Multi-dimensional Reasoning in Games: Framework, Equilibrium and Applications*

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Abstract

We develop a framework for analyzing multi-dimensional reasoning in strategic interactions, motivated by the following experimental findings: (a) in games with a large and complex space of strategies, players tend to think in terms of strategy characteristics rather than the strategies themselves, and (b) in choosing between strategies with a number of characteristics, players consider one characteristic at a time. The solution concept captures Nash-like stability of a choice of features of strategies rather than of strategies. The concept is applied to a number of economic interactions, where stable modes of behavior are identified.

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1. Introduction

The starting point of this paper is the recognition that in games in which the space of strategies is large and complex players deliberate over the space of characteristics of strategies rather than over the strategies themselves (Arad and Rubinstein (2012) and Harstad and Selten (2013, 2016)).

Arad and Rubinstein (2012) studied a version of the Colonel Blotto game and demonstrated the prevalence of multi-dimensional reasoning, i.e. thinking in terms of dimensions of strategies rather than strategies per se. In this game, each player (in the role of a colonel) allocates 120 troops across 6 battlefields ordered in a line. We argued that the main dimensions considered by the subjects were: (i) the number of reinforced battlefields, i.e. those with more than 20 troops, (ii) the location of the reinforced battlefields (for example, the outer vs. the inner battlefields), and (iii) the choice of the unit digit in the number of troops on each battlefield: for example, whether to allocate 0, 1, or 2 on a battlefield that the player is prepared to abandon. Arad and Penczynski (2016) investigated the Blotto game as well as multi-object auctions (all-pay and winner-pay) using an experimental protocol that allows eliciting the subjects' strategic considerations: subjects played in teams and were allowed to communicate before making a final decision. The analysis of the text messages revealed that almost all of the subjects thought in terms of properties of strategies, rather than strategies per se.

These findings suggest that in complex games, one should look for regularity in the combination of chosen characteristics of strategies rather than in the particular strategies played. To formally capture such a notion of stability requires a new modeling approach. We suggest such an approach below.

In the proposed model, each player classifies the strategies along a number of dimensions. He makes a decision in each dimension based on the profile of characteristics (not strategies) that he believes each of the other players will choose. After deciding on the desirable characteristics of his strategy, he picks a strategy that has all these characteristics.

For simplicity, we confine the analysis to symmetric two-player strategic interactions. The model, which we call an edited game, extends the standard model of a game by including a specification of an array of dimensions. Each strategy has a value in each dimension. The set of all strategies is partitioned by the values they receive in the various dimensions. A cell is a set of strategies which share values in all dimensions. A candidate for equilibrium in our proposed solution concept is a cell. A cell is unstable if changing one characteristic (i.e. the value in one dimension) while keeping the others fixed is desirable, in the sense that there is a strategy which differs in its characteristics from those in the cell only in that one dimension and which performs better against the uniform distribution over the strategies in the cell.

We seek an equilibrium concept that is stable in the sense of Nash equilibrium, that is: given that players think that other players behave according to the equilibrium mode of behavior, they don't have a reason to deviate. Like Nash equilibrium, such a solution fits situations in which players have accumulated experience in playing the game and have settled on a particular mode of behavior.

The proposed solution concept, which we call MD-equilibrium, is a cell that is not unstable in the aforementioned sense. In other words, the cell contains an optimal strategy against the cell from among all those that do not differ from the cell in more than one dimension. Note that the definition of an MD-equilibrium does not rule out the possibility that there are better strategies which differ in more than one dimension from the strategies in the cell. If there are no such strategies, then we refer to the MD-equilibrium as global.

Thus, an MD-equilibrium provides only a rough prediction: it does not specify which strategies will be chosen but rather points to a collection of strategies that share a profile of characteristics, one for each dimension.

Example: Consider the following edited *"settlements" game*. There are two players and five territories ordered on a line and labeled from left to right as 1,2,3,4,5. A player chooses either one territory or a pair of adjacent territories to settle in (that is, each player chooses one of the following nine possibilities: 1, 2, 3, 4, 5, 12, 23, 34, 45). In the case that the territories chosen by the two players coincide or overlap, there is a clash and neither of them gets to keep any territories. In the case that the choices don't coincide or overlap, each keeps the territory or territories he chose. The utility of holding one territory is 2 and the utility of holding two territories is 3.

