Holding a group together: non-game-theory vs. game-theory

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ABSTRACT: Each member of a group chooses a position along a line and has preferences regarding his chosen position. The group's survival depends on the profile of chosen positions meeting a specific condition. We analyse a solution concept (Richter and Rubinstein, 2019) based on the emergence of a permissible set of individual positions which plays a role analogous to that of a price system in competitive equilibrium. Given the permissible set, members self-maximize. The set is pressured to tighten if the chosen positions are inharmonious and to be relaxed if the restrictions are unnecessary. It is argued that this new equilibrium concept often yields more attractive outcomes than Nash equilibrium does in the corresponding game.

KEYWORDS: General equilibrium, Nash equilibrium, Political groups.
AEA classification: D0, C0.

1. Introduction

Each member of a group has to choose a position on the real line and has a single-peaked preference relation over the position he chooses. The group's harmony depends on the profile of chosen positions satisfying some exogenous condition. If the members' choice profile does not satisfy the condition, a crisis erupts.

We analyze this scenario under three different conditions. In each of the first two, the overall position is the median or the average of the chosen positions and a crisis ensues if one of the chosen positions is too far from the overall position. In the third scenario, a crisis occurs if less than a threshold number of members agree on a position that is determined to be the group's overall position. In all three variants, each member cares about the position he chooses and does not attempt to influence the overall position.

We employ our non-conventional equilibrium concept which is referred to here as *Y-equilibrium* (see Richter and Rubinstein (2019)). We have in mind that in such groups norms evolve and determine the permissable and forbidden positions. An equilibrium specifies the convex set of permissable positions and each member's chosen position. In equilibrium:

(i) each member's choice self-maximizes from among the permissible positions;
(ii) the profile satisfies the "harmony condition";
(iii) there is no larger set of permissible positions from which a profile satisfying (i) & (ii) can be assigned.

Thus, the equilibrium is required to be resistent to two forces. The first modifies the permissible set when the profile of chosen positions leads to a crisis. The second loosens restrictions on the permissible set when they are unnecessarily tight, that is, when agents' optimal choices after the loosening would still avoid a crisis.

We regard the permissible set as an expression of *social norms* which, like competitive prices, apply uniformly to all members of the society. This uniformity might be the outcome of a societal desire for fairness but primarily it is a simplicity condition. Norms must be simple and easy to understand and apply. An important aspect of simplicity is that the norms do not depend on personal characteristics.

The Y-equilibrium notion offers a decentralized way to obtain harmony in a society, without introducing an extraneous medium. There is no authority that dictates the permissible and forbidden sets, just as there is no authority that sets prices in the market.

The paper includes a comparison of our approach to the conventional game-theoretical approach. For each of the three conditions, we study the corresponding game in which each member selects a position, where his first priority is to avoid a crisis and his second priority is the position he takes. In a Nash equilibrium, either the group collapses and no single member can save it by changing his position, or the group survives and no member can improve his position without causing a crisis.

Our approach is closer to competitive equilibrium theory than to game theory. Does it allow a more interesting analysis than the standard game-theoretical approach? As the title suggests, we have a view about this question. We do not find the game-theoretic model appealing for the class of situations analyzed here. In particular, in the game-theoretic approach, each member faces a dual mandate: maximizing his position and avoiding a crisis. The latter responsibility requires that each member holds information about all of the other members' choices. In our approach, it is the common permissible set which avoids crises and each member simply maximizes his position from among the permissible. Besides the excess informational requirements of the game-theoretical approach, it will be shown that the three corresponding games have too many Nash equilibria, some with Pareto-inefficient outcomes and some make little sense.

2. The model

Each member of a group $N = \{1, \ldots, n\}$ chooses a position along the line $X = \mathbb{R}$. The group has an odd number of members. Each member i has continuous and single-peaked preferences over his position with a peak at $peak^i$. For simplicity, we assume that all peaks are distinct. Without loss of generality, we assume that the members are ordered by their peaks from left to right. Denote the left-most peak by L, the median by M and the right-most by R. A *profile of positions* is a vector in X^N and for ease of notation, we denote $x = (x^i)_{i \in N}$, $y = (y^i)_{i \in N}$ and $z = (z^i)_{i \in N}$. The group's *overall position* is $O(x)$ where O is a function of the positions chosen by the members. We study three aggregation schemes: the average – $avg(x)$, the median – $med(x)$ and the mode – $mode(x)$ (whenever it is well-defined).

