to the previous literature which explores other maxmin representations (see Section 5 for a detailed comparison).

**Definition:** A preference relation  $\succsim$  over  $X$  has a  $\Lambda$ -maxmin representation if: for each  $\succeq_k$  in  $\Lambda$  there is a utility representation  $U_k$  such that  $\min_k U_k(x)$  represents  $\succsim$ .

Example 3 provided such a representation for a particular context. The existence of such a representation means that we can identify each element in the set  $X$  with a vector of numbers in  $\mathbb{R}^\Lambda$  such that:

- (i) for each primitive ordering, the values that are attached to the elements in  $X$  at the corresponding coordinate respect the primitive ordering's ranking; and
- (ii) the minimum value that is attached to an alternative across the different dimensions specifies how the alternatives are ranked.

We first verify that any preference relation which has a  $\Lambda$ -maxmin representation is  $\Lambda$ -convex:

**Proposition 1:** If  $\succsim$  has a  $\Lambda$ -maxmin representation, then  $\succsim$  is  $\Lambda$ -convex.

**Proof:** Suppose that for every primitive ordering  $\succeq_k$ , there is a  $y_k \neq b$  such that  $b \succeq_k y_k$  and  $y_k \succsim a$ . Then, for every  $\succeq_l$ ,  $U_l(b) \geq U_l(y_l) \geq \min_k U_k(y_l) \geq \min_k U_k(a)$ , and therefore  $\min_k U_k(b) \geq \min_k U_k(a)$ , which implies  $b \succsim a$ . ■

The converse requires more than  $\Lambda$ -convexity. For example, the total indifference is always  $\Lambda$ -convex but typically will not have a  $\Lambda$ -maxmin representation.

Proposition 2 will show that  $\Lambda$ -strict convex preferences have  $\Lambda$ -maxmin representations. As preparation, we need one additional concept. Recall the familiar Euclidean property that for any strictly convex preference relation and any point  $x$  there is a tangent hyperplane that touches  $x$ 's indifference curve only at  $x$ . This motivates the following definition: Given a preference relation  $\succsim$ , the set  $Critical(z)$  contains every ordering  $\succeq_k \in \Lambda$  that satisfies "for every  $y \neq z$ , if  $y \succsim z$ , then  $y \succ_k z$ ". Define  $C_k = \{z \mid \succeq_k \in Critical(z)\}$ .

In the Euclidean context, the set  $Critical(z)$  is analogous to the subdifferential of a utility representation of  $\succsim$ . Recall that a standard strictly convex preference relation has a nonempty subdifferential at every point. Analogously, if  $\succsim$  is a  $\Lambda$ -strictly convex preference relation, then  $Critical(z) \neq \emptyset$  for all  $z$ . To see why, if  $Critical(z) = \emptyset$  for some  $z \in X$ , then, for every  $\succeq_k$  there would exist  $y_k \in X - \{z\}$  such that  $z \succeq_k y_k$  and  $y_k \succsim z$ , which violates trait (III).

**Proposition 2:** Let  $X$  be a finite set. Any  $\Lambda$ -strictly convex preference relation  $\succsim$  on  $X$  has a  $\Lambda$ -maxmin representation.

**Proof:** First, notice that the elements of  $C_k$  are strictly ordered identically by both  $\succeq_k$  and  $\succsim$ : given any two distinct elements  $x, y \in C_k$  where  $x \succsim y$ , we have  $x \succ_k y$  since  $y \in C_k$ . Moreover, if  $x \succ_k y$ , then it must be that  $x \succ y$  since  $x \in C_k$ .

Let  $U$  be a utility function representing  $\succsim$ . For every  $\succeq_k$ , define  $U_k(z) = U(z)$  for all  $z \in C_k$ . Since  $\succsim$  and  $\succeq_k$  give exactly the same ranking over  $C_k$ , the function  $U_k$  represents  $\succeq_k$  on  $C_k$ .

In order to expand the definitions of  $U_k$  to the entire set  $X$ , count the elements of  $C_k$  as  $c_1 \succ_k \dots \succ_k c_L$  and consider the following partition of  $X$ :  $D_0 = \{x \mid x \succ_k c_1\}$ ,  $D_i = \{x \mid c_i \succeq_k x \succ_k c_{i+1}\}$  and  $D_L = \{x \mid c_L \succeq_k x\}$ . Notice that for all  $z \in D_i \setminus \{c_i\}$ ,  $c_i \succ z$  since if  $c_i \succeq_k z$ , then  $c_i \succ z$  since  $c_i \in C_k$ . Expand  $U_k$  on  $D_i \setminus \{c_i\}$  to represent  $\succeq_k$  with

values taken from the interval  $(\max\{U(z) : c_i \succ z\}, U(c_i)]$ . Therefore,  $U_k$  represents  $\succeq_k$  for all  $k$ .

