An étude in modeling the definability of equilibrium

Ariel Rubinstein

School of Economics, Tel Aviv University
and Department of Economics, New York University

Kemal Yildiz

Department of Economics, Bilkent University

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ABSTRACT: We model equilibrium in a society with an equal number of agents and objects, together with a set of orderings over the set of agents. These orderings provide potential criteria for determining the suitability of agents to be assigned to an object. A *definable equilibrium* is an assignment of the agents to the objects and an attachment of a single criterion to each object, such that there is no agent who envies another and is more suited than him according to the criterion attached to his assigned object. We analyze the notion of definable equilibrium and its refinements.

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AEA classification: D0, C0.

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1. Introduction

An étude is a -usually short- instrumental musical composition, of considerable difficulty, which is designed to provide practice material for perfecting a musical skill (Wikipedia (2020)). This paper can be viewed as being analogous to an étude: It is a short modeling composition of considerable difficulty that is designed to provide practice material for economic theorists.

Our view is that solution concepts in economic theory (whether they refer to markets, games or decision scenarios) should be expressed in the "language" of the participants. This is a view expressed by Rubinstein (1978), Rubinstein (1998) and Rubinstein (2000) particularly for agents' preferences.

Our inquiry is built on the object assignment model. Consider a society consisting of an equal number of agents and objects. Each agent has preferences over the objects and there are no externalities. Each agent is assigned to only one object. The novel feature of the model is the inclusion of a language which in our case consists of a set of orderings over the set of agents. The orderings can be thought of as potential criteria for deciding whether an agent is suited to be assigned to an object. The main idea of the paper is that in equilibrium the assignment of an agent to an object should be "justifiable" by a statement that is expressible using the language. Specifically, a statement that justifies the assignment of an agent i to an object, where the set of candidates to be assigned to the object is denoted as I, should be of the following form: Agent i is the best-suited agent in I according to the ordering $\geq_\lambda$ (where $\geq_\lambda$ is one of the language's orderings).

Our approach is not descriptive nor do we attempt to solve any practical economic problem. Nonetheless, the basic idea of applying different criteria to allocate different types of objects can be observed in the real world. For example, some school seats are assigned to students according to their proximity to the school while others are assigned according to students' academic abilities. In public housing projects, some apartments are assigned according to socio-economic status while others are assigned according to willingness to pay.

Our main solution concept is *definable equilibrium* (D-equilibrium). A candidate for D-equilibrium is an assignment of agents to objects and an attachment of a single criterion to each object. In D-equilibrium, each agent is the best-suited within the group of agents that includes himself and every agent who envies him, according to the criterion attached to his assigned object. In other words, there is no agent who envies another's assignment and is better-suited than him according to the criterion attached to the assigned object. Note that in principle any ordering can be attached to any object.

The concept of D-equilibrium has several interpretations. For some of them, we have in mind a "decentralized economy" in which a behind-the-scenes process attaches a criterion to each object. In a D-equilibrium, the assignment of an agent to a particular object is evaluated by the criterion attached to the object in the following sense: An agent is better-suited to the assigned object than any other agent who prefers the object to the one he is assigned to. For other interpretations, we have in mind a central planner who justifies an assignment by declaring – possibly cynically – that the assigned agent is better-suited than any other agent who demands the object (i.e. he prefers the object to the one he is assigned to).

Of special interest is the class of *dichotomous languages* in which each criterion partitions the agents into those who satisfy a certain property and those who do not. Given such a language, an agent i can be singled out from a group of agents I by a statement of the form: Agent i is the only agent in I who satisfies a certain property. In this case, a D-equilibrium is an assignment of the agents to the objects and an attachment of a property to each object, such that the agent assigned to the object has the attached property while any other agent who demands the object does not.

In the first part of the paper, we define the notion of D-equilibrium, discuss its interpretations, present some examples and prove several simple propositions regarding its existence and efficiency. In the second part, we present and analyse a refinement of D-equilibrium that will be referred to as C-equilibrium, in which harmony is achieved by incorporating an additional priority ordering over the set of agents. This priority ordering can be thought of as the power relation adopted by the society or the preference relation of a central planner over the agents. If there is more than one agent who can justify his assignment to an object, then the one assigned to the object must have the highest priority.

The paper is primarily intended to demonstrate an alternative equilibrium formulation that does not involve trade, but rather requires a justification of an assignment using statements from a given language.

2. Definable equilibrium

2.1 The model and the D-equilibrium definitions

The basic model is that of a *society* that is described by a tuple $\langle N, X, (\succsim^i)_{i\in N}, \mathscr{L}\rangle$. The set of *agents* is $N = \{1, \dots, n\}$ and the set X consists of n *objects*. Each agent i has a strict preference relation $\succsim^i$ over X. Up to this point, the model is the familiar object assignment (housing economy) model without initial endowments. The new feature is the **language** $\mathscr{L}$ which is a set of complete and transitive binary relations over the set of agents N. We write $\mathscr{L} = \{\geq_\lambda\}_{\lambda\in\Lambda}$ where Λ is the index set of $\mathscr{L}$'s members. The set $\mathscr{L}$ is the stock of criteria that can be used to justify the choice of an agent from within a **group** (a nonempty subset) of agents. The choice of agent i from within the group of agents I is **justified by** $\geq_\lambda$, if i is the unique maximizer of $\geq_\lambda$ from within I. The choice of i from within I is **justifiable** (in short, i is justifiable in I) if there is $\geq_\lambda \in \mathscr{L}$ such that the choice of i from within I is justified by $\geq_\lambda$. Note that the justification for choosing agent i from within the group I is independent of the agent's identity. Let $C_{\mathscr{L}}(I)$ denote the group of agents whose choice from within a group I is justifiable. That is, $C_{\mathscr{L}}(I) = \{i \in I \mid i \text{ is the unique } \geq_\lambda\text{-maximal agent in } I \text{ for some } \geq_\lambda \in \mathscr{L}\}$. By definition, $C_{\mathscr{L}}(\{i\}) = \{i\}$.