In describing the edited game, suppose that each player has in mind two dimensions: the number of territories he settles in (i.e. one or two) and whether or not he settles in the middle territory (number 3). The following matrix presents the classification of the strategies in the edited game: each column corresponds to a value in the first dimension while each row corresponds to a value in the second.

	one	*two*
center	3	$23,34$
not center	$1,2,4,5$	$12,45$

Let us first see why $(one, not\ center)$, for instance, is not an MD-equilibrium. Any strategy in this cell obtains an expected payoff of $3/2$ when playing against the uniform distribution over the cell, whereas the strategy 3 in $(one, center)$ obtains a payoff of 2 with certainty. The only MD-equilibrium is $(two, not\ center)$ where each strategy obtains an expected payoff of $3/2$ when playing against the cell. This is not a global MD-equilibrium since the strategy 3 in $(one, center)$ would obtain a higher payoff of 2.

Note that the MD-equilibrium is different from all Nash equilibria of the game. Any mixed strategy Nash equilibrium which contains in its support the strategies 12 and 45 also assigns positive probability to the strategy 3 (assigning probability $1/7$ to the strategy 3 and $3/7$ to each of the strategies 12 and 45 is a Nash equilibrium with expected payoff of $12/7$).

It is worth emphasizing again that the MD-equilibrium concept involves the stability of characteristics of strategies rather than of the strategies themselves. When applying the model to an economic interaction, the modeler must specify the dimensions and their possible values either based on empirical evidence or by using his sense of the natural language used to organize the strategy space.

The MD-equilibrium is of course sensitive to the specification of dimensions, which is in our opinion a positive feature of the model. The way that a game is perceived by the players is an important part of the description of the strategic situation. How strategies are organized by dimensions might indeed affect behavior. The current model makes it possible to examine how different perceptions lead to different equilibria.

The applied economist who believes that players have in mind a particular set of dimensions can use the solution concept to "predict" or explain the choice of certain types of strategies that share common features, without committing himself to a prediction of specific pure or mixed strategies. We find the MD-equilibrium therefore to be an attractive equilibrium concept, especially in the case of games with a large number of strategies that do not have a pure Nash equilibrium and in which the mixed strategy equilibrium is complicated (as in the Blotto game, multi-object auctions and the two-dimensional Hotelling game analyzed in this paper).

In what follows, we present the formal model and solution concept and apply it to a number of economic interactions. At the end of the paper, we discuss an extension to asymmetric games and some existence results.

2. The model and the equilibrium concept

2.1 The Model

The basic component of the model is a symmetric two-player game (S,u) where S is each player's action set and $u(s,s')$ is a player's payoff if he chooses s when his opponent chooses s'. Assume that the set S is finite or that the function u is continuous.

The restriction of the analysis to two-player symmetric games is done for the sake of focusing on the conceptual issues. In Section 6 we provide an n-player asymmetric version of the solution concept.

We depart from the standard model by adding a description of the players' perception of the space of the strategies. We have in mind that a player thinks about the strategies in terms of K dimensions. Each strategy receives a value in each of the dimensions. A profile of values - one in each dimension - fits a set of strategies that share this combination. A player deliberates over the space of the vectors of dimensional values rather than on the space of the strategies. This leads to the following definition:

Definition: An edited symmetric game is a tuple $< S, u, (D_k)_{k=1,..,K} >$ where (S,u) is a symmetric game and each D_k is a function that assigns to every strategy s a value $D_k(s)$.

Each k stands for a "dimension". The symbol d_k denotes a generic value of the k'th dimension. The notation $d = (d_k)_{k=1,..,K}$ is used for the set of strategies s for which $D_k(s) = d_k$ for all k. We call such a set a cell. Note that each function D_k partitions the strategies by their D_k-value and the set of all cells is the join of all partitions. Denote by $u(s,C)$ the expected utility when playing the strategy s against the uniform distribution over the cell C.

We assume here that all players edit the strategy space identically. In Section 6 the model is extended to a situation in which players can have different perceptions of the strategy space.

2.2 The Solution Concept

We seek a definition of the stability of a mode of behavior that is described in terms of a cell, i.e. a vector of K values, one for each dimension. We look for a concept similar to that of Nash equilibrium but which will be defined on the space of vectors of characteristics rather than on the space of strategies. In order to do so we require a concept analogous to that of a best response in a standard game.

