Normative Equilibrium: The permissible and the forbidden as devices for bringing order to economic environments*

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ABSTRACT:

We introduce the notion of a *normative equilibrium* as a method which brings harmony to "general equilibrium" like environments where individuals make preference-maximizing choices but not every profile of choices is feasible. In an equilibrium, norms stipulate what is permissible and what is forbidden. These uniform norms play a role analogous to that of price systems in competitive equilibrium and also feature some element of "fairness" since all individuals face the same choice set. The solution concept is a maximally permissive set of alternatives that is consistent with the existence of a profile of optimal choices which is feasible. Properties of the solution concept are analysed and the concept is applied to a variety of economic settings.

KEYWORDS: Normative Equilibrium, Envy-Free, General Equilibrium, Convexity.

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1. Introduction

We, in Economics, are used to thinking about prices as a device to bring order and harmony to economic environments where joint and conflicting interests prevail and the formation of a stable arrangement requires putting restrictions on what economic agents are allowed to do.

Here, we investigate the logic of another social instrument, which sometimes brings stability into society: *social norms*. These social norms are applied uniformly to all individuals in the society and determine what is permissible and what is forbidden. Central solution concept which we introduce is *maximal normative equilibrium*. It is a maximal set of permissible actions combined with a compatible profile of permissible alternatives such that each agent's assigned alternative is optimal from among the permissible ones.

We have in mind that when faced with disharmony the social norm adjusts until harmony is achieved. We also assume that if there are unnecessarily forbidden actions such that harmony would prevail even if they were allowed, then the set of normative actions will be extended to permit them. We do not assume that there is an authority that determines the norms. We imagine that the same invisible hand that calculates equilibrium prices so well is also able to determine a minimal set of forbidden actions for which the optimal individualistic behavior is compatible.

The notion of maximal normative equilibrium has a flavor of fairness in the sense that all agents have the same set of opportunities. Agents may choose differently according to their tastes but the outcome will always be "envy-free". Note, however, that just because agents face the same set of opportunities does not make it necessarily fair. One set of normative alternatives could be most preferred by one agent and least preferred by another while a different normative set might be considered more fair because it achieves a better balance between the interests of the two agents.

There are three leading scenarios we have in mind:

1. A bundle is to be distributed among a group of people. Social norms determine the set of permissible bundles that group members are to choose from. For compatibility, the set of permissible bundles should be such that total demand does not exceed the total bundle.

An example of such a norm is derived from the notion of egalitarian competitive equilibrium: calculate the competitive equilibrium prices of the economy where all agents are initially endowed with an equal share of the total bundle. In this equilibrium, each agent faces the same budget set, which together with the profile of individually optimal choices from it constitutes a maximal normative equilibrium.

2. The survival of a group depends on their ability to reach each other within a certain time in the case of danger to any of its members. Therefore, they need to live within a certain distance from one another. The members have preferences over where they live. The equilibrium normative set specifies where they are allowed to live. An equilibrium imposes minimal restrictions so that the individuals' choices fulfill the closeness requirement.

Alternatively, one may think of the group as a political party. Each party member chooses a political position. In order to maintain party unity, the chosen positions must not be too far from each other. We look for the broadest sets of positions that maintain party harmony.

3. A group owns quantities of several types of resources. Each member of the group can choose a quantity of only one type. In equilibrium, a maximal quota is determined on each resource such that its total demand does not exceed its available quantity. We will see that in such an equilibrium some of the resources might not be exhausted and therefore the result may be Pareto-inefficient. Nonetheless, we will see that in equilibrium the amount of inefficiency is "small".

In some sense, the current paper continues our earlier work, Richter and Rubinstein (2015). In that paper, we investigated a different solution concept: *abstract equilibrium*. There, a social ranking of alternatives is evolved and agents are assigned alternatives, so that, for each agent, the assigned alternative is the best from among all the alternatives which are socially ranked weakly lower than the one assigned to him. The role of the social ranking in abstract equilibrium is analogous to the role of the relation "more expensive" which governs the standard competitive equilibrium notions. Harmony is obtained through individualistic choice sets governed by the social ranking (unlike in the current paper where the allowed set of alternatives is identical for all agents).

In the rest of the paper, we formally define the solution concept, prove some general results about its nature, especially its connection to Pareto optimality, and apply the concept to seven economic examples.

2. Maximal Normative Equilibrium

We start with the basic definitions of an economy and the basic solution concept:

Definition 1 *An **economy** is a tuple $\langle N, X, \{\succsim^i\}_{i\in N}, F\}\rangle$ where N is a finite set of agents, X is a set of alternatives, $\succsim^i$ is i's preference and F is a subset of X^N (the set of profiles). The set F contains all profiles of choices that are feasible.*

Definition 2 *A **Normative Equilibrium** (NorE) is a pair $\langle Y, (y^i)\rangle$ where $Y \subseteq X$ and (y^i) is a profile of elements in Y satisfying that:*
(i) for all i, y^i is an i-best action in Y and
(ii) $(y^i) \in F$.
*We refer to Y as a **normative set** and to (y^i) as a **NorE outcome**.*

In many of the examples, the set X is a subset of an Euclidean space. In this case, we will require that the set of normative alternatives is closed.

A central solution concept is the following:

Definition 3 *A NorE $\langle Y, (y^i)\rangle$ is a **maximal NorE** if there is no NorE $\langle Z, (z^i)\rangle$ such that $Z \supset Y$. We refer to Y as a **maximal normative set**.*

The maximality of the normative set among those which induce harmony in the society captures a stability concept. If a NorE evolves and is too strict, in the sense that releasing constraints does not interfere with feasibility, then the society will extend the set of permissible alternatives.

In this solution concept, all agents face the same choice set and since all individuals are rational, no agent envies the alternative chosen by another. This property is called "envy-freeness" (see Foley (1967)) and is defined formally as follows:

Definition 4 *A profile (y^i) is **envy-free** if for all $i \neq j$, $y^i \succsim^i y^j$.*
*A profile (y^i) is **strictly envy-free** if for all $i \neq j$, $y^i \succ^i y^j$.*

The following claim demonstrates that the maximal Normative Equilibrium outcomes are precisely the profiles that are Pareto-efficient among the set of envy-free profiles.