A candidate for a definable equilibrium is a pair $\langle (x^i)_{i\in N}, p\rangle$ where $(x^i)_{i\in N}$ is an **assignment** that maps each agent to an exclusive object, and $p : X \to \Lambda$ is a **labeling function** that attaches a criterion $\geq_{p(x)} \in \mathscr{L}$ to each object x. For the sake of brevity, we write (x^i) instead of $(x^i)_{i\in N}$. For each assignment (x^i), an agent j **envies** agent i if $x^i \succ^j x^j$. The **label** $p(x)$ is the criterion used to justify assigning an agent to the object x. In a definable equilibrium, every agent who envies an agent i must be inferior to i according to the criterion $\geq_{p(x^i)}$, which is attached to the object that i is assigned to. In other words, each agent i is "definable" by the criterion $\geq_{p(x^i)}$ from among the group consisting of himself and all agents who envy him.

Definition 1 A **definable equilibrium** (D-equilibrium) is a pair $\langle (x^i), p\rangle$ where (x^i) is an assignment and $p : X \to \Lambda$ is a labeling function such that for every $i, j \in N$, if j envies i then $i >_{p(x^i)} j$.

Example A (The jungle) Consider a society with a language $\mathscr{L}$ consisting of a single strict ordering $\geq$. Then, in the unique D-equilibrium, $\geq$ is attached to all objects and the D-equilibrium assignment is the jungle equilibrium à la Piccione and Rubinstein (2007) in which the power relation is $\geq$. This is the assignment obtained by running the serial dictatorship according to $\geq$.

Example B (Identical preferences) Assume that all agents share the same preferences $a_1 \succ a_2 \succ \cdots \succ a_n$. Let $\mathscr{L}$ be a set of strict orderings. Inductively pick a sequence of agents such that i_l is the unique maximizer of some $\geq_{\lambda_l} \in \mathscr{L}$ in $N \setminus \{i_1, \ldots, i_{l-1}\}$. Then, the assignment of a_l to i_l combined with the labeling function that attaches λ_l to a_l is a D-equilibrium. Any D-equilibrium can be obtained by following this procedure.

2.2 Dichotomous languages

A dichotomous language consists of properties (unary relations) that an agent may or may not satisfy. Formally, a **dichotomous language** consists of orderings with two indifference sets, a top set and a bottom set, where every agent in the top set is superior to every agent in the bottom set. For each $\lambda \in \Lambda$, we identify the ordering $\geq_\lambda$ with a proposition λ in the sense that agent i satisfies λ if i is in the top set of $\geq_\lambda$ and fails to satisfy λ if he is in the bottom set of $\geq_\lambda$. We treat total indifference in two ways: If all agents are in the top set of $\geq_\lambda$, then they all satisfy λ, and if all agents are in the bottom set of $\geq_\lambda$, then none of them do. We can represent a dichotomous language as a profile $(\phi^i)_{i\in N}$ where ϕ^i is a nonempty subset of propositions in Λ that are valid for agent i.

The following proposition shows that the label $p(x)$ can be thought of as the "price" required from any agent to obtain the object x, and a D-equilibrium specifies a system of prices such that each agent is either assigned to his most preferred object in his "budget set", or an object that he prefers more and is a "free good" in the sense that no other agent prefers it to the object he is assigned to.

Proposition 1 *Let $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ be a society with a dichotomous language. Then, $\langle (x^i), p\rangle$ is a D-equilibrium if and only if for every $i \in N$,*

(i) x^i is the $\succsim^i$-best object in $\{x \mid p(x) \in \phi^i\}$, or

(ii) i prefers x^i to any object in $\{x \mid p(x) \in \phi^i\}$ and no agent envies i.

Proof. First, assume that $\langle (x^i), p\rangle$ is a D-equilibrium. Then, for every $i \in N$, $x^i \succsim^i x$ for every object x with $p(x) \in \phi^i$. Otherwise, if $x^j \succ^i x^i$ and $p(x^j) \in \phi^i$, then agent i envies agent j and $i \geq_{p(x^j)} j$, contradicting that the pair is a D-equilibrium. Therefore, if $p(x^i) \in \phi^i$ then (i) holds; otherwise, no agent envies i and (ii) holds. Conversely, suppose that $\langle (x^i), p\rangle$ is not a D-equilibrium. Then, there exists $i, j \in N$ such that i envies j and $i \geq_{p(x^j)} j$. Now, if $p(x^j) \in \phi^i$, then since $x^j \succ^i x^i$, x^i is not the $\succsim^i$-best object in $\{x \mid p(x) \in \phi^i\}$ (violating (i) for agent i) and is not better than every object in $\{x \mid p(x) \in \phi^i\}$ (violating (ii) for agent i). If $p(x^j) \notin \phi^i$, then $i \geq_{p(x^j)} j$ implies $p(x^j) \notin \phi^j$. Thus, (i) is violated for agent j, and since i envies j, (ii) is violated for agent j. □