Definition: The value e_k is a *proper response in the k'th dimension to the cell d*, denoted by $e_k \in PR_k(d)$, if the cell (d_{-k}, e_k) contains a best response to the uniform distribution over d.

The definition of a proper response captures a key assumption of the model. A player considers one dimension at a time, while keeping the characteristics in the other $K-1$ dimensions fixed. A player views the value l_k as a proper response in the k^{th} dimension to a cell d if from among all strategies which share with d all characteristics besides that in the k^{th} dimension, a best response strategy to the uniform distribution over d has l_k as its k^{th} characteristic. We view the existence of such an optimal strategy as a justification for defining l_k as a proper response to d in the k^{th} dimension.

This definition is a particular formalization of the proper response mode of reasoning. We believe that in real life players often use heuristics in the process of proper response reasoning and do not necessarily implement the exact calculations behind the formal definition of a proper response. Thus, the definition can be viewed as a reasonable approximation of the natural reasoning process in complex settings. In some instances, a transparent consideration lies behind the proper response operator and persuades us (without calculating it explicitly) that the best response to a cell must be in a particular cell. In what follows we will emphasize that the concept of proper response often captures intuitive considerations which also guide us in characterizing the MD-equilibrium.

We now arrive at the definition of the solution concept called Multi Dimensional equilibrium:

Definition: An *MD-equilibrium* of the edited symmetric game $< S, u, (D_k)_{k=1,..,K} >$ is a non-empty cell $d^* = (d_k^*)$ such that $d_k^* \in PR_k(d^*)$ for all k.

Thus, a candidate for MD-equilibrium is a cell, i.e. a vector that specifies a value in each dimension. A player considers the dimensions one at a time and finds each value d_k^* to be a proper response to the equilibrium cell. We view the combination of values that characterize the cell as a "stable norm of behavior" analogous to the standard symmetric Nash equilibrium strategy.

An equivalent definition of the MD-equilibrium is a cell $d^* = (d_k^*)$ satisfying that there is a strategy in the cell that maximizes $u(s, d^*)$ over $\{s|\ D_k(s) = d_k^*$ for all k except for at most one$\}$ (namely, the set of strategies with values that differ from the cell values in only one dimension). In other words, a player considers using only strategies that share with the equilibrium cell all the dimensions but one. In an MD-equilibrium, a player does not find any such strategy to be better than all the others in the cell when playing against the uniform distribution over the cell.

We envision that, using the concept of proper response, a player decides over the cell only. The concept is silent about what brings a player to play a particular strategy in the equilibrium cell. In particular, it does not imply that the strategies in the equilibrium cell are chosen with equal probability. The uniformity is used only for approximating the players proper response reasoning process.

The concept of MD-equilibrium is a mix of categorical thinking and dimensional thinking, but one can think about the two issues separately. Thus, one can focus only on dimensional thinking by applying the MD-equilibrium concept to "product games" where each strategy is a vector, a player decides on each component separately but each cell is a single strategy (see Section 6). Alternatively, we might wish to focus only on categorical thinking and allow a player to reason in terms of characteristics of a strategy though he does so simultaneously for all dimensions. In that case, we can apply the following definition:

Definition: An MD-equilibrium is *global* if the equilibrium cell contains a strategy that is a best response to the uniform distribution over the cell from among all strategies in S. In other words, an MD-equilibrium is not global if some strategy which differs from it in at least two dimensions, is a better response to the uniform distribution on the MD-equilibrium cell than all strategies in the cell itself.

2.3 Comments about the model and the solution concept

(i) We view the solution concept as the specification of a stable type of strategy, a type which is expressible as a profile of characteristics. A type can be described also as the set of strategies which have all those characteristics. However, intuitively, the stability of the norm of behavior is stated in the language of the dimensions.

(ii) The proper response operator captures the justification used by a player for following the equilibrium. A player evaluates the equilibrium cell according to the maximal expected utility he can obtain if he chooses a strategy in it and the strategy is played against some arbitrary strategy in that cell. Two players' expectations may often be inconsistent with the payoffs they would achieve if both chose a best response strategy against the uniform distribution over the cell.

(iii) In the economic examples that follow we will show that an MD-equilibrium exists. In section 6 we will present an existence theorem for super-modular games where the strategies are classified along a unique dimension. However, as in the case of pure Nash equilibrium, MD-equilibrium does not always exist. We will interpret the non-existence of an MD-equilbrium in a particular edited game as an indication that the language of the edited game does not facilitate a stable mode of behavior.