Proposition 1 *A profile is a maximal NorE outcome if and only if it is Pareto-efficient among all feasible envy-free profiles.*

Proof. Let $\langle Y,(y^i)\rangle$ be a maximal NorE outcome. The profile (y^i) is envy-free. If it is Pareto inefficient among the feasible envy-free profiles, then there is a feasible envy-free profile (z^i) which Pareto dominates (y^i). Clearly, $\langle Y\cup\{z^1,..,z^n\},(z^i)\rangle$ is a NorE and for at least one agent i, $z^i \succ^i y^i$ and therefore $z^i \notin Y$ contradicting the maximality of $\langle Y,(y^i)\rangle$.

Let (y^i) be Pareto-efficient among the feasible envy-free profiles. Define $Y=\cup_i\{y^i\}\cup \{x|$ for all i, $y^i \succsim^i x\}$. Clearly, $\langle Y,(y^i)\rangle$ is a NorE. If $\langle Z,(z^i)\rangle$ is a NorE with $Z\supset Y$ then $z^i \succsim^i y^i$ for all i and (z^i) is envy-free. Moreover, for any $x\in Z-Y$ there is an agent i so that $x \succ^i y^i$ and therefore, $z^i \succsim^i x \succ^i y^i$. Therefore, (z^i) is an envy-free profile which Pareto dominates (y^i), contradicting (y^i) being Pareto-efficient among the envy-free profiles. Thus, no such $\langle Z,(z^i)\rangle$ can exist and $\langle Y,(y^i)\rangle$ is a maximal NorE. $\square$

An obvious consequence of Proposition 1 is that any profile which is both envy-free and Pareto-efficient (among *all* feasible profiles) is a maximal NorE outcome. However a maximal NorE might be Pareto inefficient: To see this, consider the "housing economy" where $N=\{1,2\}$, $X=\{a,b,c\}$ (the set of houses) and F is the set of all profiles that assign each house at most once. Suppose that the agents' preferences are $a\succ^1 b\succ^1 c$ and $a\succ^2 c\succ^2 b$. Every Pareto-efficient allocation assigns a to one of the agents. But in any NorE $\langle Y,(y^i)_{i\in N}\rangle$ it must be that $a\notin Y$ (otherwise the two agents would demand it). The unique NorE (which is also the unique maximal NorE) is $Y=\{b,c\}$, $y^1=b$, $y^2=c$, which is Pareto-inefficient.

One set of economies for which any maximal normative equilibrium outcome is Pareto-efficient is given in the following proposition:

Proposition 2 *If F has the property that $x=(x^i,x^{-i})\in F\Rightarrow (x^j,x^{-i})\in F$, then the set of maximal normative equilibrium outcomes is the set of all Pareto-efficient profiles.*

Proof. Let $\langle Y,(y^i)\rangle$ be a maximal NorE with a Pareto inefficient outcome. Then, there is a feasible profile (z^i) which Pareto dominates (y^i). We will define an envy-free feasible profile (x^i) that Pareto dominates (y^i) and then use it to construct a larger NorE.

Assign x^1, a $\succsim^1$-maximal alternative from $\{z^1,...,z^N\}$, to agent 1. The profile (x^1,z^{-1}) is weakly Pareto-superior to (z^i) and agent 1 does not envy any other agent. Assign x^2, a most preferred alternative for agent 2 from $\{x^1,z^2,\dots,z^N\}$, to agent 2, and so on to form the profile (x^i). In this construction: 1) the profile selected at each stage is feasible and weakly Pareto-dominates the previous one and thus (x^i) weakly Pareto-dominates (z^i) and 2) at stage j, no agent $i \le j$ envies any other agent. Therefore, (x^i) is envy-free and Pareto dominates (y^i) (because (z^i) does), violating Claim 1.

On the other hand, under the condition on F every Pareto-efficient profile is envy-free and therefore is efficient among the envy-free allocations. By Proposition 1, it is an outcome of a maximal NorE. □

Example A *Cakes splitting economy*

There are two goods, 1 and 2, with total bundle (α,β). Each agent can choose a quantity of only one of the two goods. Thus, the set of alternatives X consists of all objects of the type $(a,0)$ and $(0,b)$. A profile is feasible if for each good the sum of the agents' assignments of that good does not exceed its total. Agents have continuous and strictly monotonic preferences. A social norm determines what quantities are allowed to be chosen from each good and in equilibrium it guarantees that the total demands do not exceed the available quantities.

In this example, any maximal Normative Equilibrium set is downward closed. The following result shows that there is a unique specification of maximal "quotas" in which the demand for each of the goods does not exceed the supply. Note that the result applies equally to any economy of K goods where each agent consumes only one type of good.

Claim A *(i) There is a unique maximal Normative set.*

(ii) In any maximal NorE, at least one of the goods is fully consumed.

(iii) In any maximal NorE, each good is allocated in a fixed quantity and if a good is not fully allocated, then the unallocated portion is smaller than one allocated portion.

Proof. **Step 1:** If $\langle Y,(y^i)\rangle$ and $\langle Z,(z^i)\rangle$ are NorE, then there is a NorE with the normative set $Y \cup Z$.

Let $\langle Y,(y^i)\rangle$ and $\langle Z,(z^i)\rangle$ be NorE. For any closed subset W of X, let $a_W = \max(a : (a,0)\in W)$ and $b_W = \max(b : (0,b)\in W)$. If $a_Y \geq a_Z$ and $b_Y \geq b_Z$, then $\langle Y\cup Z,(y^i)\rangle$ is a NorE. Similarly, if $a_Z \geq a_Y$ and $b_Z \geq b_Y$, then $\langle Y\cup Z,(z^i)\rangle$ is a NorE.

Otherwise, without loss of generality, we have $a_Y > a_Z$ and $b_Z > b_Y$. Then, $a_{Y\cup Z} = a_Y$ and $b_{Y\cup Z} = b_Z$. Total consumption of the first good given the normative set $Y\cup Z$ is then bounded above by $\#\{i : (a_Y,0)\succsim^i (0,b_Z)\} * a_Y \leq \#\{i : (a_Y,0)\succ^i (0,b_Y)\} * a_Y \leq \alpha$. An analogous argument applies to the second good as well, and thus the total consumption of both goods given $Y\cup Z$ is less than the total endowment.

Step 2: Existence of a maximal normative set.