Example C (Permissible vs. forbidden) Consider a language $\mathscr{L}$ that contains the two versions of total indifference: $\geq_{all}$ and $\geq_{null}$. Suppose that $\phi^i = \{all\}$ for every agent i. Then, the notion of D-equilibrium is identical to the equilibrium notion proposed by Richter and Rubinstein (2020), where for each object x, the label $p(x) = all$ is interpreted as the object x is *permissible* and $p(x) = null$ as the object x is *forbidden*. Since the number of objects is equal to the number of agents, a D-equilibrium exists for this language only if every agent has a distinct most-preferred object. Obviously, the model allows for richer languages for which the notion of D-equilibrium is not degenerate.

Example D (Nested dichotomous languages) Consider a dichotomous language where the sets of orderings at which agents are in the top sets are *nested*, in the sense that $\phi^n \subset \phi^{n-1} \cdots \subset \phi^1$. For each preference profile, the associated society has a unique D-equilibrium assignment obtained by running the serial dictatorship according to the ascending order.

Example E (A dichotomous language with pairwise intersections) Let $N = \{1,2,3\}$ and $X = \{x, y, z\}$, with the dichotomous language $\phi^1 = \{\alpha, \beta\}$, $\phi^2 = \{\beta, \gamma\}$, and $\phi^3 = \{\alpha, \gamma\}$. For each agent i, let b^i be his most preferred object.

If the most-preferred objects are distinct, then the assignment $[b^1, b^2, b^3]$ with $p(b^1) = \alpha$, $p(b^2) = \beta$, and $p(b^3) = \gamma$ is a D-equilibrium. The labeling function $p(b^1) = \gamma$, $p(b^2) = \alpha$, and $p(b^3) = \beta$ is part of a D-equilibrium (with a Pareto dominated assignment) if and only if the agents' second bests are distinct.

If $b^1 = b^2 = x$ and $b^3 = y$, then the assignment $[x,z,y]$ with $p(x) = \alpha$, $p(y) = \alpha$, and $p(z) = \beta$ is a D-equilibrium. If $b^1 = b^2 = b^3$, then no D-equilibrium exists since for each labeling function p, there are two agents who can choose the same most-preferred object.

2.3 Interpretations of D-equilibrium

The D-equilibrium notion has several interpretations:

(i) Analogous to competitive equilibrium

In various versions of the object assignment model, the harmony in the society is obtained through endogenously determined prices. In a D-equilibrium, it is not a price but an ordering over the set of agents that is attached endogenously to each object to obtain harmony in the society. We presented this analogy more precisely for societies with dichotomous languages in Section 2.2.

Our model does not explicitly specify initial endowments but an analogy to the initial endowments is buried in the language. In a society, an agent's strength partially originates from his rankings in the language's orderings. For example, if an agent is the top-ranked agent according to all orderings, then he has the highest priority to be assigned to each object, and if another agent is the second-ranked agent according to all orderings, then he has the highest priority to be assigned to each object only if the top-ranked agent foregoes his right to be assigned to the object.

(ii) Endogenous formation of consideration sets

Given a society with a dichotomous language, consider a D-equilibrium such that $p(x^i) \in \phi^i$ for every agent i. Then, we can interpret the label $p(x)$ as a trigger that attracts the attention of every agent i for whom $p(x) \in \phi^i$. Each agent first shortlists the objects that have a label that attracts his attention, and then chooses his most preferred object in this "consideration set", that is the set of objects that attract his attention (see Masatlioglu et al. (2012)). As illustrated in Proposition 1, in a D-equilibrium, consideration sets are endogenously formed – via the labeling function – such that every agent is assigned to the object he demands.

(iii) Assignment of a power relation to each object

The ordering attached to an object via a labeling function can be thought of as a power relation used to determine who will be the "winner" whenever some agents "fight" for the object . In equilibrium, whenever an agent j envies agent i, then agent i is stronger than agent j according to the power relation attached to the object to which i is assigned. As we have seen in Example A, when the language contains a single ordering, the D-equilibrium boils down to the jungle equilibrium à la Piccione and Rubinstein (2007).

2.4 Existence and efficiency of D-equilibrium

We say that a language is *justification-friendly* if for every group of agents there is at least one agent who is justifiable in the group, i.e. there is an ordering for which that agent is the unique maximizer within the group. A dichotomous language is justification-friendly if for each group of agents there is a particular proposition satisfied by a unique agent in the group.

Definition 2 A language $\mathscr{L}$ is **justification-friendly** if for every group of agents I, there exists an agent $i \in I$ who is justifiable in I.