Some of the ingredients of the MD-equilibrium concept are shared by other economic concepts. In particular:

(1) There are several game-theoretic models in which the solution concept is a set of strategies rather than a single strategy. In particular, see Basu and Weibull (1991) who (adjusted for the symmetric case) search for minimal sets of strategies for which all best responses to any belief on the set are inside the set (see also the discussion in Myerson and Weibull (2015), p. 950).

(2) Previous examples of categorical beliefs (i.e. players' beliefs that are framed in terms of categories rather than strategies) include numerous models with analogy-based reasoning due to Philippe Jehiel (see for example, Jehiel (2005)) as well as Piccione and Rubinstein (2003).

(3) A player in our model can be thought of as a team in the sense of Marschak and Radner (1972), such that all members share the same target and each is responsible for choosing one characteristic of the team's decision. A manager collects these choices and makes a decision that is consistent with the chosen array of characteristics.

(4) In his proper response calculation, a player assumes a uniform distribution over the strategies in the cell. This is an application of the widely used "principle of indifference", which goes back centuries to Jacob Bernoulli and Pierre Simon Laplace.

3. Colonel Blotto Game: Reasoning on the number of reinforced battlefields and their location

In this section, we apply the solution concept to a variant of the famous Colonel Blotto game which originally appeared in Borel (1921). In this variant there are two generals and each has N troops at his disposal. (For simplicity we confine ourselves to values of N that are multiplies of 6.) Each general allocates his troops among three battlefields denoted 1,2 and 3 (we refer to field 2 as the center and to fields 1 and 3 as the edges). The set of strategies for each general is $S = \{(x_1,x_2,x_3)|\ x_i$ is a non-negative integer and $\sum_{i=1,2,3} x_i = N\}$. The set contains $\frac{(N+2)!}{N!\cdot 2}$ strategies. When a player uses a strategy $x = (x_1,x_2,x_3)$ against a strategy $y = (y_1,y_2,y_3)$ he scores one point in field i if $x_i > y_i$, half a point if $x_i = y_i$ and 0 otherwise. His score is the sum of the points he scores in the three fields. A match between two strategies can yield only one of three scores: 2:1 (a win), 1.5:1.5 (a draw), 1:2 (a loss). Each general wishes to maximize his expected number of points.

The Blotto game has received widespread attention due to its interpretation in the Political Economics literature as a game between two presidential candidates who allocate their limited budgets among campaigns in the “battlefield” states (see, for example, Brams (1978)). Myerson (1993) suggested an alternative interpretation of the Blotto game as a vote-buying game. The game can be also interpreted as an R&D race between two firms who compete by allocating their limited resources among a number of projects.

As mentioned in the Introduction, experimental evidence indicates that the Blotto game triggers multi-dimensional reasoning. In particular, the vast majority of players choose their strategy after deliberating on the number of fields in which to concentrate their resources. We define the number of reinforced fields as the first dimension in the edited game.

Experimental results suggest that participants take into account the order of the battlefields. Therefore, the second dimension is defined as the ordering of the troops allocated among the three battlefields. In other words, does the player assign the troops in increasing order, in decreasing order or not according to any of the two orderings.

Formally, the edited game involves the following two dimensions:

(i) The number of fields (1,2 or 3) in which the player reinforces his troops, where field i is reinforced in x if $x_i \geq N/3$.

(ii) The order of the troop assignments: this dimension receives the value "$\nearrow$" if the assignments are in increasing order ($x_1 \leq x_2 \leq x_3$ with at least one strict inequality); "$\searrow$" if the assignments are in decreasing ($x_1 \geq x_2 \geq x_3$ with at least one strict inequality). Otherwise, it gets the value "*other*".

To illustrate, following is the classification of the 28 strategies for the case of $N = 6$ (the notation abc stands for the strategy (a,b,c)):

	$1st\ dimension$		
$2nd\ dimension$	1	2	3
$decreasing \searrow$	411, 510, 600	321, 330, 420	–
$increasing \nearrow$	006, 015, 114	024, 033, 123	–
$other$	051, 060, 105 141, 150, 501	**042, 132, 204, 213, 231** **240, 303, 312, 402**	222

The cell $(1, \searrow)$, for example, is not an MD-equilibrium since $other$ is a proper response in the second dimension to $(1, \searrow)$: the best strategy against $(1, \searrow)$ within the cell is $(4,1,1)$ (which achieves one win and two draws against the three strategies in $(1, \searrow)$ while $(0,5,1) \in (1, other)$ does better with two wins and one draw. Intuitively, abandoning the first field, reinforcing the second and placing a positive number of troops on the third guarantees scoring a point on the second field and often scoring a point on the third, which is the weakest of the $(1, \searrow)$ strategies.