For all  $x \in C_k$ ,  $U_k(x) = U(x)$ ; and for all  $x \notin C_k$ ,  $U_k(x) > U(x)$ . Since,  $Critical(x)$  is always non-empty, it follows that  $X = \cup C_k$ , and so  $\min_k U_k(x) = U(x)$  for all  $x$ . Recall that  $U$  represents  $\succeq$ , and thus,  $\succeq$  has a  $\Lambda$ -maxmin representation. ■

Notice that for weak preferences, Propositions 1 and 2 do not form a complete if-and-only-if characterization because Proposition 1 demonstrates  $\Lambda$ -convexity and Proposition 2 requires  $\Lambda$ -strict convexity. However, recall that for strict preferences, the concepts of  $\Lambda$ -convexity and  $\Lambda$ -strict convexity are equivalent (VIII). Thus, for strict preferences, Propositions 1 and 2 together provide an exact equivalence between  $\Lambda$ -convexity and the existence of a  $\Lambda$ -maxmin representation.

**Example 4:** (Monotonic Preferences over Menus) Let  $Z$  be a finite set of alternatives and  $X$  be the set of all non-empty menus of  $Z$ . Given a utility function over alternatives  $u : Z \rightarrow \mathbb{R}$ , the preference relation  $\succeq_u^{\max}$  is defined over  $X$  by  $A \succeq_u^{\max} B$  if  $\max_{z \in A} u(z) \geq \max_{z \in B} u(z)$ . In words, each menu is evaluated by its  $u$ -best alternative. Let  $\Gamma^{\max}$  consist of all such induced orderings over  $X$ .

**Observation:**

(i) A preference on  $X$ ,  $\succeq$ , is  $\Gamma^{\max}$ -strictly convex if and only if it is strictly monotonic in the sense that  $B \supset A$  implies  $B \succ A$ .

(ii) If  $\succeq$  is a strictly monotonic preference over menus, then there exists a set  $U$  of functions from  $Z$  to  $\mathbb{R}$  such that:

$$A \succeq B \text{ if and only if } \min_{u \in U} \max_{z \in A} u(z) \geq \min_{u \in U} \max_{z \in B} u(z).$$

**Proof:** (i) Let  $\succeq$  be a  $\Gamma^{\max}$ -strictly convex preference relation. For any two nested menus  $B \supset A$ , it is the case that  $B \succeq_k A$  for every  $\succeq_k \in \Gamma^{\max}$  and thus  $B \succ A$  (by (II), the strong Pareto property).

For the other direction, let  $\succeq$  be a strictly monotonic preference on  $X$  and  $A, B$  be two menus. Suppose that for every  $\succeq_k \in \Gamma^{\max}$ , there exists  $Y_k \neq B$ , such that  $B \succeq_k Y_k$  and  $Y_k \succeq A$ . Take  $\succeq_u^{\max} \in \Gamma^{\max}$  where  $u(z) = 0$  for all  $z \in B$  and  $u(z) > 0$  for all  $z \notin B$ . This ordering bottom-ranks  $B$  and all of its subsets and ranks all other sets above it. Thus,  $B \succeq_u^{\max} Y_u$  implies that  $B \supset Y_u$  (inclusion is strict because  $Y_u \neq B$ ) and  $B \succ Y_u$  by the strict monotonicity of  $\succeq$ . Therefore,  $B \succ Y_u \succeq A$  which implies that  $B \succ A$ .

(ii) By part (i),  $\succeq$  is  $\Gamma^{\max}$ -strictly convex. Then, for every ordering  $\succeq_k$  in  $\Gamma^{\max}$  pick

one utility function  $u_k$  on  $Z$  which represents it. By Proposition 2 there exist a strictly increasing function  $f_k$  such that  $\min_k(f_k(\max_{z \in A} u_k(z)))$  represents  $\succeq$ . But then  $f_k \circ u_k$  also represents  $\succeq_k$  and  $\min_k \max_{z \in A} f_k \circ u_k(z)$  represents  $\succeq$ . ■

This kind of representation can be thought of as a state-dependent maxmin utility. The agent currently does not know his future preferences over  $Z$ , but will know them when he chooses from the menu. He evaluates each menu by its worst possible state. This conclusion was proved by Gorno and Natenzon (2018) who in fact shows that any weakly monotonic menu preference  $\succeq$  can be represented in this manner. Notice the difference from Kreps (1979) who requires weak monotonicity and an additional submodularity axiom to derive a representation of the form  $\sum_u \pi(u) \max_{z \in A} u(z)$  where  $\pi$  is a distribution over utility functions.