Notice that if every agent has the same preferences $\succsim$ and a D-equilibrium exists, then the language must be justification-friendly. To see this, let $\langle (x^i), p\rangle$ be a D-equilibrium for such a society. Let I be a group of agents and let $i \in I$ be the agent who is assigned to the $\succsim$-best object from among the objects assigned to the members of I. It must be that $i >_{p(x^i)} j$ for every $j \in I$, and therefore the language is justification-friendly. We next verify that if the language is justification-friendly, then for every preference profile there exists a D-equilibrium with a Pareto efficient assignment.

Proposition 2 *Let $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ be a society with a justification-friendly $\mathscr{L}$. Then, the society has a D-equilibrium with a Pareto efficient assignment.*

Proof. Since $\mathscr{L}$ is justification-friendly there exists an agent, without loss of generality take him to be agent 1, who is justifiable in N by $\geq_{\lambda^1}$. Continuing with $N \setminus \{1\}$ and using a similar process, we obtain a sequence $(\lambda^i)_{i\in\{1,\dots,n\}}$ such that $i >_{\lambda^i} j$ for every $j > i$. By applying the serial dictatorship in ascending order, we obtain an assignment (x^i) such that for every agent i, x^i is $\succsim^i$-best within $X \setminus \{x^1, \ldots, x^{i-1}\}$. Define $p(x^i) = \lambda^i$. To see that $\langle (x^i), p \rangle$ is a D-equilibrium, note that for every distinct $i, j \in N$, if $x^i \succ^j x^j$ then $i < j$, which implies that $i >_{\lambda^i} j$. $\square$

A special case of a justification-friendly language is a language of strict orderings $\mathscr{L}$, consisting of complete, transitive, and anti-symmetric binary relations over N. With such a language, an agent i is justifiable in a group I if and only if i is $\geq_\lambda$-maximal in I for some $\geq_\lambda \in \mathscr{L}$.

Proposition 3 *Let $\langle N, X, (\succsim^i), \mathscr{L} \rangle$ be a society with a language $\mathscr{L}$ of strict orderings. For every labeling function p, there exists a weakly Pareto efficient assignment (x^i) (that is, there is no assignment (z^i) such that $z^i \succ^i x^i$ for every $i \in N$) such that $\langle (x^i), p \rangle$ is a D-equilibrium and (x^i) Pareto dominates every other assignment (y^i), such that $\langle (y^i), p \rangle$ is a D-equilibrium.*

Proof. Consider the marriage problem $\langle N, X, (\succsim^i)_{i \in N}, (\geq_\lambda)_{\lambda \in \Lambda} \rangle$, where the two sides of the market are N and X. Each $i \in N$ has the preference relation $\succsim^i$ over X and each $x \in X$ has the preference relation $\geq_{p(x)}$ over N.

An assignment (x^i) is *stable* in this marriage problem if and only if $\langle (x^i), p \rangle$ is a D-equilibrium. This directly follows from the observation that there exists an agent j who envies i in (x^i) with $j >_{p(x^i)} i$ if and only if (j, x^i) is a *blocking pair* in the marriage problem. Gale and Shapley (1962) showed that there exists a stable assignment (x^i) in the marriage problem, which is obtained by running the *deferred acceptance algorithm* and that for every other stable assignment (y^i), we have $x^i \succsim y^i$ for all $i \in N$. Thus, $\langle (x^i), p \rangle$ is a D-equilibrium and (x^i) Pareto dominates every other assignment (y^i) such that $\langle (y^i), p \rangle$ is a D-equilibrium. Finally, by Theorem 3 of Gale and Sotomayor (1985), the assignment (x^i) is weakly Pareto efficient. $\square$

When the language consists of weak orderings:
(i) not every labeling function is part of a D-equilibrium (see Example D); but
(ii) for every labeling function p such that a D-equilibrium with p exists, we obtain a counterpart of Proposition 3.

Proposition 4 *Let $\langle N, X, (\succsim^i), \mathcal{L}\rangle$ be a society with a language $\mathcal{L}$, and let p be a labeling function such that a D-equilibrium with p exists. Then, there exists an assignment (x^i) such that $\langle (x^i), p\rangle$ is a D-equilibrium and (x^i) Pareto dominates every other assignment (y^i) such that $\langle (y^i), p\rangle$ is a D-equilibrium.*

Proof. Step 1: If both $\langle (x^i), p\rangle$ and $\langle (y^i), p\rangle$ are D-equilibria and $y^j = x^i = z$, then it cannot be that j envies i in (x^i) and i envies j in (y^i). Otherwise, both $i >_{p(z)} j$ and $j >_{p(z)} i$.

Step 2: Let $\langle (x^i), p\rangle$ and $\langle (y^i), p\rangle$ be distinct D-equilibria such that (x^i) and (y^i) are not Pareto comparable. The assignment (y^i) is obtained from (x^i) by a *disjoint collection of minimally-sized sets of agents*, denoted by $I_1, I_2, \dots I_K$, such that for each k the members of I_k form a trade cycle among themselves. That is, for every k there is a permutation σ_k of I_k of order $|I_k|$ such that $y^i = x^{\sigma_k(i)}$.

Step 3: For each trade cycle I_k, either $(x^i)_{i\in I_k}$ dominates $(y^i)_{i\in I_k}$, (i.e., for each $i \in I_k$ we have $x^i \succsim^i y^i$) or (y^i) dominates (x^i) (i.e., for each $i \in I_k$ we have $y^i \succsim^i x^i$). If not, then there must be a pair of agents $i, j \in I_k$ such that $y^j = x^i = z$, and therefore, by Step 1, it cannot be that j envies i in (x^i) and i envies j in (y^i).