Unlike many familiar models, here a member does not care about the group's overall position. While in Hotelling (1929), a member cares only about the group's position and in Downs (1957) he also cares about his chosen position, here we go a step further and analyse a model in which a member cares only about his chosen position.

The last component of the model is a set $F \subseteq X^N$, which is the set of choice profiles that avoid a crisis. We will study three sub-models, each characterized by a function O and a set F. In all three, the function O and the set F are anonymous.

2.1 The equilibrium concept

We now come to describe the Y-equilibrium concept (in Richter and Rubinstein (2019) it was called a convex equilibrium). An equilibrium consists of two ingredients. The first is a permissable set, namely the set of alternatives that all members consider choosing from. It is a reflection of the social norms that dictate the limits on the positions that are acceptable. The second is a profile of choices from the permissible set, one choice for each member.

The following three conditions define an equilibrium:
(i) Each member's position is his most preferred from the permissable set.
(ii) The profile of choices is feasible.
(iii) Expanding the permissable set is inconsistent with a feasible profile of optimal choices.

Formally:

Definition 1 *A **Y-equilibrium** is a convex set Y and a profile of choices (y^i) satisfying:*
(i) for all i, y^i is a $\succsim^i$-maximal position in Y;
(ii) $(y^i) \in F$; and
(iii) for no convex set $Z \supset Y$ is there a profile $(z^i) \in F$ such that z^i is a $\succsim^i$-maximal alternative in Z for all i.

We require that the permissible set is *convex*. There are two motivations for this requirement. First, there is a natural asymmetry between the permissable and the forbidden. For example, it is conceivable that a club would only admit people aged 30-45 but it would sound strange if the age limit is above 45 or below 30. "Forbidden" is associated with extreme conditions and "permissable" with moderate conditions. Second, convexity on the line implies that permitted behavior is defined by a lower bound and an upper bound. An interval is the simplest way to describe a subset of the line and simplicity is a merit, and perhaps necessary, for a public norm to be understood and internalized by a large group of members.

The permissible set is applied *uniformly* to all members. Although one can imagine situations in which the norms are different for different members, fair normative principles tend to be uniform. Moreover, uniformity has the merit of simplicity.

Given that the set Y is required to be convex and that all preference relations are single-peaked, each member's optimal choice is unique. Therefore, a Y-equilibrium can be specified simply by the permissible set. Thus, we often refer to a Y-equilibrium $\langle Y,(y^i)\rangle$ by its permissible set Y.

It is worthwhile comparing the Y-equilibrium concept with that of standard competitive equilibrium applied to an exchange economy:

(a) The structure of equilibrium is similar. The permissible set in our solution is analogous to a system of prices in the competitive equilibrium model in that both determine the choice sets.

(b) The permissible set is uniform as is the competitive price system. In the standard competitive markets model, all members face the same (linear) set of "net trades".

(c) The competitive markets model, unlike ours, has initial endowments, which leads to nonuniform final choice sets (even though the net trade sets are uniform).

(d) There is no analogy in competitive equilibrium to our requirement that the permissible set be maximal in the family of permissible sets that are consistent with a feasible profile of optimal choices.

We view Y-equilibrium as a decentralized concept. In competitive equilibrium, prices are evolved when there is excess supply or demand in order to bring feasibility of the agents' optimal choices (market clearing). Likewise, the permissable set is evolved by analogous equilibrating forces. If the profile of optimal choice induces members to choose an incompatible profile of positions, then there will be a pressure that tightens the limits of the permissible set. On the other hand, if the limits can be loosened and the members' updated optimal choices are feasible then there is a pressure to relax the limits.

2.2 *The corresponding game*

The following is the *strategic game* that corresponds to our model. The players are the members of the group. The set of actions for each player is the set of positions X. Each member j has a preference relation $\succsim_*^j$ on the set of choice profiles, such that $x = (x^i) \succsim_*^j y = (y^i)$ if either:

(i) $x \in F$ and $y \notin F$, or

(ii) both $x, y \in F$ or both $x, y \notin F$ and $x^j \succsim^j y^j$.

In other words, every member strictly prefers every feasible profile to every infeasible one. When comparing two feasible (or two infeasible) profiles, a member only cares about his position and ranks the profiles accordingly.