**Example 5:** (Betweenness Preferences over Menus) Let  $Z$  be a finite set of alternatives and  $X$  be the set of non-empty menus of  $Z$ . Given a function  $u : Z \rightarrow \mathbb{R}$  the preference relation  $\succeq_u^{avg}$  over menus is defined by  $A \succeq_u^{avg} B$  if  $avg_{z \in A} u(z) \geq avg_{z \in B} u(z)$ . Let  $\Gamma^{avg}$  consist of all such induced orderings over  $X$ .

A non-empty sequence of proper subsets of  $A$  (the sequence may contain repetitions) is an *equal cover* of  $A$  if there is some positive number  $m$  such that each alternative in  $A$  is contained in exactly  $m$  of the subsets. We say that a preference  $\succeq$  satisfies the *Equal Covering property* if for every equal cover of  $A$ , at least one of the sets in the sequence is strictly inferior to  $A$ . Clearly, the monotonicity property of Example 4 implies the equal covering property.

**Observation:**

For any preference  $\succeq$  over  $X$ :

- (i) The preference  $\succeq$  is  $\Gamma^{avg}$ -strictly convex if and only if it satisfies the Equal Covering property.
- (ii) If  $\succeq$  satisfies the Equal Covering property, there exists a set  $\{U_k\}$  of functions from  $Z$  to  $\mathbb{R}$  and a set of increasing functions  $\{V_k\}$  such that:

$$A \succeq B \text{ if and only if } \min_{u_k \in U} V_k(avg_{z \in A} u_k(z)) \geq \min_{u_k \in U} V_k(avg_{z \in B} u_k(z)).$$

- (iii) If  $\succeq$  satisfies Gul and Pessendorfer (2001)'s set-betweenness axiom [ $\forall A, B \subseteq Z$  such that  $B \succeq A$ , it is the case that  $A \cup B \succeq A$  and  $B \succeq A \cup B$ ], then it is  $\Gamma^{avg}$ -convex.

**Proof:** (i) Assume that  $\succeq$  is  $\Gamma^{avg}$ -strictly convex. To show that it satisfies the Equal Covering property, let  $(A_1, \dots, A_n)$  be an equal cover of a set  $A$  and WLOG assume

that  $A_2, \dots, A_n \succsim A_1$ . Then  $\text{avg } u(A)$  is a convex combination of  $\{\text{avg } u(A_i)\}$ . (Let  $m = |\{i : x \in A_i\}|$ . Then, for any given  $u$ ,  

$$m|A|\text{avg } u(A) = m \sum_{a \in A} u(a) = \sum_{i=1}^n \sum_{a \in A_i} u(a)$$

$$= \sum_{i=1}^n |A_i| \text{avg } u(A_i) \text{ and } \sum_{i=1}^n \frac{|A_i|}{m|A|} = 1.)$$
 Thus,  $\text{avg } u(A) \geq \text{avg } u(A_i)$  for at least one  $A_i$ , so  $A \succeq_u^{\text{avg}} A_i$  and  $A_i \succsim A_1$ . Therefore, by  $\Gamma^{\text{avg}}$ -strict convexity,  $A \succ A_1$ .

For the other direction, let  $\succsim$  be a preference satisfying the Equal Covering property. Suppose that for every  $\succeq_k \in \Gamma^{\text{avg}}$ , there exists  $Y_k \neq B$ , such that  $B \succeq_k Y_k$  and  $Y_k \succsim A$ . Our goal is to now show that  $B \succ A$ . Index all elements in  $Z$  as  $z_1, \dots, z_{|Z|}$  and attach to each set  $A \subseteq X$ , a vector  $v(A)$  where  $v(A)_i = 1/|A|$  if  $z_i \in A$ , and  $v(A)_i = 0$  otherwise. Notice that for any utility function  $u$ ,  $\text{avg}_{z \in A} u(z) = \hat{u} \cdot v(A)$  where  $\hat{u} = (u(z_1), \dots, u(z_{|Z|}))$ . Thus,  $B \succeq_u^{\text{avg}} Y_u$  means that  $\hat{u} \cdot v(B) \geq \hat{u} \cdot v(Y_u)$  and since this holds for every  $\hat{u}$ , it must be that  $v(B)$  is a convex combination of  $\{v(Y_k)\}$ . All of the  $v(Y_k)$  are rational vectors and by a theorem of the alternative (Fishburn (1971), Theorem A),  $B$  can be equally covered by a sequence of the  $Y_k$  (possibly with repetitions). Therefore, by the Equal Covering property for at least one  $Y_k$ ,  $B \succ Y_k$  and thus  $B \succ A$ .