Let $a^* = \sup\{a_Y : Y \text{ is a normative set in some NorE}\}$ and similarly define b^*. We now show that $M = \{(x_1,x_2)\in X : x_1 \leq a^*,\ x_2 \leq b^*\}$ is a normative set in some NorE. By definition, there are sequences of NorE normative sets (Y_n) and (Z_n) such that $a_{Y_n} \to a^*$ and $b_{Z_n} \to b^*$. By Step 1, $W_n = Y_n \cup Z_n$ is a sequence of normative sets and $(a_{W_n}, b_{W_n}) \to (a^*,b^*)$. This sequence has a subsequence in which a fixed set of agents Q choose $(a_{W_n},0)$ and the remainder $N-Q$ choose $(0,b_{W_n})$. Finally, $\langle M,(m^i)\rangle$ is a NorE where $m^i = (a^*,0)$ if $i\in Q$ and $m^i = (0,b^*)$ if $i \in N-Q$. To verify feasibility of (m^i), notice that $a_{Wn} * |Q| \leq \alpha$ and so $a^* * |Q| \leq \alpha$ and similarly for the other good. To verify individual optimality, notice that since $(a_{Wn},0)\succsim^i (0,b_{W_n})$ for all $i\in Q$, then by continuity $(a^*,0)\succsim^i (0,b^*)$ for all $i\in Q$. Similarly, $(0,b_{W_n})\succsim^i (a_{Wn},0)$ for all $i\in N-Q$.

Step 3: M is the unique maximal normative set.

Given any NorE $\langle Y,(y^i)\rangle$, by the definition of a^* and b^*, it is the case that $a^* \geq a_Y$ and $b^* \geq b_Y$. Thus, $Y \subseteq M$.

Step 4: In a maximal normative equilibrium, at least one of the goods is fully consumed.

Assume otherwise. Let $\langle Y,(y^i)\rangle$ be any NorE where k agents are allocated $(a_Y,0)$ while $N-k$ agents are allocated $(0,b_Y)$ and no good is fully consumed. It is the case that $ka_Y < \alpha$ and $(N-k)b_Y < \beta$. Take a',b' so that $a_Y < a'$, $b_Y < b'$ and $ka' < \alpha$ and $(N-k)b' < \beta$. Now, define $Y^\lambda = \{(x_1,x_2) : (x_1,x_2) \leq (\lambda a' + (1-\lambda)a_Y, \lambda b_Y + (1-\lambda)b')\}$. When $\lambda = 0$, at least $N-k$ agents prefer $(0, b_{Y^0} = b')$ to $(a_{Y^0} = a_Y, 0)$. When $\lambda = 1$, at least k agents prefer $(a_{Y^1} = a', 0)$ to $(0, b_{Y^1} = b_Y)$. By continuity, there is some intermediate

λ where at least k agents weakly prefer $(\lambda a' + (1-\lambda)a_Y, 0)$ to $(0, \lambda b_Y + (1-\lambda)b')$ and at least $N-k$ agents prefer $(0, \lambda b_Y + (1-\lambda)b')$ to $(\lambda a' + (1-\lambda)a_Y, 0)$. Then, Y^λ is a larger normative set and therefore Y could not have been maximal.

Step 5: For any maximal normative equilibrium, if a good is not fully consumed, then its unallocated portion is smaller than each allocated piece of that good.

Let $\langle Y, (y^i)\rangle$ be a maximal NorE where k agents are allocated $(a_Y, 0)$ and $\alpha - k a_Y > a_Y$. Suppose that every agent who is allocated $(0, b_Y)$ strictly prefers $(0, b_Y)$ to $(a_Y, 0)$. Then, a_Y can be slightly increased without changing consumption patterns, thus violating the maximality of Y. On the other hand, if for at least one i, $y^i = (0, b_Y)$ and $(a_Y, 0) \sim_i (0, b_Y)$, then $\langle Y, (z^i = (a_Y, 0), z^{-i} = y^{-i})\rangle$ is also a NorE where no good is fully consumed. By Step 4, there is a NorE with a larger normative set, contradicting the maximality of $\langle Y, (y^i)\rangle$. $\square$

As is shown in Proposition 1, any maximal NorE outcome is Pareto-optimal within the set of envy-free profiles. In the current example, a maximal NorE outcome may be Pareto inefficient. For example, consider the case where the total endowment is $(1,1)$ and there are two agents who both have preferences satisfying $(0,1) \succ (1,0) \sim (0,3/4)$. Then, $\langle Y, (y^i)\rangle$ where $Y = \{(x_1, x_2) : (x_1, x_2) \leq (1, 3/4)\}$, $y^1 = (1,0)$ and $y^2 = (0, 3/4)$, is a maximal NorE, but (y^i) is Pareto inefficient. Notice that the unallocated portion is $1/4$ of the second good which is indeed smaller than the allocated portion.

Example B *Lower threshold clubs economy*

Consider an economy where X is a finite set of clubs. Feasibility requires that each club x is either empty or occupied by at least m_x agents ($m_x \leq N$). If each agent would choose his beloved club, then typically there will be insufficiently occupied clubs. The role of the maximal Normative Equilibrium is to allow coordination with minimum restrictions on the agents.

Many normative equilibria exist (for example, any set $Y = \{x\}$ combined with all agents choosing x). Since there are finitely many normative sets, a maximal normative equilibrium always exists as well.

Claim B *In some lower threshold clubs economies, every maximal normative equilibrium is Pareto inefficient.*

Proof. Consider the lower threshold clubs economy with $N = 6$, $X = \{a, b, c\}$, and $m_x = 3$ for all x. Suppose that agents 1 and 2 have the preferences $a \succ b \succ c$, agents 3 and 4 have the preferences $b \succ c \succ a$ and agents 5 and 6 have the preferences $c \succ a \succ b$.

In any feasible allocation, there can be only 1 or 2 active clubs. Any profile with one active club, say b, is Pareto inefficient since there are four agents who prefer a to b, leading to a Pareto improvement where exactly three of them move to a. Any profile with two active clubs cannot be a NorE outcome since feasibility requires that three agents choose each club, but in that case individual optimality is violated because one of the clubs is only preferred by two agents. Thus, in this example, every NorE outcome is Pareto inefficient. □

Restrictions on which clubs are allowed to operate encourage agents to group together. In this way, lower bound feasibility conditions can be achieved with a normative set of clubs. On the other hand, in a setting with upper bounds on club membership, often there is no normative equilibrium. For example, if there are two clubs and two agents with identical preferences, say $a \succ b$, and F requires that at most one member chooses each club, then there is no NorE.

Of special interest are the *Euclidean economies* where the set of alternatives is embedded in an Euclidean space. In such economies, we introduce standard restrictions of closedness and convexity on the parameters of the model (the set of alternatives, the preference relations and the feasibility set):

Definition 5 *An economy $\langle N, X, \{\succsim^i\}_{i \in N}, F\} \rangle$ is a **Euclidean economy** if*

(i) The set X is a closed convex subset of some Euclidean space.