The standard Nash equilibrium is applied to this game. A *crisis equilibrium* is a Nash equilibrium that is a profile outside F. If it exists, then it must be the profile in which each member chooses his peak since otherwise a member could deviate to his peak, which he prefers, regardless of whether that results in feasibility.

Note that the notion of (a non-crisis) Nash equilibrium in this game is identical to that of social equilibrium in Debreu (1952).

3. The near-median model

In the first scenario, the overall position $O(x)$ is the median of the choices, (x^i), and feasibility requires that all chosen positions are "near the median", specifically within distance 1. Formally, $F = \{x \mid d(x^j, O(x)) \le 1 \text{ for all } j\}$. We assume that $L + 1 < M < R - 1$ (when these inequalities don't hold, then there are fewer types of Y-equilibria).

3.1 *The Y-Equilibrium*

Proposition 1 characterizes the Y-equilibria. There is a multiplicity of equilibria. The Y-equilibrium $[M-1, M+1]$ is special. It is the only one with overall position M. It is also the only one with a Pareto-efficient choice profile and the only one that involves choices both to the right and to the left of the median choice. All other equilibria are Pareto-inefficient and "unbalanced" in the sense that the majority of choices are at one of the endpoints of the permissible set.

Proposition 1

(i) The Y-equilibria of the near-median model are the following sets:

(a) $[M-1, M+1]$ *(with overall position M),*

(b) $[t-1, t]$ *with* $L+1 < t < M$*, or,* $[t, t+1]$ *with* $M < t < R-1$ *(with overall position t),*

(c) $(-\infty, L+1]$ *or* $[R-1, \infty)$ *(with overall position* $L+1$ *or* $R-1$*).*

(ii) The only Pareto-efficient equilibrium is (a).

Proof. *(i)* We first verify that the above are indeed Y-equilibria.

(a) $[M-1, M+1]$. Member 1 (with peak L) chooses $M-1$, but would rather go further right and member n (with peak R) chooses $M+1$ and would rather go further left. Thus, from any larger permissible set, the right-most and left-most choices would be more than 2 apart, and thus at least one member must be further than 1 from the median (which does not change). Therefore, there is no larger permissible set with a feasible profile of optimal choices.

(b-c) $[t-1, t]$ with $L+1 < t < M$. A majority of members choose t and member 1 chooses $t-1$. Any right extension of the set would move the median to the right and any left extension would move the choice of member 1 to the left. Thus, the profile of optimal choices from any larger permissible set is infeasible. A similar argument holds for the other bounded case, $[t, t+1]$ with $M < t < R-1$, and the unbounded permissible set cases of *(c)*.

We now show that there are no other Y-equilibria:

First, we show that in any other Y-equilibrium $\langle Y, y\rangle$ the overall position is in $[L+1, R-1]$. Observe that Y necessarily contains an alternative which is to the left of $R-1$ because it is not a subset of the equilibrium $[R-1, \infty)$. Then, because a majority of members have peaks to the left of $R-1$, a majority of members will choose a position to the left of $R-1$ and therefore $O(y) \leq R-1$. Similarly, $L+1 \leq O(y)$.

Second, we show that there are no further Y-equilibria with overall positions in $[L+1, R-1]$. Suppose that there are two different Y-equilibria $\langle Y, y)\rangle$ and $\langle Z, z\rangle$ with $O(y) = O(z)$. If $M < O(y)$, then the left endpoint of Y and Z is $O(y)$. Therefore, the sets Y, Z are nested, contradicting the smaller one being an equilibrium. Similarly, it cannot be that $O(Y) < M$. Thus, $O(Y) = M$, but then $Y, Z \subseteq [M-1, M+1]$, which is an equilibrium, and thus $Y = Z = [M-1, M+1]$, a contradiction.

(ii) The Y-equilibrium $[M-1, M+1]$ is Pareto-efficient. To see why, all interior members are at their peak and cannot be improved. Thus, any Pareto improvement must move members at the right endpoint to the right, or members at the left endpoint to the left, or both. Such a Pareto improvement does not change the median position. Since both endpoints are 1 away from M, then any new position is further than 1 away from M, and therefore the Pareto improvement is infeasible.