(ii) By part (i),  $\succsim$  is  $\Gamma^{\text{avg}}$ -strictly convex. Then, for every ordering  $\succeq_k$  in  $\Gamma^{\text{avg}}$  pick one utility function  $u_k$  on  $Z$  such that  $\text{avg } u_k$  represents it. By Proposition 2 there exist a strictly increasing function  $V_k$  such that  $\min_k (V_k(\text{avg}_{z \in A} u_k(z)))$  represents  $\succsim$ .

(iii) By induction, the first half of set-betweeness implies the stronger condition: for any sequence of proper subsets of  $A$  which covers  $A$  (not necessarily an equal cover),  $A$  is weakly preferred to at least one of the subsets. ■

To demonstrate the above "min avg" representation, here are two preferences which satisfy the Equal Covering Property and their representations with  $Z = \{a, b\}$ :

Preferences	Underlying Utilities
$\{b\} \succ \{a, b\} \succ \{a\}$	$u(a) = 1, u(b) = 2$
$\{a, b\} \succ \{a\} \sim \{b\}$	$u(a) = 1, u(b) = 2; v(a) = 2, v(b) = 1$

#### 4. A Maxmin Representation Theorem for compact metric spaces

In Proposition 2 we proved that when  $X$  is finite, any  $\Lambda$ -strictly convex preference relation has a  $\Lambda$ -maxmin representation. Proposition 4 below is an analogous result (with additional continuity-type restrictions) for compact metric spaces. That result requires a significant amount of technical machinery and therefore we first present Proposition 3 which illustrates some of the key ideas in a simpler two-dimensional Euclidean setting.

**Proposition 3:** Let  $X$  be a compact convex subset of  $\mathbb{R}^2$  and let  $\Psi$  be the set of all algebraic linear orderings on  $X$ . If  $\succeq$  is a continuous  $\Psi$ -strictly convex preference relation (not necessarily monotonic), then it has a  $\Psi$ -maxmin representation.

**Proof:** We first need to derive some properties of the set  $C_k$  the set of critical points of  $\succeq_k$ .

(i) The set  $C_k$  contains an element  $a_k$ , which is a  $\succeq$ -maximal element in the set of  $\succeq_k$ -minimal elements. To see it, note that the set of  $\succeq_k$ -minimal elements is convex and compact and since  $\succeq$  is strictly convex and continuous, an element  $a_k$  which is a  $\succeq$ -maximal element in the set of  $\succeq_k$ -minimal elements exists and is strictly preferred to all other  $\succeq_k$ -minimal elements. Therefore, for any different  $z \succeq a_k$ , it is the case that  $z >_k a_k$  and thus  $a_k$  belongs to  $C_k$ .

(ii) There are no two distinct  $x, y \in C_k$  such that  $x \sim_k y$  (WLOG  $x \succeq y$  and then  $y \in C_k$  leads to  $x >_k y$ ).

(iii) The set  $C_k$  is connected.

(iv) Define  $\overline{C_k}$  to be the set of all  $x$  such that  $x \sim_k y$  for some  $y \in C_k$ . Any element  $x \notin \overline{C_k}$  satisfies  $x >_k a_k$  by definition of  $a_k$  and satisfies  $x >_k y$  for all  $y \in C_k$  (since  $C_k$  is connected and  $\succeq_k$  is continuous).

We now define for each  $k$ , a function  $U_k$  that represents  $\succeq_k$ . In the construction, we use  $U : X \rightarrow [0, 1]$ , a continuous utility representation of  $\succeq$  (whose existence is guaranteed by the continuity of  $\succeq$ ). For every  $x \in \overline{C_k}$ , define  $U_k(x) = U(y)$  where  $y$  is the unique element in  $C_k$  for which  $x \sim_k y$ . The function  $U_k$  represents  $\succeq_k$  on  $\overline{C_k}$ . (Let  $x, y \in \overline{C_k}$  and let  $\hat{x}, \hat{y} \in C_k$  satisfying  $x \sim_k \hat{x}$ ,  $y \sim_k \hat{y}$ . If  $x >_k y$ , then  $\hat{x} >_k \hat{y}$  and since  $\hat{x} \in C_k$ , it must be that  $\hat{x} \succ \hat{y}$ , and therefore  $U_k(x) = U(\hat{x}) > U(\hat{y}) = U_k(y)$ . If  $x \sim_k y$ , then by (ii),  $\hat{x} = \hat{y}$  and  $U_k(x) = U(\hat{x}) = U_k(y)$ .) For  $X - \overline{C_k}$ , extend  $U_k$  to represent  $\succeq_k$  with values above 1. Figure 1 illustrates the construction.