The cell $(2, other)$ is a (global) MD-equilibrium. Several strategies, including $(3,0,3) \in (2, other)$, are optimal against $(2, other)$ and achieve an expected score of 1.72 (4 wins and 5 draws against the 9 strategies in $(2, other)$). This is higher than the score of any strategy outside the set, which at most achieves an expected score of 1.56 (for example, $(0,2,4)$ yields 4 wins, 2 draws and 3 losses).

Proposition 1: **In the edited Blotto game with 3 battlefields and N as a multiple of 6**, $(2, other)$ **is the only MD-equilibrium and is global**.

Proof: See Appendix.

The proof that $(2, other)$ is the only MD-equilibrium can be shown intuitively by means of the proper responses in the two dimensions.

Note that "*other*" is the union of the "peak $\cap$" strategies ($x_2 > x_1, x_3$) and the "sink $\cup$" strategies ($x_2 < x_1, x_3$).

$PR_1(2, other) = 2$: Reinforcing one field against a strategy in $(2, other)$ will win on one field if the reinforced field is matched against a non-reinforced one, and scores 1.5 points on average otherwise. The strategy of three reinforced fields against $(2, other)$ typically wins on only one of them. On the other hand, in any cell, one can find a strategy that scores at least 1.5 points against the cell.

$PR_2(2, other) = other$: Against the peak strategies in this cell, a player can find a successful sink strategy with two reinforcements that sacrifices the center, leaving relatively more troops to fight on the edges. By balancing between the edges, he wins against a non-reinforced assignment and has a good chance of winning the other edge as well. Against the sink strategies in this cell, a player can find a successful decreasing-order strategy with two reinforcements which wins with a medium-size assignment on the center field, often wins one edge with a relatively high assignment and scarifies the other edge. Sink strategies in $(2, other)$ do well against peak strategies in the cell and tie against sink strategies, whereas increasing/decreasing strategies with two reinforcements do worse against peak strategies because they waste a medium-size assignment against the highest assignment in the peak strategies.

$PR_1(1, other) \neq 1$: If a player believes that his opponent is concentrating his troops on one field, then splitting his troops among all three fields ($(3, other)$) will yield a win for certain.

$PR_1(3, other) \neq 3$: Concentrating on the two edges and sacrificing the center ($(2, other)$) will win against the strategy that splits the troops equally among the three fields.

$PR_2(2, \nearrow) \neq \nearrow$: A player can find a peak strategy with a medium size assignment to the first field and a high assignment to the second, while sacrificing the third field ($(2, other)$). This will win against most strategies of increasing order. Similarly $PR_2(2, \searrow) \neq \searrow$.

$PR_1(1, \nearrow) \neq \nearrow$ since $(1, other)$ contains a pick strategy that abandons the third field, always wins the second and has a good chance of winning the first. Similarly $PR_1(1, \searrow) \neq \searrow$.

Comparison to Nash equilibrium: The equilibrium of the Blotto game's continuous version was characterized by Roberson (2006), while that of the discrete case was characterized by Hart (2008). Both concluded that in equilibrium, players treat the battlefields symmetrically and the marginal distribution of the troops among the battlefields is essentially uniform in the interval $[0, 2N/3]$. The MD-equilibrium cell is very different from the support of the mixed strategy Nash equilibrium. In addition to the order dimension, all of the strategies in the MD-equilibrium cell have two reinforced fields while in Nash equilibrium the expected number of reinforced fields is 1.5.

Discussion: We believe that the only reasonable type of prediction in such a game is a coarse description of behavior. Proposition 2 suggests that the following coarse rule of behavior is stable: Reinforce two battlefields and do not monotonically order the three single-field assignments. The cell $(2, other)$ appears to be immune to deviations when applying intuitive coarse thinking, which is supported by fine calculations.

Interestingly, the MD-equilibrium in this game is in the spirit of the most sophisticated strategies observed in Arad and Rubinstein (2012)'s experiment of the one-shot Blotto game: the most successful strategies reinforced two-thirds of the battlefields and did not use monotonic ordering.