The Y-equilibria in (b) and (c) are Pareto-inefficient. Consider, for example, any Y-equilibrium $[t-1, t]$ with overall position $t < M$. A majority of members are at t and would like to move to the right. Thus, a Pareto-improvement can be achieved by moving a single member (or any number less than a majority) closer to his peak. □

3.2 The near-median game

We will now show that $[M-1, M+1]$ is the only equilibrium whose choice profile is a Nash equilibrium of the near-median game. We also show that $[L+1, R-1]$ is the set of Nash equilibria overall positions which is also the set of Y-equilibria overall positions. However, many of the Nash equilibria are unintuitive.

Proposition 2 *(i) The only Y-equilibrium of the near-median model in which the profile of positions is a Nash equilibrium of the near-median game is* $[M-1, M+1]$.
(ii) The set of Nash equilibria overall positions of the near-median game is $[L+1, R-1]$.

Proof. (i) The position profile chosen from the Y-equilibrium $[M-1, M+1]$ is a Nash equilibrium of the game. A player who chooses an internal point is at his first-best. A player who chooses an endpoint, say $M+1$, does not prefer to move to the left since his peak is to the right and does not prefer to move to the right since he will then be further away from M, making the profile infeasible.

On the other hand, consider the Y-equilibria $(-\infty, L+1]$ or $[t-1, t]$ with $L+1 < t < M$. A majority of members choose the right endpoint of the permissible set which is also the overall position. Each of them has a profitable deviation, a move of size less than 1 towards his ideal position, which does not change the overall position and preserves feasibility.

(ii) We now construct a Nash equilibrium with overall position $t \in [L+1, M)$. Assign each i with $peak^i \leq t$ his most preferred position in $[t-1, t]$. The number m of such members satisfies $1 \leq m < (n+1)/2$. Assign t to the $(n+1)/2 - m$ most rightist members. Assign each other i his most preferred position in $[t, t+1]$. This assignment is feasible and its median is t. No member at his peak wishes to move. Members at $t-1$ only wish to move leftward, while members at $t+1$ (if there are any) only wish to move rightward; but any such move would destroy feasibility. Each of the rightist members at t desires to move to the right, but each is pivotal and doing so would destroy feasibility.

There is no Nash equilibrium with an overall position $t < L+1 < M$. If there were, then the left-most position would not be $t-1$, since any member at this point would prefer to move right and can do so without disturbing feasibility. As a majority of members choose a position in $(-\infty, t]$ and a majority of members have peaks in $[M, R]$, there is a member who wishes to move to the right and can do so without disturbing feasibility. $\square$

Comment: Suppose that there are three members with peaks at 0, 5 and 10. The Nash equilibrium constructed in the above proof is (3½, 5, 4½) with the overall position 4½. This Nash equilibrium is unintuitive because it is not monotonic in agents' peaks.

3.3 Comparing the solution concepts

When preferences are sufficiently diverse, and in particular if $L+2 < M < R-2$, then the game has a crisis equilibrium where (as in any crisis equilibrium) all members choose their peaks. In terms of overall positions, our approach and Nash equilibrium yield the same result. For both equilibrium concepts, the set of overall positions is $[L+1, R-1]$. However, as demonstrated in Proposition 2, the Nash equilibria may be unreasonable. Some members with extreme ideals are trapped. They wish to move further out as other agents are, but they uphold the median and would cause some other agents to be too far from the new median, causing a crisis. In contrast, our Y-equilibrium concept is based on choice from the same set and therefore is always monotonic in the peaks.

While the sets of equilibrium overall positions are identical, our notion is substantially more structured: each equilibrium overall position is supported by a unique permissible set, whereas there are many Nash equilibria with the same overall position.

4. The near-average model

In the second scenario, the overall position is the average of the choices. Feasibility requires that all chosen positions be "near the average". Formally, $F = \{x \mid d(x^j, O(x)) \leq 1 \text{ for all } j\}$. We assume that the ideal positions are diverse, specifically that $R - L > 2$.

4.1 The Y-Equilibrium

We will show that there is a continuum of Y-equilibria. The left limit of an equilibrium permissible set is either $-\infty$ or a point in $[L, L']$ where $L' < R$. The right limit is either ∞ or a point in $[R', R]$ where $L < R'$. No two equilibria have the same overall position.