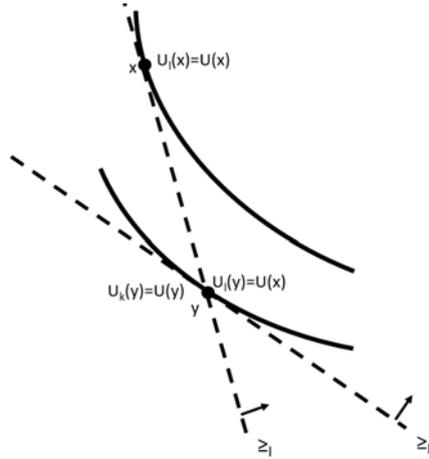


Figure 1: The construction of  $U_k, U_l$

It remains to be shown that for every  $x \in X$ ,  $\min_k U_k(x) = U(x)$ . As mentioned earlier for each  $x \in X$ , there is some  $\geq_k$ , such that  $x \in C_k$ , and for this ordering  $U_k(x) = U(x)$ . For any  $l$  such that  $x \in \overline{C_l} \setminus C_l$ ,  $x \sim_l \hat{x}$  for some  $\hat{x} \in C_l$ . Since  $x \neq \hat{x}$  and  $x \sim_l \hat{x}$ , it must be that  $U(\hat{x}) > U(x)$ , and therefore  $U_l(x) = U(\hat{x}) > U(x)$ . Finally, for any  $l$  such that  $x \notin \overline{C_l}$ ,  $U_l(x) > 1 \geq U(x)$ . Thus,  $\min_k U_k(x) = U(x)$ . ■

As mentioned in Example 2, when  $\Lambda$  is the set of all algebraic linear orderings, a continuous preference relation is  $\Lambda$ -strictly convex if and only if it is strictly convex in the standard sense. Therefore, Proposition 3 demonstrates that any continuous strictly convex preference relation on a compact convex subset of  $\mathbb{R}^2$  has a  $\Lambda$ -maxmin representation of the form  $\min_k F_k(x \cdot e_k)$  where  $e_k$  is a vector in  $\mathbb{R}^2$  that points in the direction of  $\geq_k$  and  $F_k$  is a strictly increasing function. For Euclidean settings with the standard convexity, Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2011) establish a similar result that any continuous convex preference relation (not necessarily strict) has a representation using weakly increasing  $F_k$ . That is, they represent a larger class of preferences for  $\mathbb{R}^N$ , but are not  $\Lambda$ -maxmin representations since the  $F_k$  are  $\geq_k$ -weakly increasing but do not represent  $\geq_k$ .

We now prove the existence of a  $\Lambda$ -maxmin representation when  $X$  is a compact metric space and  $(X, \Lambda)$  satisfies the following *Betweenness* condition:

For every  $x, y \in X$  and ordering  $\geq_l \in \Lambda$ , if  $y >_l x$ , then there exists  $z \in X$  such that:

(i)  $y >_l z >_l x$  and (ii)  $z \geq_k x$  or  $z \geq_k y$  for all other  $\geq_k$ .

This condition is inspired by the Euclidean setting. In any convex subset of

Euclidean space with any collection of linear orderings  $\Lambda$ , an even stronger property holds: for any  $x$  and  $y$  and any point  $z$  on the line segment between them,  $z$  is sandwiched between  $x$  and  $y$  according to every algebraic linear ordering. An example of a non-convex set that satisfies the Betweenness condition with  $\Lambda = \Psi_+$  is a hollow square. The only closed sets in  $\mathbb{R}^2$  that satisfy Betweenness with  $\Psi$ -convexity are the standard convex sets.

**Proposition 4:** Let  $X$  be a compact metric space and  $\Lambda$  be a set of continuous primitive orderings satisfying Betweenness. Then, any continuous  $\Lambda$ -strictly convex preference relation  $\succsim$  has a  $\Lambda$ -maxmin representation.