(ii) The preferences $\{\succsim^i\}_{i \in N}$ are continuous, strictly convex and differentiable.

(iii) The feasibility set F is closed and convex.

Proposition 1 characterizes maximal normative equilibrium outcomes as precisely the Pareto-efficient allocations among the envy-free allocations, but does not guarantee overall Pareto-efficiency. In discrete environments, we have examples of maximal NorE outcomes that are Pareto inefficient. Furthermore, example A was an economy with Euclidean alternatives where a maximal NorE outcome may be Pareto inefficient. When this occurred, there was an agent who was indifferent between his assigned alternative and someone else's. The next proposition shows that for Euclidean economies, any maximal NorE outcome without such indifferences is Pareto efficient. Thus, for Euclidean economies, the gap between maximal NorE outcomes and Pareto-efficiency lies in the set of envy-free allocations with indifferences.

Proposition 3 *In an Euclidean economy, any strictly envy-free maximal normative equilibrium outcome is Pareto-efficient.*

Proof. Assume $\langle Y,(y^i)\rangle$ is a maximal NorE and $y^i \succ^i y^j$ for all $i \neq j$. If (y^i) is Pareto inefficient, then there is (z^i) such that $z^i \succsim^i y^i$ for all i with at least one inequality. By the continuity of the agents' preferences, for $\varepsilon > 0$ small enough, $\varepsilon z^i + (1-\varepsilon)y^i \succ^i \varepsilon z^j + (1-\varepsilon)y^j$ for all $i \neq j$. By the strict convexity of the agents' preferences, $\varepsilon z^i + (1-\varepsilon)y^i \succ_i y^i$ for each agent i for whom $z^i \neq y^i$. Thus, for each i, the bundle $\varepsilon z^i + (1-\varepsilon)y^i$ is i's best bundle in $Z = Y \cup \{\varepsilon z^i + (1-\varepsilon)y^i\}_{i\in N}$. Finally, since $(y^i),(z^i) \in F$ and F is convex the profile $(\varepsilon z^i + (1-\varepsilon)y^i)$ is in F. Thus, $\langle Z,(\varepsilon z^i + (1-\varepsilon)y^i)\rangle$ is a larger NorE, violating the maximality of $\langle Y,(y^i)\rangle$. □

3. Convex Normative Equilibrium

In this section, we deal with economies in which the set of alternatives is some subset of an Euclidean space. We introduce convexity into the solution concept by requiring that the normative set is convex (in addition to the requirement that the normative set is closed). This convexity requirement captures either simplicity of "what is allowed" or a sentiment that "if a and b are allowed then anything between them must also be allowed as well".

Definition 6 *A **convex normative equilibrium** (for economies where X is a subset of an Euclidean space) is a NorE $\langle Y,(y^i)\rangle$ such that Y is closed and convex. A maximal convex normative equilibrium is a convex NorE $\langle Y,(y^i)\rangle$ such that there is no other convex normative equilibrium $\langle Z,(z^i)\rangle$ with $Z \supset Y$.*

One direction of Proposition 1 implies that a Pareto-efficient profile which is a normative equilibrium outcome is also a *maximal* normative equilibrium outcome. Proposition 4 below is analogous: if a Pareto-efficient profile is a convex normative equilibrium outcome, then it is also a maximal convex normative outcome. The proof actually demonstrates a stronger statement: it is sufficient for a profile to be Pareto-efficient among the convex normative equilibrium outcomes in order to be a maximal convex normative outcome. The other direction of Proposition 1 states that any maximal normative outcome is Pareto-efficient among the envy-free profiles. This direction generally cannot be extended to the case of maximal convex normative equilibria, as seen later.

Proposition 4 *If (y^i) is Pareto-efficient and a convex NorE outcome, then (y^i) is a maximal convex NorE outcome.*

Proof. We first show that that there is a set Y^* such that $\langle Y^*,(y^i)\rangle$ is a convex NorE and is maximal among all convex Y for which $\langle Y,(y^i)\rangle$ is a NorE. To do so, we apply Zorn's Lemma. (A reminder: A chain is a completely ordered subset of P. Given a partially ordered set P , if every chain in P has an upper bound in P, then the set P has at least one maximal element.) Here, P consists of all sets Y for which $\langle Y,(y^i)\rangle$ is a convex NorE and the partial order is $\supseteq$.

In order to show that any chain C of elements in P has an upper bound in P, it suffices to show that $\overline{U}$, the closure of the union of the sets in C is in P. To be in P means that $\overline{U}$ is a closed convex set for which $\langle \overline{U},(y^i)\rangle$ is a NorE. By definition, $\overline{U}$ is closed. To show that $\overline{U}$ is convex, it suffices to show that U, the union of the sets in C, is convex. Given any two points x,y in U, there is some $Y \in C$ so that $x,y \in Y$ and therefore all points between x and y are in Y and therefore in U. To see that $\langle \overline{U},(y^i)\rangle$ is a NorE, by continuity of preferences it suffices to show that for each i the element y^i is $\succsim^i$ top-ranked in U. Suppose that there is an $x \in U$ so that $x \succ^i y^i$. Then, there is some $Y \in C$ such that $x \in Y$, contradicting that $\langle Y,(y^i)\rangle$ is a NorE.

The proof of the maximality of Y^* is parallel to that of Proposition 1. Suppose that there is another convex NorE $\langle Z, (z^i)\rangle$ such that $Z \supset Y^*$. As $\langle Z, (z^i)\rangle$ is a NorE, it must be that $z^i \succsim^i y^i$ for all i. If $z^i \sim^i y^i$ for all i , then $\langle Z, (y^i)\rangle$ is a NorE, contradicting the maximality of Y^*. On the other hand, if $z^i \succsim^i y^i$ for all i with at least one strict inequality, then the profile (z^i) is a convex NorE outcome and it Pareto dominates (y^i), contradicting (y^i) being Pareto-efficient. □

We now turn to the structure of maximal convex normative sets for Euclidean economies. Recall that any convex set is the intersection of the infinite family of half-spaces that contain it. Proposition 5 states that for Euclidean economies, any maximal convex normative set is not just convex but is also a polygon (an intersection of a finite set of half-spaces), with at most one half-space per agent.

Proposition 5 *Let $\langle Y, (y^i)\rangle$ be a maximal convex NorE in an Euclidean economy and let $J = \{i |\ y^i$ is not $\succsim^i$ -global maximum in $X\}$. Then, there is a profile of closed half-spaces $(H^i)_{i\in J}$, such that $Y = \cap_{i\in J} H^i$.*

Proof. By the differentiability and strict convexity of the agents' preference relations, for every $i \in J$ there is a unique closed half-space H^i containing y^i so that all other elements in H^i are dispreferred to y^i.