Step 4: Since there exists a D-equilibrium with p, there also exists a D-equilibrium $\langle (x^i), p\rangle$ such that there is no D-equilibrium assignment with p that Pareto dominates (x^i). We show that (x^i) Pareto dominates every other assignment that is a D-equilibrium with p. By contradiction, suppose that there is another D-equilibrium $\langle (y^i), p\rangle$ such that (x^i) and (y^i) are not Pareto comparable. Then, it follows from Steps 2 and 3 that (y^i) is obtained from (x^i) by a set of disjoint minimally-sized trade cycles $I_1, I_2, \dots I_K$ such that for each I_k either $(x^i)_{i\in I_k}$ dominates $(y^i)_{i\in I_k}$ or vice versa. For every I_k, if $(x^i)_{i\in I_k}$ dominates $(y^i)_{i\in I_k}$, then define $z^i = x^i$; if $(y^i)_{i\in I_k}$ dominates $(x^i)_{i\in I_k}$, then define $z^i = y^i$. If $\{i\}$ is a (degenerate) trade cycle, i.e. $x^i = y^i$, then define $z^i = x^i$.

Since (x^i) and (y^i) are not Pareto comparable there is at least one trade cycle in which (x^i) dominates (y^i) and at least one in which (y^i) dominates (x^i). Therefore, the assignment (z^i) Pareto dominates both (x^i) and (y^i). To see that $\langle (z^i), p\rangle$ is a D-equilibrium, let $i, j \in N$. If j envies i in (z^i), since $z^i \in \{x^i, y^i\}$ and (z^i) Pareto dominates (x^i) and (y^i), then j envies i in either (x^i) or (y^i). Since $\langle (x^i), p\rangle$ and $\langle (y^i), p\rangle$ are D-equilibria, then $i >_{p(z^i)} j$. This contradicts that there is no D-equilibrium assignment with p that Pareto dominates (x^i). □

3. Refinements of the D-equilibrium

3.1 C-equilibrium

The set of D-equilibria is typically large. We propose a refinement called C-equilibrium. For a D-equilibrium to be part of a C-equilibrium, there needs to be an additional *priority ordering* over the group of agents, such that if an agent j envies agent i and j is justifiable in the group that includes all agents who envy i as well as agent i, then j must have a *lower priority* than i. The additional priority ordering can be thought as a power relation that facilitates the achievement of harmony in the society. Unlike the model of the jungle, the use of the power relation is restricted such that a stronger agent can exercise his power with respect to an object only if he can justify being assigned to the object by one of the criteria recognized as legitimate by the society.

Another interpretation of C-equilibrium is that of a "central planner" who has in mind a priority ordering of the agents. He will be satisfied only if the agent assigned to an object is justifiable in the group of agents who envy him, and the agent has a higher priority than other justifiable agents in the group. We chose to refer to this concept as C-equilibrium since the central planner interpretation fits the concept better.

A candidate C-equilibrium is a triple $\langle (x^i), p, \trianglerighteq \rangle$ where $\langle (x^i), p \rangle$ is a D-equilibrium and $\trianglerighteq$ is a **priority ordering**, which is a complete, transitive and anti-symmetric binary relation on N. The statement $i \triangleright j$ means either that agent i has a higher priority than agent j or that i is stronger than j. It is convenient to restrict the priority relation so that it is **reflective** of the language $\mathscr{L}$, in the sense that if an agent i is better-suited than agent j according to all the criteria in $\mathscr{L}$, then i must have a higher priority than j, i.e. for every $i, j \in N$, if $i \geq_k j$ for every $\geq_k \in \mathscr{L}$ with at least one strict inequality, then $i \triangleright j$. In the case of a dichotomous language, this is equivalent to requiring that for every $i, j \in N$, if $\phi^j \subset \phi^i$ then $i \triangleright j$.

Definition 3 A triple $\langle (x^i), p, \trianglerighteq \rangle$ is a **centralized equilibrium** (C-equilibrium) if $\langle (x^i), p \rangle$ is a D-equilibrium and for every agent i, either no one envies him or i is the $\trianglerighteq$-maximizer in $C_{\mathscr{L}}(D((x^i), i))$ (the set of justifiable agents in $D((x^i), i)$).

Example F (Justification by "I am who I am") Consider a society $\langle N, X, (\succsim^i)_{i\in N}, (\phi^i)_{i\in N}\rangle$ with the dichotomous language $\phi^i = \{m^i\}$ for every $i \in N$. The statement m^i stands for "my name is i". Then, each assignment (x^i) is a D-equilibrium assignment with the labeling function $p(x^i) = m^i$ for every $i \in N$. Since every agent is justifiable in every group of agents, C-equilibrium assignments are the ones obtained by running serial dictatorship according to some ordering $\trianglerighteq$. Since any Pareto efficient assignment can be obtained by running serial dictatorship according to some order, it follows that the C-equilibrium assignments are the Pareto efficient ones.