Proposition 3 *In the near-average model:*
(a) There is a continuous and strictly increasing function $r : [L, L'] \to [R', R]$ (with $L < L'$ and $R' < R$) such that $r(L) = R'$ and $r(L') = R$ such that the Y-equilibria are: $(-\infty, R']$, $[L', \infty)$ and for each $l \in (L, L')$ the set $[l, r(l)]$.
(b) There are no two Y-equilibria with the same overall position.
(c) Every Y-equilibrium profile of positions is Pareto-efficient.
(d) There can be a Pareto-efficient profile of positions that is not a Y-equilibrium profile.

Proof. (a) For any $s, t \in \mathbb{R}$ with $s \leq t$, let $x(s,t)$ be the profile where $x^i(s,t)$ is member i's most preferred location in $[s,t]$ and $\Phi(s,t) = \max_i d(x^i(s,t),\ avg(x(s,t)))$. The function Φ is continuous and $\Phi(s,s) = 0$ for all s. The function $\Phi(s,t)$ is strictly decreasing in s when $s \in [L,R]$ and constant when $s \notin [L,R]$; it is strictly increasing in t when $t \in [L,R]$ and constant when $t \notin [L,R]$.

Since $R - L > 2$, it follows that $\Phi(L,R) > 1$. Thus, there is a unique $R' < R$ with $\Phi(L,R') = 1$. Similarly, there is a unique L' such that $\Phi(L',R) = 1$. For every $l \in (L,L')$, we have $\phi(l,R) > 1$, and so there is a unique point $r(l)$ such that $\Phi(l, r(l)) = 1$. Then, $(-\infty, R']$, $[L', \infty)$ are Y-equilibria. Also, for every $l \in (L, L')$ the set $[l, r(l)]$ is a Y-equilibrium and there are no other equilibrium with left endpoint l. Any set $[l, r]$ where $-\infty < l < L$ will not be an equilibrium since $\Phi(l, R') = 1$ and therefore, if $R' < r$ then choices from it are infeasible and if $r \leq R'$ then $[l, r]$ is strictly included in $Y = (-\infty, R']$. Similarly, there is no equilibrium permissible set with $R < r < \infty$.

(b) Notice that $O(x(l,r(l)))$ is an increasing function of $l \in [L, L']$ since the intervals move to the right, the members' choices weakly move to the right and the members at L and R strictly move to the right. Thus, no two equilibria have the same average choice.

(c) Consider an equilibrium of the type $[l,r]$ (a similar argument applies to the two unbounded equilibria). Since $L < l$ and $r < R$, some members choose l and others choose r. Let $t = O(x(l,r))$. By definition, $t = l+1$ or $t = r(l)-1$.

Assume that $t = l+1$ (a similar argument applies to $t = r(l)-1$). Members with peaks to the left of l choose l and desire to move further left. Members with peaks between l and $r(l)$ choose their peaks. Members with peaks to the right of $r(l)$ choose $r(l)$ and desire to move further right. Suppose there is a Pareto-improvement with average t'. Members with peaks in $[l,r]$ remain at their peak. If members move only to the right, then $t' > t$, the members who choose l do not move and $d(l,t') > 1$, violating feasibility. Otherwise, some members move to the left. Let λ be the largest move leftward. Since not all members move to the left, the average decreases by less than λ and therefore $d(l-\lambda, t') > 1$, violating feasibility.

For illustration, suppose that there are three members with peaks at 1, 4 and 7. The equilibria are depicted in Figure 1. Notice that $R' = 5/2$ and $L' = 11/2$.

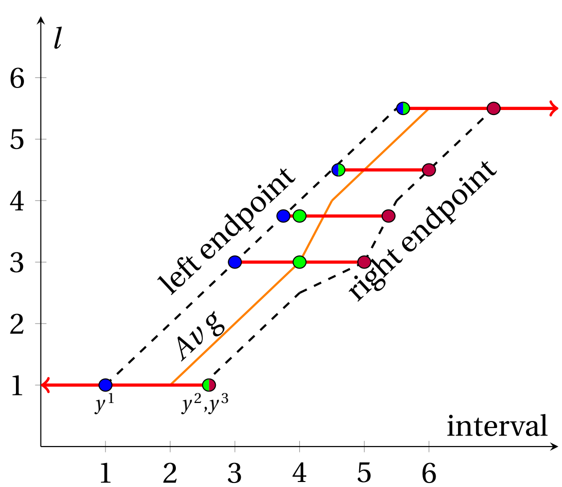

Figure 1: The two infinite equilibrium permissible sets, $(-\infty, 5/2]$ and $[11/2, \infty)$, are depicted by the red arrows. The bounded permissible sets have left endpoints l ranging from 1 to 11/2. Each bounded equilibrium is a horizontal interval between the dashed lines. Three such equilibria are depicted by the red line segments. The average of each permissible set is depicted by the orange curve. Choices are respectively depicted by blue, green and purple dots.