**Proof:** See the Appendix.

## 5. Comments:

(i) **Maxmin models:** Maxmin functions have a long history, originating with Wald (1950). It is interesting to compare our maxmin representation with the familiar but different maxmin representation of Gilboa and Schmeidler (1989). Let  $S$  be a finite set of states and  $Z$  a set of outcomes. An act is a function  $f : S \rightarrow Z$ . Gilboa and Schmeidler (1989) prove that if a preference relation over the set of acts satisfies certain axioms, then there is a function  $u : Z \rightarrow \mathbb{R}$  and a set  $C$  of probability measures (priors) over  $S$  such that the preference relation is represented by  $U(f) = \min_{p \in C} \{p \times [u \circ f]\}$ . By this approach, an act is transformed subjectively into a point  $u \circ f \in \mathbb{R}^S$ . Each  $p_k \in C$  can be thought of as the algebraic linear function  $p_k \times f$  over  $\mathbb{R}^S$ , and the utility of an act is the minimal value it receives according to these functions.

In order to obtain a related but different representation in our framework, one can take the alternatives to be objective vectors  $f \in \mathbb{R}^S$  and the set  $\Lambda$  to be a set of orderings represented by functions of the type  $p_k \times f$ . Then, by Proposition 4, an agent's  $\Lambda$ -strictly convex preferences can be thought of him having in mind a set of increasing functions  $\{U_k\}$  which he applies to the values  $p_k \times f$  and then judges alternatives by  $\min_k \{U_1(p_1 \times f), \dots, U_K(p_K \times f)\}$ .

Thus, in our setting the set of probability measures  $C$  is taken as given, in contrast to Gilboa and Schmeidler (1989)'s framework. However, this is not the key difference since any utility function  $U_k$  can be taken above the minimum so as to

render the associated probability measure ineffective. The main difference between these two representations is the order in which the functions  $U$  and  $p_k \times f$  are applied. More importantly, we study a general notion of convex preferences according to which the primitive orderings are not necessarily algebraic linear functions and where the set of alternatives need not be Euclidean.

(ii) **Social Choice:** Methods for constructing preference relations are the focus of Social Choice Theory where the social preferences are determined as a function of the individuals' preferences. The notion of  $\Lambda$ -convexity can also be thought of as a social welfare function requirement. We say that the social welfare function  $F$  is *convex* if for every profile  $P$  the social preference  $F(P)$  is  $\Lambda$ -convex where  $\Lambda$  consists of all preferences that appear in the profile  $P$ . Note that the concept of  $\Lambda$ -convexity is an intra-profile condition. Thus our analysis can be thought as within the single-profile approach in social choice where a preference relation is built on a specific profile of preference relations without requiring consistency in its definition across various profiles.

Recall that for finite  $X$  and strict primitive rankings,  $\Lambda$ -convexity requires that at the bottom of the social ranking lies an element which is at the bottom of one of the individuals' rankings (see (V)). However, the principle by which a convex SWF picks one of the bottom-ranked elements may vary from one profile to another. Here are two convex social welfare functions that additionally satisfy the standard neutrality, monotonicity, and anonymity conditions:

(1) The "uniform max-min" SWF is defined by  $U(x) = \min_i u_i(x)$  where  $u_i(x) = -rank(x, \geq_i)$ . This SWF bottom-ranks all elements that are ranked last by at least one individual. Above them it places those that are ranked next to last by at least one individual but were not ranked last by any agent, and so on.

(2) A "recursive bottom element" SWF: Let  $X_1 = X$  and define inductively  $M_j = \{x \in X \mid \text{there is an individual } i \text{ for whom } x \text{ is } \succ_i \text{ -minimal in } X_j\}$  and let  $X_{j+1} = X_j - M_j$ . Define  $class(x) = l$  if  $x \in M_l$ . The SWF ranks  $x$  at least as high as  $y$  if  $class(x) \geq class(y)$ .

This SWF bottom-ranks all elements which are ranked last by at least one individual and then above them it places all the elements which are ranked last by at least one individual among the remaining alternatives and so on.

Note that the Borda rule is a typical SWF which does not necessarily convex.

(iii)  **$\Lambda$ -Concavity:** Dual to our notion of  $\Lambda$ -convexity is the following concept, which we call  $\Lambda$ -concavity:

A preference relation  $\succsim$  on  $X$  is  $\Lambda$ -concave ( $\Lambda$ -strictly concave) if for every  $a, b \in X$  the following condition holds:

If for every  $\succeq_k \in \Lambda$ , there is a  $y_k \neq a$  such that  $b \succsim y_k$  and  $y_k \succeq_k a$ , then  $b \succ a$  ( $b \succ a$ ).