Suppose that for some $i \in J$, there is an element $w^i \in Y \backslash H^i$. By the differentiability and strict convexity of i's preferences, for small ε, $\varepsilon w^i + (1-\varepsilon) y^i \succ_i y^i$ and by convexity of Y, $\varepsilon w_i + (1-\varepsilon) y^i \in Y$. Therefore, y^i is not top $\succ_i$-ranked in Y, a contradiction. Thus, $Y \subseteq \cap_{i\in J} H^i$.

Since $\langle Y, (y^i)\rangle$ is a maximal convex NorE it is sufficient to show that $\langle \cap_{i\in J} H^i, (y^i)\rangle$ is a convex NorE. This follows from:

(i) the set $\cap_{i\in J} H^i$ is closed and convex.

(ii) for each j, $y^j \in Y$ and by the previous stage of the proof $Y \subseteq \cap_{i\in J} H^i$ and thus $y^j \in \cap_{i\in J} H^i$.

(iii) for each $j \notin J$, y^j is a global maximum and therefore preferred by agent j to all other alternatives in $\cap_{i\in J} H^i$.

(iv) for each $j \in J$, y^j is $\succsim^j$-preferred to all other alternatives in H^j and therefore y^j is $\succsim^j$-preferred to all alternatives in $\cap_{i \in J} H^i$. $\square$

4. Examples

We now consider a variety of economic examples illustrating the maximal convex NorE concept and its relationship to Pareto-efficiency.

Example C *Exchange economy*

An **exchange economy** $\langle N, X, \{\succsim^i\}_{i \in N}, F\}\rangle$ is an Euclidean economy such that:

(i) The set $X = \mathbb{R}^n_+$ is the set of non-negative bundles.

(ii) Agents' preferences $\{\succsim^i\}_{i \in N}$ are monotonic, continuous, strictly convex and differentiable.

(iii) There is some bundle $e \in \mathbb{R}^n_+$ such that $(x^i) \in F$ if and only if $\sum x^i = e$.

The next claim demonstrates the special status of the egalitarian equilibrium allocation: it is a maximal convex NorE outcome and it is uniquely so among the Pareto-efficient interior profiles. But, this uniqueness does not extend to the boundary, and there can be Pareto-efficient non-interior allocations which are also maximal convex normative outcomes.

Claim C *For an exchange economy* $\langle N, X, \{\succsim^i\}_{i \in N}, F\}\rangle$

(i) The egalitarian equilibrium allocation is a maximal convex NorE outcome.

(ii) If a profile is a maximal convex NorE outcome, Pareto-efficient and interior, then it is the egalitarian equilibrium profile.

(iii) There can exist a non-interior Pareto-efficient profile which is a maximal convex NorE outcome, but is not the egalitarian equilibrium outcome.

Proof. (i) Let $\langle p^*, (y^i)\rangle$ be the competitive equilibrium in the exchange economy where each agent is initially endowed with e/n. Then, the pair $\langle B(p^*, e/n), (y^i)\rangle$ is a convex NorE where $B(p^*, e/n)$ is the budget set given by the equilibrium price vector p^* and the initial endowment e/n. By Proposition 4, since (y^i) is Pareto-efficient, it is also a maximal convex NorE outcome.

(ii) Let $\langle Y,(y^i)\rangle$ be a maximal convex NorE and (y^i) be interior. For each agent i, the chosen alternative y^i is not $\succsim^i$-globally maximal and thus by Proposition 5, $Y = \bigcap_{i\in N} H^i$ where each H^i is an half-space. Because the assignments are interior and the allocation is Pareto-efficient the half spaces must be parallel (otherwise, any two agents on non-parallel half spaces could make a Pareto improving local exchange). By monotonicity, the half spaces must be identical and equal to $Y = \{x|\lambda x \leq w\}$ for some positive vector λ and a number w. For each i, the bundle y^i is optimal in Y and by monotonicity, $\lambda y^i = w$. Therefore, $\sum_{i\in N} \lambda y^i = nw$, which implies that $\lambda y^i = w = \lambda(\sum_{i\in N} y^i/n) = \lambda(e/n)$. Thus, (y^i) is a competitive equilibrium allocation for the market where each agent is initially endowed with e/n.

(iii) Consider the exchange economy with three agents, $e = (5,5)$ and preferences represented by the following utility functions (a slight modification of the preferences will strengthen the weak convexity of the preference relations to strict convexity.):

$u^1(x_1,x_2) = 5x_1 + 1x_2$

$u^2(x_1,x_2) = 1x_1 + 1x_2$

$u^3(x_1,x_2) = 1x_1 + 5x_2$.

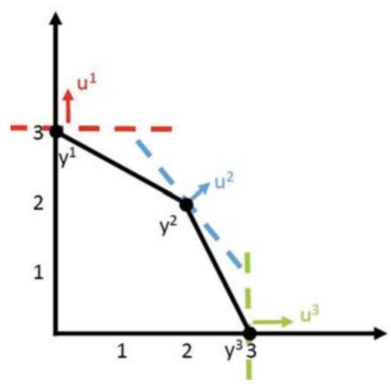

Figure 1.

Let Y be the set of all bundles below the kinked line connecting $y^1 = (0,3)$, $y^2 = (2,2)$ and $y^3 = (3,0)$. The allocation (y^i) is Pareto-efficient since any allocation (z^i) which dominates (y^i) satisfies that $z_1^i + z_2^i \geq y_1^i + y_2^i$ for all i with at least one inequality and thus $\Sigma_i(z_1^i + z_2^i) > \Sigma(y_1^i + y_2^i) = 10$ which is not feasible. Since (y^i) is a convex NorE outcome and Pareto-efficient, then by Proposition 4, it is also a maximal convex NorE outcome. $\square$

Claim C demonstrates that for the exchange economy, among interior Pareto-efficient allocations, the maximal convex NorE concept is much more selective than the maximal NorE concept: the egalitarian equilibrium allocation is the unique maximal convex NorE outcome, whereas *any* envy-free Pareto-efficient allocation is a maximal NorE outcome.

Example D *Give and take economy.*

Let $X = [-1, 1]$, where a positive x represents a withdrawal of x from a social fund and a negative x represents a contribution of $-x$ to the fund. Feasibility requires that the social fund is balanced, that is $(x^i) \in F$ iff $\sum_i x^i = 0$. All agents have strictly convex and continuous preferences (single-peaked) with their ideal denoted by $peak^i$.