Example G *In a society with a language of strict orderings, there may exist a reflective priority ordering that is not part of any C-equilibrium*. Let $N = \{1,2,3\}$ and $X = \{a,b,c\}$. The preference profile $(\succsim^i)$, the language $\{\geq_\alpha, \geq_\beta\}$ and the priority order $\trianglerighteq$ are specified below.

$\succ^1$	$\succ^2$	$\succ^3$
a	b	a
b	a	c
c	c	b

$\geq_\alpha$	$\geq_\beta$	$\trianglerighteq$
1	2	3
3	3	1
2	1	2

Assume that $\langle (x^i), p, \trianglerighteq\rangle$ is a C-equilibrium. Agent 1 does not envy 2 , since otherwise $1 \in C_{\mathscr{L}}(D((x^i),2))$ and $1 \triangleright 2$. Then, 3 does not envy 2, since otherwise $3 \in C_{\mathscr{L}}(D((x^i),2))$ and $3 \triangleright 2$. We are left with $(x^i) = [a,b,c]$, for which $D((x^i),1) = \{1,3\} = C_{\mathscr{L}}(D((x^i),1))$ and $3 \triangleright 1$. Thus, the society does not have a C-equilibrium with the priority relation $\trianglerighteq$.

3.2 C-equilibrium with $\mathscr{L}$-concave priority orderings

In Example G, we showed that not every priority relation can be part of a C-equilibrium even if it is *reflective*. A condition on the priority relation $\trianglerighteq$ that guarantees the existence of a Pareto efficient C-equilibrium assignment is that $\trianglerighteq$ is $\mathscr{L}$ - *concave* (as defined by Richter and Rubinstein (2019)): If for every criterion in $\mathscr{L}$, there is an agent who is superior to j by that criterion and inferior to i by $\trianglerighteq$, then $i \triangleright j$.

Definition 4 A priority ordering $\trianglerighteq$ is $\mathscr{L}$**-concave** if the following condition holds:
If for every $i, j \in N$ and $\geq_\lambda \in \mathscr{L}$, there exists $i_\lambda \in N$ such that $i_\lambda \geq_\lambda j$ and $i \triangleright i_\lambda$, then $i \triangleright j$.

We will show that if a priority ordering is $\mathscr{L}$-concave, then for every preference profile, there exists a Pareto efficient C-equilibrium assignment with that priority ordering. Since each ordering in $\mathscr{L}$ is $\mathscr{L}$-concave, it follows that a C-equilibrium exists for every society.

Proposition 5 *Let $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ be a society and $\trianglerighteq$ be an $\mathscr{L}$-concave priority ordering. Then, there is a Pareto efficient C-equilibrium assignment with $\trianglerighteq$. If $\mathscr{L}$ is a language of strict orderings, then there is a unique C-equilibrium assignment with $\trianglerighteq$.*

Proof. Let (x^i) be the assignment obtained by running the serial dictatorship according to $\trianglerighteq$. Then, for each $i \in N$, we have $D((x^i), i) \subseteq \{j | i \trianglerighteq j\}$. We will show that there exists a labeling function p such that $\langle (x^i), p\rangle$ is a D-equilibrium. If $i \notin C_{\mathscr{L}}(D((x^i), i))$, then for each $\lambda \in \Lambda$, there exists $j_\lambda \in D((x^i), i) \setminus \{i\}$ such that $j_\lambda \geq_\lambda i$ and $i \triangleright j_\lambda$. Since $\trianglerighteq$ is $\mathscr{L}$-concave, we get $i \triangleright i$. Therefore, $i \in C_{\mathscr{L}}(D((x^i), i))$, which implies that there is $\lambda^i \in \Lambda$ such that i is the unique $\geq_{\lambda^i}$-maximal agent in $D((x^i), i)$. It follows that $\langle (x^i), p\rangle$ is a D-equilibrium in which $p(x^i) = \lambda^i$ for every $i \in N$ and $\langle (x^i), p, \trianglerighteq\rangle$ is a C-equilibrium. Since (x^i) is obtained by means of running a serial dictatorship, (x^i) is Pareto efficient.

To show uniqueness, suppose there exists another assignment (y^i) such that $\langle (y^i), q, \trianglerighteq\rangle$ is a C-equilibrium. Then, there exists $i, j \in N$ such that $i \triangleright j$ and i envies j in (y^i). Since $\langle (y^i), q, \trianglerighteq\rangle$ is a C-equilibrium, $i \notin C_{\mathscr{L}}(D((y^i), j))$. Therefore, for each $\lambda \in \Lambda$, there exists $j_\lambda \in C_{\mathscr{L}}(D((y^i), j))$ such that $j_\lambda >_\lambda i$. Since $\langle (y^i), q, \trianglerighteq\rangle$ is a C-equilibrium, we have $j \trianglerighteq j_\lambda$ for every $\lambda \in \Lambda$. But then, the $\mathscr{L}$-concavity of $\trianglerighteq$ implies $j \triangleright i$, a contradiction. $\square$

On the other hand, in a society with a language of strict orderings, if the priority ordering is not $\mathscr{L}$-concave, then we can find a preference profile such that this priority ordering cannot be part of any C-equilibrium with a Pareto efficient assignment.