Recall that the "persuading argument" for  $b \succ a$  which lies behind the notion of  $\Lambda$ -convexity is the existence for any criterion of an alternative which is ranked weakly below  $b$  by the criterion and still is weakly superior to  $a$ . The persuading argument behind the notion of  $\Lambda$ -concavity is the existence for each criterion of an alternative which is ranked weakly above  $a$  by the criterion and still is weakly inferior to  $b$ . Both arguments are sound, but apparently it is the former which fits to the standard notion of convexity. In the context of choice, the  $\Lambda$ -convexity conditions are arguments for choosing  $b$  whereas  $\Lambda$ -concavity provides arguments for not choosing  $a$ .

The reader will now be expecting an attempt to connect the notion of  $\Lambda$ -strictly concavity to dual representations in the spirit of Propositions 1-4, and we shall not disappoint. For simplicity, we only do so for Proposition 2. We say that a preference relation  $\succsim$  over  $X$  has a  $\Lambda$ -maxmax representation if  $\max_k U_k(x)$  represents  $\succsim$  where  $U_k$  is a utility representation of  $\succeq_k$ .

**Proposition 2 (Dual):** Let  $X$  be a finite set. Any  $\Lambda$ -strictly concave preference relation  $\succsim$  on  $X$  has a  $\Lambda$ -maxmax representation.

**Proof:** For any binary relation  $R$ , define the converse binary relation  $R^T$ , as  $bR^T a$  if  $aRb$ . If  $\succsim$  is  $\Lambda$ -strictly concave, then  $\succsim^T$  is  $\Lambda^T$ -strictly convex where  $\Lambda^T = \{\succeq_k^T : \succeq_k \in \Lambda\}$ . By Proposition 2, there exists  $\{V_k\}$  such that  $V_k$  represents  $\succeq_k^T$ , and  $V(x) = \min V_k(x)$  represents  $\succsim^T$ . Therefore, for every  $x \in X$ ,  $-V(x) = -\min V_k(x) = \max -V_k(x)$ , and  $-V_k, -V$  represents  $\succeq_k, \succsim$ , respectively. Thus,  $\succsim$  has a  $\Lambda$ -maxmax representation. ■

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## Appendix

**Proposition 4:** Let  $X$  be a compact metric space and  $\Lambda$  be a set of continuous primitive orderings satisfying Betweenness. Then, any continuous  $\Lambda$ -strictly convex preference relation  $\succsim$  has a  $\Lambda$ -maxmin representation.

**Proof:** Let  $U$  and  $V_k$  be continuous functions representing  $\succsim$  and  $\geq_k$ , respectively, each with a range of  $[0, 1]$ . Recall that for every  $\geq_k \in \Lambda$ , the set  $C_k$  is defined as  $C_k = \{x \mid \geq_k \in \text{Critical}(x)\}$ . Notice that there cannot be  $x \sim_k y$  such that  $x, y \in C_k$ . To see why, WLOG suppose  $x \succsim y$ , then  $x >_k y$  because  $y \in C_k$ . Let  $cl(C_k)$  denote the topological closure of  $C_k$  and define  $\overline{cl(C_k)} = \{y \mid y \sim_k x \text{ for some } x \in cl(C_k)\}$ .

We now define for each  $k$  a function  $U_k$  that represents  $\geq_k$ , such that  $\min_k U_k(x)$  represents  $\succsim$ .

**Step 1:** Defining  $U_k$  on  $\overline{cl(C_k)}$ .

For each  $x \in \overline{cl(C_k)}$ , define  $U_k(x) = \max\{U(y) \mid y \sim_k x\}$ . Notice that this definition implies that for all  $x \in C_k$ ,  $U_k(x) = U(x)$ . To see this, note that since  $x \sim_k x$  we have  $U_k(x) \geq U(x)$  and by the definition of  $C_k$  there is no  $y$  such that  $U(y) > U(x)$  and  $y \sim_k x$ .

**Step 2:** If  $x >_k y$  where  $x \in cl(C_k)$  and  $y$  is arbitrary, then  $x \succ y$ .

Suppose by contradiction that  $y \succsim x$ . By Betweenness, there exists  $w$  such that  $x >_k w >_k y$  and for all other  $l$ ,  $w \geq_l x$  or  $w \geq_l y$ .

Case (i):  $x \succsim w$ . Then  $y \succsim x \succsim w$  and for every  $l$ , either  $w \geq_l x$  or  $w \geq_l y$  which implies by strict convexity that  $w \succ w$ , a contradiction.

Case (ii):  $w \succ x$ . Since  $x \in cl(C_k)$ , take a sequence  $x_n \rightarrow x$  such that  $x_n \in C_k$ . By the continuity of  $\succsim$  and  $\geq_k$ , for  $n$  large enough, it is true that  $w \succ x_n$  and  $x_n >_k w$ , violating  $x_n \in C_k$ .