The following claim characterizes the maximal convex NorE when there is an aggregate preference for taking, that is $\sum peak^i > 0$. (The opposite case is analogous.)

Claim D *Consider a give and take economy with $\sum peak^i > 0$. There is a unique maximal convex NorE $\langle Y, (y^i)\rangle$. In it ,Y takes the form $[-1, m]$ and (y^i) is Pareto-efficient.*

Proof. Given any set $[-1, m]$, every agent who wants to give will select his peak, and every agent who wants to take is either at his peak or cannot reach his peak and instead makes do with taking m instead. There is a unique m such that $[-1, m]$ is a normative set. To see why, denote by $D(m)$ the net amount given and taken by all agents when the normative set is $[-1, m]$. The function D is weakly increasing and continuous. Moreover, $D(0) \leq 0$, $D(1) = \sum peak^i > 0$, and D is strictly increasing whenever $m < \max\{peak^i\}$. Thus, there is a unique $m^* \geq 0$ for which $D(m^*) = 0$. The set $[-1, m^*]$ combined with the agents' optimal choices is a convex NorE.

To see that the above is a maximal convex NorE, notice that any larger closed convex set must be of the form $[-1, m]$ where $m > m^*$. However, for such an m, $D(m) > 0$ and therefore, $[-1, m]$ is not a normative set because the agents would take too much.

We now show that any other convex normative set $[x, y]$ is smaller than $[-1, m^*]$. In order for the social fund to be balanced, it must be that $x \leq 0 \leq y$. In equilibrium, agents who wish to give will do so at either their peak or at x if $peak^i < x$. Therefore, the total giving in $[x,y]$ is not more than the total giving in $[-1,m^*]$. Since the social fund is balanced, the total taking in $[x,y]$ is also less than or equal to the total taking in $[-1,m^*]$, and therefore $y \le m$. Thus, $[x,y] \subseteq [-1,m^*]$.

The equilibrium outcome (y^i) is Pareto-efficient since if (z^i) Pareto dominates (y^i) it must be that $y^i \le z^i$ for all i with strict inequality for at least one agent violating the feasibility constraint. $\square$

The requirement that the normative set is convex was necessary for the previous claim. There is a give and take economy with a maximal normative equilibrium outcome which is Pareto inefficient. To illustrate, consider a two-agent give and take economy with utilities depicted in Figure 2 where $peak^1 = 1/2$ and $peak^2 = 3/4$. Claim D establishes that $[-1,0]$ is the unique maximal convex normative set.

The economy has a maximal NorE which is inefficient: $Y = \{-1,1\}$ and $y^1 = -1$, $y^2 = 1$. To see its maximality, suppose that there is a NorE $\langle Z,(z^i)\rangle$ with $Z \supset Y$. Feasibility requires that $z^1 = -z^2$. It must be that $|z^1| \ne 1$ since 1 and -1 are agent 1's two least preferred alternatives in Y, and since Z is larger it contains a better alternative. It is impossible that $0 < |z^1| < 1$, because both agents prefer $|z^1|$ to $-|z^1|$. Finally, $z^1 \ne 0$ since if $z^1 = 0$ then $z^2 = 0$, but agent 2 prefers 1 over 0.

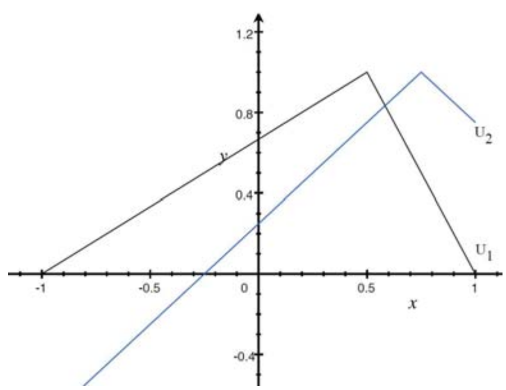

Figure 2.

Example E *The Keeping Close Economy*

Consider an Euclidean economy with X being a closed convex set of locations (geographical or political). Motivated by example 2 in the introduction, feasibility requires that each pair of locations is within a distance of 1 from each other.

The set of Pareto-efficient profiles is non-empty (for example, the serial dictatorship algorithm leads to a Pareto-efficient profile). It follows from Proposition 2 (as F satisfies the condition there) that the set of maximal NorE outcomes is equal to the set of Pareto-efficient profiles. We will now see that the set of maximal convex NorE outcomes includes all Pareto-efficient profiles but may also include Pareto inefficient ones.

Claim E *For a keeping close economy:*

(i) Any Pareto-efficient allocation $(y^i) \in F$ is a maximal convex normative equilibrium outcome.

(ii) A maximal normative convex equilibrium may be Pareto inefficient.

Proof. (i) Given a Pareto efficient allocation (y^i), define Y to be the convex hull of the set $\{y^1, ..., y^n\}$. Notice that since (y^i) is feasible, any pair of locations in Y is within distance 1 of each other. Thus, any profile within Y is feasible (this follows from the inequality $d(\lambda x + (1-\lambda)y, z) \leq \max\{d(x,z),\ d(y,z)\}$). By the Pareto-efficiency of (y^i) , it must be that y^i is i's best alternative within Y. Therefore, $\langle Y, (y^i)\rangle$ is a convex NorE. By Proposition 4, (y^i) is also a maximal convex NorE outcome.

(ii) An example of a Pareto inefficient maximal normative convex outcome is now given for a two-agent economy with alternatives $X = \mathbb{R}^2$, $U^1(x) = -d(x, (-\frac{1}{2}, -\frac{1}{2}))$ and $U^2(x) = -d(x, (\frac{1}{2}, \frac{1}{2}))$ where d denotes the standard Euclidean distance (see Figure 3). The agents want to be closest to their ideal points $peak^1 = (-\frac{1}{2}, -\frac{1}{2})$ and $peak^2 = (\frac{1}{2}, \frac{1}{2})$. A maximal convex NorE is the profile $\langle Y, (y^i)\rangle$ where Y is the x-axis, $y^1 = (-\frac{1}{2}, 0)$ and $y^2 = (\frac{1}{2}, 0)$. There are Pareto-superior profiles, such as $w^1 = (-0.3, -0.3)$ and $w^2 = (0.3, 0.3)$, but there is no larger convex NorE $\langle Z, \{z^i\}\rangle$. This is because Z must be convex and closed and so Z must be a horizontal strip including the x-axis. Therefore, $z^1 = (-\frac{1}{2}, a)$ and $z^2 = (\frac{1}{2}, b)$ where $b \geq 0 \geq a$ with at least one strict inequality, and any such z^1, z^2 are strictly more than distance 1 apart.