Proposition 6 *Let $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ be a society with a language $\mathscr{L}$ of strict orderings. If the priority ordering $\trianglerighteq$ is not $\mathscr{L}$-concave, then there is a preference profile $(\succsim^i)$ such that the society $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ does not have a Pareto efficient C-equilibrium assignment with $\trianglerighteq$.*

Proof. Since $\trianglerighteq$ is not $\mathscr{L}$-concave, there exist $i, j \in N$ such that for every $\lambda \in \Lambda$, there exists $j_\lambda \in N$ such that $j_\lambda >_\lambda i$ and $j \triangleright j_\lambda$, but $i \trianglerighteq j$. Thus, for every $\lambda \in \Lambda$, we have $i \triangleright j_\lambda$ and $i \neq j_\lambda$. Let $I = \{i_\lambda\}_{\lambda \in \Lambda} \cup \{i\}$. The set $C_{\mathscr{L}}(I)$ consists of agents in I who are the maximizers of $\geq_\lambda$ for some $\lambda \in \Lambda$. Since $\geq_\lambda$ is a strict ordering for every $\lambda \in \Lambda$, for each $j \in I \setminus \{i\}$, if $j \notin C_{\mathscr{L}}(I)$ then $C_{\mathscr{L}}(I) = C_{\mathscr{L}}(I \setminus \{j\})$. Let $J = \{i, j_1, \ldots, j_m\}$ be a subset of I such that $C_{\mathscr{L}}(J) = J \setminus \{i\}$. We assume without loss of generality that $j_1 \rhd j_2 \rhd \cdots j_m$.

Let $Y = \{y_0, y_1, \ldots, y_m\}$ be a set of distinct alternatives. Define a preference profile $(\succsim^i)$ such that:

- every $j \in J$ prefers every alternative in Y to every alternative in $X \setminus Y$ and every $j \in N \setminus J$ prefers every alternative in $X \setminus Y$ to every alternative in Y, and
- the preferences of every $j \in J$ restricted to Y are as follows:

$\succ^i$	$\succ^{j_1}$	$\succ^{j_2}$	$\succ^{j_3}$	$\cdots$	$\succ^{j_m}$
y_0	y_0	y_1	y_2		y_{m-1}
y_m	y_1	y_0	y_0		y_0
$\cdot$	$\cdot$	y_2	y_3		y_m
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$

Let (x^i) be a Pareto efficient C-equilibrium assignment. Then, $x^j \in Y$ for every $j \in J$. We arrive at a contradiction by showing that there is no $j \in J$ with $x^j = y_0$. There are three cases and in each we arrive at a contradiction to (x^i) being a C-equilibrium assignment with $\trianglerighteq$:

- Suppose that $x^i = y_0$. It must be that $x^{j_k} \neq y_k$ for some k, since otherwise $D((x^i), i) = J$ and $i \notin C_{\mathscr{L}}(J)$. Let k be the minimal index for which $x^{j_k} \neq y_k$. Let $j' \in J \setminus \{j_k\}$ with $x^{j'} = y_k$. Then, $D((x^i), j') \subseteq \{j_k, j_{k+1}, \ldots, j_m\}$. Since $j_k \in C_{\mathscr{L}}(J)$, we have $j_k \in C_{\mathscr{L}}(D((x^i), j'))$. Since $j_k \rhd j$ for every $j \in D((x^i), j')$, it must be that $x^{j_k} = y_k$, a contradiction.
- Suppose that $x^{j_1} = y_0$. Since (x^i) is Pareto efficient, it must be that for every $k \in \{1, \ldots, m\}$, $x^{j_k} = y_{k-1}$ and $x^i = y_m$. Therefore, $D((x^i), j_1) = \{i, j_1\}$. Since $i \rhd j_1$ and $\trianglerighteq$ is *reflective*, there exists $\lambda \in \Lambda$ such that $i >_\lambda j_1$. Thus, $i \in C_{\mathscr{L}}(\{i, j_1\})$ and $i \rhd j_1$, contradicting that $x^{j_1} = y_0$.
- Suppose that $x^{j_k} = y_0$ for some $k \neq 1$. Then j_1 envies j_k. Since $j_1 \in C_{\mathscr{L}}(J)$, we have $j_1 \in C_{\mathscr{L}}(D((x^i), j_k))$ and $j_1 \rhd j_k$, contradicting that $x^{j_k} = y_0$. □

Note that $\mathscr{L}$-concavity requires a stronger relation between the language and a priority ordering than *reflectivity*. In a society with a language of strict orderings, by Propositions 5 and 6, this type of relation is the one needed to guarantee the existence of a Pareto efficient C-equilibrium assignment with that priority ordering.

3.3 Constrained efficient D-equilibrium

In order to better understand how C-equilibrium is a refinement of D-equilibrium, we adopt the notion of *constrained efficiency* for both of them and focus on the connection between them. A D-equilibrium $\langle (x^i), p\rangle$ is **constrained efficient** if there is no D-equilibrium $\langle (y^i), q\rangle$ such that (y^i) Pareto dominates (x^i). Similarly, a C-equilibrium $\langle (x^i), p, \trianglerighteq\rangle$ is constrained efficient if there is no C-equilibrium $\langle (y^i), q, \trianglerighteq'\rangle$ such that (y^i) Pareto dominates (x^i). We show that, in a society with a language of strict orderings, any constrained efficient D-equilibrium assignment is also a C-equilibrium assignment.