**Step 3:**  $U_k$  represents  $\geq_k$  on  $\overline{cl(C_k)}$ .

Let  $x, y \in \overline{cl(C_k)}$ . By definition, if  $x \sim_k y$ , then  $U_k(x) = U_k(y)$ . Now, suppose that  $x >_k y$ , and consider  $w \in cl(C_k)$ , such that  $w \sim_k x$ , and some  $z \in X$ , such that  $z \sim_k y$  and  $U_k(y) = U(z)$ . Then,  $w >_k z$ . Since  $w \in cl(C_k)$ , then by step 2,  $w \succ z$ . Thus,  $U_k(x) = U_k(w) \geq U(w) > U(z) = U_k(y)$ .

**Step 4:** Extension of  $U_k$  for  $x \notin \overline{cl(C_k)}$ .

Since  $cl(C_k)$  is a closed subset of a compact set and  $V_k$  is continuous, the set of numbers  $V_k(cl(C_k))$  is also closed and  $V_k(\overline{cl(C_k)}) = V_k(cl(C_k))$  is therefore closed

as well. Thus, the set  $[0, 1] \setminus V_k(\text{cl}(C_k))$  is a collection  $I_k$  of disjoint open intervals of the form  $(a, b)$ ,  $[0, b)$  or  $(a, 1]$ .

Case (i) Take  $x \notin \overline{\text{cl}(C_k)}$  which according to  $\geq_k$  is neither strictly above nor strictly below all members of  $\overline{\text{cl}(C_k)}$ . Then,  $V_k(x)$  lies on a member of  $I_k$  of the type  $(a, b) = (V_k(\alpha), V_k(\beta))$  where  $\alpha, \beta \in \text{cl}(C_k)$ . Define  $W(V_k(x)) = \max\{U(y) : x \sim_k y\}$ . Let  $\overline{W} : (a, b) \rightarrow (U_k(\alpha), U_k(\beta))$  be the upper convex envelope of  $W$  on  $[a, b]$ . To see that  $\overline{W}$  is strictly increasing, since  $W$  is upper hemi-continuous by the theorem of the maximum, it suffices to show that if  $\beta >_k x >_k \alpha$ , then  $U_k(\beta) > W(V_k(x))$ . To see this, take  $y$  such that  $x \sim_k y$  and  $W(V_k(x)) = U(y)$ . As  $\beta \in \text{cl}(C_k)$  and  $\beta >_k x \sim y$ , then by step 2,  $U(\beta) > U(y)$ . Therefore,  $U_k(\beta) \geq U(\beta) > U(y) = W(V_k(x))$ . Define  $U_k(x) = \overline{W}(V_k(x))$ . The function  $U_k$  represents  $\geq_k$  for any  $x, y$ , such that  $b \geq V_k(x), V_k(y) \geq a$  since  $U_k$  is a strict monotonic transformation of  $V_k$ . Furthermore,  $U_k(x) = \overline{W}(V_k(x)) \geq W(V_k(x)) \geq U(x)$ .

Case (ii) There is no  $x \notin \overline{\text{cl}(C_k)}$  which according to  $\geq_k$  is strictly below all members of  $\overline{\text{cl}(C_k)}$ . This is because a  $\succeq$ -maximal element of  $V_k^{-1}(0)$  is necessarily in  $C_k$ .

Case (iii) Consider  $x \notin \overline{\text{cl}(C_k)}$  which according to  $\geq_k$  is strictly above all members of  $\overline{\text{cl}(C_k)}$ . For interval  $(a, 1] \in I_k$ , we can simply define  $W(1) = 2$  and then allow  $\overline{W}$  to be the upper convex envelope of  $W$  on  $[a, 1]$  where  $W$  is defined as before. Define  $U_k(x) = \overline{W}(V_k(x))$ . Since  $1 \geq W$ , the function  $\overline{W}$  is strictly increasing and therefore  $U_k$  represents  $\geq_k$  for  $x, y$ , such that  $1 \geq V_k(x), V_k(y) \geq a$  and  $U_k(x) = \overline{W}(V_k(x)) \geq W(V_k(x)) \geq U(x)$ .

**Step 5:**  $U(x) = \min_k U_k(x)$ .

By construction, for all  $k$ ,  $U_k(x) \geq U(x)$ . Because  $\succeq$  is  $\Lambda$ -strictly convex, for every  $x$  there is an ordering  $\geq_k$  such that  $x \in C_k$  and  $U_k(x) = U(x)$ . Consequently, for all  $x$ ,  $U(x) = \min_k U_k(x)$ . ■