$\square$

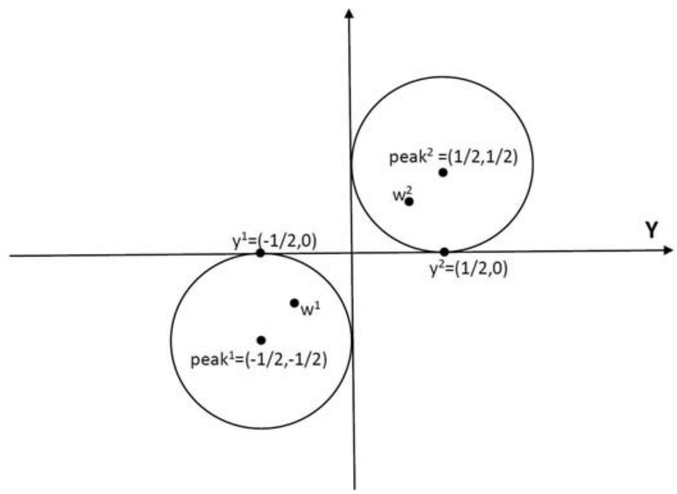

Figure 3.

Example F *Consensus Economy*

A consensus economy is one where the set X is a convex set of positions. The set F consists of all the unanimous ones where all agents make the same choice. This F satisfies the condition of Proposition 2 and thus a profile is a maximal normative equilibrium outcome if and only if it is Pareto-efficient.

The relationship between Pareto efficiency and maximal convex normative equilibrium outcomes is less clear. The next claim shows that when X is a subset of the real line and agents have convex single-peaked preferences, then a profile is a maximal convex normative equilibrium outcome if and only if it is Pareto-efficient, namely a profile of the form $\{y^i = \alpha\}$ where α is between the left-most and right-most peaks. However, in multidimensional Euclidean economies, maximal convex normative equilibrium outcomes may be Pareto inefficient.

Claim F *: (i) For any consensus Euclidean economy with $X = [-1, 1]$, a profile is a maximal convex NorE profile if and only if it is Pareto-efficient.*

(ii) There is a multidimensional consensus Euclidean economy which has a Pareto inefficient maximal convex NorE outcome.

Proof. (i) If (y^i) is Pareto-efficient, then there exists $L \le y^* \le R$ such that $y^i = y^*$ for all i. The pair $\langle \{y^*\}, (y^i = y^*) \rangle$ is a convex NorE and by Proposition 4, $(y^i = y^*)$ is also a maximal convex NorE outcome.

If $\langle Y, (y^i) \rangle$ is a maximal convex NorE, then there is a y^* such that $y^i = y^*$ for all i. It cannot be that $y^* > R$, because then $R \notin Y$, and so the pair $\langle [R,1], (z^i = R) \rangle$ would be a larger convex NorE, contradicting Y's maximality. Similarly, y^* cannot be below L.

(ii) Consider the two agent consensus economy with $X = R^2$ and agents' preferences represented by $U^1(x_1,x_2) = 2x_2 - |x_2 - x_1|$ and $U^2(x_1,x_2) = 2x_2 - |x_2 + x_1|$. Let $Y = \{(x_1,x_2) : x_2 \le 0\}$. From Y, both agents most prefer $y^1 = y^2 = (0,0)$. The pair $\langle Y, (y^i) \rangle$ is a convex NorE. If there were a larger convex normative set Z, it would have to be of the form $\{(x_1,x_2) : x_2 \le z\}$ with $z > 0$. But, from Z, agent 1 most prefers (z,z) and agent 2 most prefers $(-z,z)$, and this profile is not in F. However, $(0,0)$ is Pareto inefficient as both agents prefer $(0,1)$ to $(0,0)$. □

Example G *Near Average Economy*

Consider an Euclidian economy where X, a set of positions, is the real line. Agents have single-peaked preferences and harmony requires that no agent is an outlier, in the sense that his position will not be more than distance 1 from the average position. Thus, denoting $Avg(x^i) = \sum x^i/n$, we have $F = \{(x^i) | d(x^i, Avg(x^i)) \le 1\}$. The set F does not satisfy the condition of Proposition 2; given a feasible profile, moving to the position of another agent may cause a third agent to become an outlier. Notice that agents choose their positions without knowing what the average will be. This is in contrast to what would be assumed in a game theoretic setting.

We will now see that for this economy maximal convex normative equilibria exist and that their outcomes are a (possibly strict) subset of the set of Pareto-efficient profiles.

Claim G *For the near average economy*

(i) There is a maximal convex normative equilibrium.

(ii) Every maximal convex normative equilibrium outcome is Pareto-efficient.

(iii) There can be Pareto-efficient profiles which are not (even) convex normative equilibrium outcomes.

Proof. (i) For any $a \in X$, let $x^i(a)$ denote agent i's most preferred location in $(-\infty, a]$. An agent i chooses either $x^i(a) = peak^i$ or $x^i(a) = a$. Let $\Phi(a) = \max_i d(x^i(a), \sum x^i(a)/n)$. If $a \leq \min(peak^i)$, then $x^i(a) = a$ and $\Phi(a) = 0$. The function Φ is continuous and strictly increasing on $(\min(peak^i), \max(peak^i))$. If $\Phi(\max(peak^i)) \leq 1$, then $Y = R$ together with $y^i = peak^i$ for all i is a maximal convex NorE. If $\Phi(\max(peak^i)) > 1$, then let b be the unique real number such that $\Phi(b) = 1$. The set $Y = (-\infty, b]$ together with $y^i = x^i(b)$ for all i is a maximal convex NorE.

(ii) Let $\langle Y, (y^i)\rangle$ be a maximal convex NorE, and let $a = \min(y^i)$ and $b = \max(y^i)$. Denote by $L = \{j : peak^j < a\}$ the set of individuals with peaks to the left of a (those agents choose a) and similarly, denote by R the set of individuals with peaks to the right of b (those agents choose b). Let $M = N - L - R$ be the "middle" agents.

If $L = R = \emptyset$, then $Y = X$, all agents are at their peaks and there is no possibility of Pareto improvement. If $L = \emptyset$ and $R \neq \emptyset$, then the analysis of $\langle Y, (y^i)\rangle$ is similar to part (a) below.