Proposition 7 *Let $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ be a society with a language of strict orderings. Then, $\langle (x^i), p\rangle$ is a constrained efficient D-equilibrium if and only if there exists a priority ordering $\trianglerighteq$ such that $\langle (x^i), p, \trianglerighteq\rangle$ is a constrained efficient C-equilibrium.*

Proof. Only if part: Let $\langle (x^i), p\rangle$ be a constrained efficient D-equilibrium. Recall that $D((x^i), i) = \{j \mid x^i \succ^j x^j\} \cup \{i\}$ for every $i \in N$. For every $i, j \in N$, define $i\ P\ j$ if $j \in D((x^i), i) \setminus \{i\}$ and j is justifiable in $D((x^i), i)$.

Step 1: The relation P is acyclic.

Suppose by contradiction and without loss of generality that $1\ P\ 2\ P \cdots\ m\ P\ 1$. For $i = 1$, we identify $i-1$ with m. By the definition of P, for every $i \in \{1, \dots, m\} = I$, $x^{i-1} \succ^i x^i$ and there exists $\alpha^i \in \Lambda$ such that i is justified in $D((x^i), i-1)$ by $\geq_{\alpha^i}$. Define (y^i) by $y^i = x^{i-1}$ for every $i \in I$ and $y^j = x^j$ for every agent $j \notin I$. Define the labeling function q such that $q(y^i) = \alpha^i$ for every $i \in I$ and $q(x^j) = p(x^j)$ for every agent $j \notin I$. The assignment (y^i) Pareto dominates (x^i). We will now show that $\langle (y^i), q\rangle$ is a D-equilibrium.

If $j \in N$ envies $i \in I$ in (y^i), then j envies $i-1$ in (x^i). Since i is justified in $D((x^i), i-1)$ by $\geq_{\alpha^i}$, we have $i >_{\alpha^i} j$. Since $q(x^i) = \alpha^i$, we have $i >_{q(x^i)} j$.

If $j \in N$ envies $i \notin I$ in (y^i), then j also envies i in (x^i). Since $\langle (x^i), p\rangle$ is a D-equilibrium, we have $i >_{p(x^i)} j$. Since $q(x^i) = p(x^i)$, we have $i >_{q(x^i)} j$.

Step 2: There exists $\trianglerighteq$ such that $\langle (x^i), p, \trianglerighteq\rangle$ is a C-equilibrium.

Let $\trianglerighteq$ be a completion of the transitive closure of P. To see that $\langle (x^i), p, \trianglerighteq \rangle$ is a C-equilibrium, suppose that there are agents who envy an agent i. Then, if an agent $j \neq i$ is justifiable in $D((x^i), i)$, then $i\ P\ j$ and therefore $i \rhd j$.

Step 3: $\langle (x^i), p, \trianglerighteq \rangle$ is a constrained efficient C-equilibrium.
Suppose that there is a C-equilibrium $\langle (y^i), q, \trianglerighteq' \rangle$ such that (y^i) Pareto dominates (x^i). Then, $\langle (y^i), q \rangle$ is a D-equilibrium, contradicting that $\langle (x^i), p \rangle$ is a constrained efficient D-equilibrium.

If part: By contradiction, suppose that $\langle (x^i), p, \trianglerighteq \rangle$ is a constrained efficient C-equilibrium, but $\langle (x^i), p \rangle$ is not a constrained efficient D-equilibrium. Let $\langle (y^i), q \rangle$ be a constrained efficient D-equilibrium such that (y^i) Pareto dominates (x^i). Then, by the only if part of this proposition, there exists a priority ordering $\trianglerighteq'$ such that $\langle (y^i), q, \trianglerighteq' \rangle$ is a C-equilibrium, contradicting that $\langle (x^i), p, \trianglerighteq \rangle$ is a constrained efficient C-equilibrium. □

4. Final comments

4.1 Relationship to Richter and Rubinstein (2015)

In our setup, a society achieves harmony by means of a mechanism that depends on $\mathscr{L}$, the set of relevant considerations, an approach related to that adopted by Richter and Rubinstein (2015). However, their "primitive orderings" in $\mathscr{L}$ are over the set of positions rather than the set of agents, as is the case here. In a D-equilibrium, an ordering is attached to each object. Their central solution concept, namely *primitive equilibrium*, is an assignment and a single primitive ordering interpreted as the prestige ranking of the objects, such that every agent's assigned object is the one he most-prefers from the set of objects that are not more prestigious than his assigned object. In their model, the set $\mathscr{L}$ is also to constrain an agent's preferences to be $\mathscr{L}$-convex while in our model, $\mathscr{L}$ is used to constrain the priority ordering in a C-equilibrium to be $\mathscr{L}$-concave.

4.2 What is internally determined and what is externally determined?

A society is defined by $\langle N, X, (\succsim^i)_{i \in N}, \mathscr{L} \rangle$ and a D-equilibrium is a pair $\langle (x^i), p \rangle$. Alternatively, we could take the labeling function p to be an additional component of the society and look for the equilibrium assignments. In the case of a language of strict orderings, there would be a D-equilibrium for any labeling function p. In the case of a language consisting of weak orderings, this would not always be the case. The same is true for C-equilibrium, in which case we can view the priority ordering as given. Both options appear to be equally attractive.

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