(d) Let $peak^1 = peak^2 = peak^3 = -1$ and $peak^4 = 2$. The profile $y^1 = y^2 = -1$ and $y^3 = y^4 = 1$ is not an equilibrium outcome since member 3 prefers y^1 to y^3. However, the profile (y^i) is Pareto-efficient. To see this, note that any Pareto-improving profile (z^i) must satisfy $z^1 = z^2 = -1$, $z^3 \leq 1$ and $z^4 \geq 1$. Notice that feasibility requires that no two choices be further than 2 from each other and therefore $z^4 = 1$. However, then $avg(z^i) = 0$ and therefore $z^3 = 1$. Thus, $(z^i) = (y^i)$, a contradiction. □

4.2 The near-average game

We will now see that every Y-equilibrium profile is also a a Nash equilibrium profile for the near-average game. However, when there are at least 5 members, there are many more Nash equilibria profiles, and in fact any position can be an overall position.

Proposition 4 *(i) Let $\langle Y, (y^i)\rangle$ be an equilibrium of the near-average model. Then (y^i) is a Nash equilibrium of the near-average game.*

(ii) When $n \geq 5$, any position, even those outside of $[L, R]$, is an overall position of some Nash equilibrium of the near-average game.

Proof. *(i)* If (y^i) is not a Nash equilibrium, then there is some j and z^j such that $(z^j, y^{-j}) \succ^j_* y$. By the definition of $\succ^j_*$, it must be that $(z^j, y^{-j}) \in F$ and (z^j, y^{-j}) is a Pareto improvement of (y^i), violating Proposition 3.

(ii) Consider a profile of choices in which one player chooses t and the other $n-1$ players split equally between $t-1$ and $t+1$. This is a Nash equilibrium of the near-average game regardless of the members' preferences, since any individual move will cause a crisis. □

4.3 Comparing the solution concepts

In contrast to the near-median model, every possible position, including those outside $[L, R]$ is an overall Nash equilibrium position. In particular, notice that Nash equilibria can be Pareto-dominated. For example, any Nash equilibrium with overall position $R+12$ is dominated by any Nash equilibrium with overall position $R+10$.

An equilibrium like the one described in *(ii)* depends on a high degree of coordination between the players. A player's optimization solely depends on knowledge of all other players' choices. Tragic coordination might cause players to be stuck at positions far from the peaks.

5. A voting model

In this section, we discuss an alternative model with the feasibility constraint being that at least τ of the n members propose the same position. The voting threshold τ may be any number from simple majority, $(n+1)/2$, to full unanimity, n. A profile of choices, x, is feasible if there is a position that at least τ members choose. For a feasible profile, the unique majority-chosen position is taken to be the overall position. Recall that the members are ordered so that $peak^1 < peak^2 < \ldots < peak^n$.

5.1 Y-equilibrium

Proposition 5 characterizes all Y-equilibria for every τ. The set of equilibrium overall positions is $[peak^{n+1-\tau}, peak^{\tau}]$. In particular, for the case of simple majority, the only Y-equilibrium overall position is M. It follows that *higher voting thresholds, rather than promoting compromise, support additional extreme outcomes*.

Regarding permissible sets, for every threshold above simple majority, each supported overall outcome is part of a unique Y-equilibrium. The case of simple majority is special: there are two Y-equilibria which support the unique overall position M.

Proposition 5 *(i) For the case of simple majority, there are two Y-equilibria with overall position M.*
(ii) For any other threshold, there is a unique Y-equilibrium with an overall position t for each $t \in [peak^{n+1-\tau}, peak^{\tau}]$,

Proof. *(i)* If a permissible set contains points both to the left and to the right of M, then no alternative attracts majority support: one agent chooses M, a minority chooses to the right of M and a minority chooses to the left of M. Therefore, given that a majority supports M from $[M,\infty)$, the set $[M,\infty)$ is a Y-equilibrium (the same applies to $(-\infty, M]$).