We now consider the case where both $L \neq \emptyset$ and $R \neq \emptyset$. Assume that (z^i) Pareto dominates (y^i). It must be that $z^i = y^i$ for every $i \in M$, $z^i \leq y^i$ for every $i \in L$ and $z^i \geq y^i$ for every $i \in R$, with at least one strict inequality.

Define $\delta_L = \sum_{i \in L}(y^i - z^i)$ and $\delta_R = \sum_{i \in R}(z^i - y^i)$. Consider three cases:

(a) $0 = \delta_L < \delta_R$. Suppose the agents face the normative set $[a, b']$ instead of Y=[a,b] where $b' > b$. In this case, an agent $i \in R$ would optimally choose $w^i(b') = \min\{peak^i, b'\}$ and all other agents would optimally choose $w^i(b') = y^i$. Let β be the number such that $\sum_{i \in R} \min\{peak^i, \beta\} = \sum_{i \in R} z^i$ and therefore $Avg(w^i(\beta)) = Avg(z^i)$. Notice that $\max(z^i) \geq \beta$. Therefore, $1 \geq \max\{z^i - Avg(z^i)\} \geq \beta - Avg(w^i(\beta))$ and thus $(w^i(\beta)) \in F$. The pair $\langle [a, \beta], (w^i(\beta))\rangle$ is a larger convex normative equilibrium contradicting the maximality of $\langle Y, (y^i)\rangle$.

(b) $0 < \delta_L = \delta_R$. Then, $Avg(x^i) = Avg(z^i) = Avg(y^i)$ where x^i=$b + \varepsilon/|R|$ for $i \in R$, $x^i = 0$ for any $i \in M$ and $x^i = a - \varepsilon/|L|$ for any $i \in L$ where $0 < \varepsilon$ is a number small enough such that it is (i) smaller than $\min_{i \in R}(peak^i - b)$ and $\min_{i \in L}(a - peak^i)$ and (ii) smaller than $max_{i \in R}\{z^i - b\}$ and $max_{i \in L}(a - z^i)$. The set $W = [a - \varepsilon/|L|, b + \varepsilon/|R|]$ with $w^i = a - \varepsilon/|L|$ for $i \in L$, $w^i = a + \varepsilon/|R|$ for $i \in R$ and $w^i = x^i$ for $i \in M$ is a larger convex normative equilibrium, contradicting the maximality of $\langle Y, (y^i)\rangle$.

(c) $0 < \delta_L < \delta_R$. Let (x^i) be a modification of (z^i) such that $x^i = y^i$ for all $i \in L \cup M$ and $b < x^i < z^i$ such that $\delta_R - \delta_L = \sum_{i \in R}(x^i - b) > 0$. The profile (x^i) Pareto dominates (y^i) and we are back to case (a).

(iii) Let $peak^1 = peak^2 = -1$, $peak^3 = 0$ and $peak^4 = 2$. The profile $y^1 = y^2 = -1$ and $y^3 = y^4 = 1$ is Pareto-efficient: any Pareto-improving profile (z^i) must have $z^1 = z^2 = -1$ (because 1 and 2 are at their peaks) and $z^3 \leq 1$, but feasibility then requires that $z^4 \leq 1$, and (z^i) is not a Pareto improvement.

The profile (y^i) is not a convex normative equilibrium outcome because if $-1, 1 \in Y$, then $0 \in Y$, and then $y^3 = 1$ is not 3's best alternative in Y. □

5. Summary

This paper is a part of our grand project exploring the logic of "price-like" institutions which bring order into "general equilibrium" environments.

In Richter and Rubinstein (2015), an equilibrium consists of a public ordering and a profile of choices. The public ordering had the interpretation of a prestige ranking on the space of alternatives. The profile was required to be feasible and each agent's choice was required to be personally optimal from among the set of alternatives that are weakly less prestigious than the one assigned to the agent. For the main solution concept, *primitive equilibrium*, the ordering was required to be a primitive ordering, i.e. a member of a set of basic orderings that all agents use in the formation of their preferences.

It is tempting to think about the normative equilibrium concept as a degenerate case of the concept of primitive equilibrium. In order to do so, one would take the public ordering to be such that all admissible alternatives are equally bottom-ranked, all forbidden alternatives are ranked above the bottom and agents would be assigned bottom-ranked alternatives only. However, note the differences: (i) In general, such an ordering is not a primitive ordering. (ii) The constraint that the set of forbidden alternatives is minimal is not present in the previous framework. (iii) In the earlier paper, we required the public ordering to be convex. The convexity of the public ordering is analogous to a requirement that the set of forbidden elements is convex. In contrast, we require here that the permissible set is convex.

In this paper, we derived general results regarding normative equilibrium outcomes and Pareto-efficiency.

For the concept of **maximal normative equilibria**, we presented several **first welfare theorem** results (i.e. equilibrium outcomes are Pareto-efficient). Proposition 1 states that any maximal NorE outcome is Pareto-efficient among the feasible envy-free profiles. Proposition 2 states a condition on the set of feasible profiles F which guarantees that a NorE outcome is Pareto-efficient. Proposition 3 states that in an Euclidean economy, a maximal NorE outcome which is strictly envy-free is also Pareto-efficient. We also presented some **second welfare theorem** results (i.e. a Pareto-efficient profile is an equilibrium outcome). Proposition 1 states that a profile which is Pareto-efficient among the envy-free profiles (although not necessarily overall Pareto-efficient) is a maximal NorE outcome. Proposition 2 states that under a condition on the set F every Pareto-efficient profile is an equilibrium outcome.

With regard to the concept of **maximal convex normative equilibrium** we did not arrive at a general **first welfare theorem** result. Nevertheless, the first welfare theorem holds in examples C, D and G, while it may be violated in examples E and F. Regarding the **second welfare theorem**, Proposition 4 states that every Pareto-efficient profile which is a convex NorE outcome is also a maximal convex NorE. The second welfare theorem holds in Examples E and F but fails in examples C, D and G.

As to the relationship between maximal NorE outcomes and maximal convex NorE outcomes, for Pareto-efficient profiles, the notion of maximal convex NorE is weakly stricter: if a Pareto-efficient profile is a maximal convex NorE outcome, then it is necessarily envy-free and by Proposition 1, also a maximal NorE outcome. Furthermore, in some examples (such as C and D), the set of maximal NorE outcomes is generally strictly larger. However, this relationship need not hold among the Pareto-inefficient profiles: there may be maximal convex NorE outcomes that are not maximal NorE outcomes (see examples E and F).

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