(ii) We say that a convex set is a d-set if it contains two points in $(peak^{n+1-\tau}, peak^{\tau})$. A key observation is that from any d-set, no alternative is chosen by at least τ members. This is because if $peak^{n+1-\tau} < s < t < peak^{\tau}$ are in a permissible set, then at least $n+1-\tau$ members prefer s to any alternative above s and at least $n+1-\tau$ members prefer t to any alternative below t.

The set $[peak^{\tau}, \infty)$ is a Y-equilibrium with the overall outcome $peak^{\tau}$: members $1, \ldots, \tau$ all choose $peak^{\tau}$ while all others choose their peaks. Any larger set $[l, \infty)$ is a d-set. A similar argument applies to $(-\infty, peak^{n+1-\tau}]$. No other Y-equilibrium Y could have the overall position $peak^{\tau}$ since Y cannot be a subset of $[peak^{\tau}, \infty)$ and thus would be a d -set. An analogous argument applies for $(-\infty, peak^{n-\tau+1}]$.

For any position $peak^{n+1-\tau} < t < peak^{\tau}$, the set $\{t\}$ is a Y-equilibrium (any larger set is a d-set) and is also the unique Y-equilibrium with overall position t.

There is no Y-equilibrium Y with an overall position larger than $peak^{\tau}$ since then either Y is a subset of $[peak^{\tau}, \infty)$ or is a d-set. An analogous argument rules out Y-equilibria with overall positions to the left of $peak^{n-\tau+1}$. $\square$

Notice that any position outside $[L, R]$ is not an Y-equilibrium overall position for any threshold. In the simple majority case, we have a "median voter theorem": M is the only Y-equilibrium overall position. While the unique overall position is M, which is often taken to be an ideal compromise outcome, the permissible set is not in the spirit of compromise since it allows choosing positions only on one side of M.

5.2 The voting game

A crisis equilibrium typically exists (except for the case $n = 3$ and $\tau = 2$). The Nash equilibrium notion allows for any overall position, even those outside of $[L, R]$. All equilibria have a specific structure: τ members choose an overall position and the rest choose their peaks.

Proposition 6 *In every Nash equilibrium of the voting game (besides the crisis equilibrium), a bare majority τ selects a position t, while all other members select their peaks.*

Proof. Clearly these are Nash equilibria. To see that there are no others, consider a Nash equilibrium in which at least τ members select a common position t. Every member who is not at t is not instrumental for feasibility, and so must select his peak. If strictly more than τ members select t, then none of them is instrumental for feasibility, and at least one of them is not at his peak and would deviate. $\square$

5.3 Comparing the solution concepts

The difference between the two solution concepts is starkest for the simple majority case. While the Y-equilibrium concept yields M as the only overall position, any position, including outside the range $[L, R]$, is a Nash equilibrium overall position.

The Nash equilibria of this game require extreme coordination between the members because exactly τ members must support the overall position while all others choose their peak. In our equilibrium, the permissible set handles the coordination just as competitive prices coordinate supply and demand.

6. Final comments

This paper is a part of a larger project exploring "price-like" though "non-price" institutions that can bring harmony to conflicting social environments (see Piccione and Rubinstein (2007), Richter and Rubinstein (2015, 2019) and Rubinstein and Wolinksy (2019)).

We focused on three models in which each member of a group chooses a position along a line and its stability depends on the combination of positions satisfying a specific exogenous constraint: "not being too far from the median", "not being too far from the average" or "having enough support for one of the positions". We characterized the Y-equilibrium concept of Richter and Rubinstein (2019) and compared it to Nash equilibrium. We arrive at four insights:

(a) In all models, there is a much larger multiplicity of Nash equilibria than Y-equilibrium choice profiles.

(b) Many of the Nash equilibria require an unreasonable degree of coordination between the members in order for the members to jointly satisfy the feasibility constraints. Our solution concept only requires members to know the social restrictions, just as in the marketplace individuals need only know prices but not other agents' actions.

(c) Unreasonable overall positions, even outside the range of the members' peaks, are supported by Nash equilibria. This is impossible in a Y-equilibrium.
(d) Even when Nash equilibria support reasonable overall positions, the profile may be unreasonable: all three games have non-monotonic Nash equilibria.

Needless to say, we are not arguing that the common game-theoretical approach is "wrong" or "valueless". But we do urge the reader to put a question mark before applying Nash-equilibrium-like concepts and to consider alternative solution concepts in the spirit of the one presented here.

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