4. A Three-object all-pay auction: Thinking in terms of categorical (high/low) bids

The following game is inspired by Rosenthal and Szentes (2003). There are three objects up for sale and two bidders. The bidders are expected-payoff maximizers and each receives a payoff M if he wins any two of the objects. No additional benefit is obtained from winning a third object and there is no benefit from winning only one object. Each bidder is allowed to make three bids. He pays what he bids for an object regardless of whether his is the winning bid. For each object, the highest bid wins and a tie is broken randomly. In the case that no one bids on an object, each bidder receives the object with probability $1/2$.

A strategy is a triple of three non-negative integers, each between 0 and T where T is an even number. We denote a strategy of three bids by $x = (x_1, x_2, x_3)$. However, we have in mind a situation in which the objects are not naturally ordered and a player does not distinguish between one strategy and another that permutes the bids.

For simplicity, we exclude $(0,0,0)$ from the set of strategies and focus on parameters of the model (i.e. M and T) for which the equilibrium expected payoff is non-negative.

The game has two main interpretations: that of a multi-object auction conducted by a government where the bidders' interest in one object depends on the availability of another (such as in the case of oil leases and spectrum licenses); and that of an election game in which each of two candidates seeks to win a majority of districts and the votes a candidate receives in each district depend on the relative investment of the two candidates in the district. We believe that in such circumstances even sophisticated players think in terms of categories. For example, a player in the election game might decide on the number of districts in which he will spend his campaign funds and the relative amounts he will spend in those districts (high amounts in all districts, low amounts in all districts or a mix of low amounts in some districts and high amounts in others). Experimental evidence for multi-object auctions suggests that both these dimensions - the number of objects to bid on and the relative size of the bids - are frequently considered by players (Arad and Penczynski (2016)).

We are interested in constructing an edited game that can be used to determine whether there are stable descriptions of behavior in which: (i) players choose whether to bid on all the objects or only on a partial set of objects (without explicitly deciding on which objects), and (ii) players decide whether to place an high or low bid on each object (without deciding on the exact size of each bid). Thus, we specify the dimensions as follows:

(i) The number of positive bids, which can take a value of either 1,2 or 3.

(ii) The mix of high and low bids (among the positive bids). We divide the set of non-zero bids into $Low = \{1,\ldots,T/2\}$ and $High = \{T/2+1,\ldots,T\}$. It is assumed that this dimension can receive three values: "L" (all positive bids are in Low), "mix" (at least one high and one low bid) and "H" (all positive bids are in $High$).

Thus, for example, in the case of $T = 4$ the strategies $(1,4,4)$ and $(2,3,1)$ are in $(3,mix)$, the strategy $(1,2,0)$ is in $(2,L)$ and the strategy $(0,0,4)$ is in $(1,H)$.

We now characterize the MD-equilibria for a range of parameters in which: (1) the prize is large enough to justify choosing the highest possible bid on two objects for some beliefs, and (2) small enough such that it is not always beneficial for a player to increase his bid by one unit if this has a positive effect on the probability of winning. In other words, in this domain a player faces a real trade-off between decreasing the costs of the bids and increasing the probability of winning the prize.

Proposition 2: **In the edited 3-object two-bidder simultaneous all-pay auction with $2T < M < T^2/2$, the cells $(3,mix)$ and $(2,H)$ are the only MD-equilibria. The cells are non-global unless $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$** (that is, with the exception of the case in which M/T is around 3).

Proof: We focus here on the intuitive strategic considerations and direct the reader to the appendix for further details.

Notation: For a cell C and a strategy $x = (x_1,x_2,x_3)$, denote by $W(x,C)$ the probability that the strategy wins M against the uniform distribution over C. The marginal increase in the probability of winning M by adding one unit of investment to the first component of the strategy is denoted by $\Delta(x,C) = W((x_1+1,x_2,x_3),C) - W((x_1,x_2,x_3),C)$. The three components are symmetric and thus the calculation of the marginals on one component is valid for the others as well. A player's expected payoff from playing x against the uniform distribution over C is $u(x,C) = W(x,C)M - \sum_{i=1,2,3} x_i$. Let $prob_C(statement$ about a strategy $x)$ denote the proportion of strategies in C satisfying the statement.

The following Lemma provides an explicit expression for $\Delta((x_1,x_2,x_3),C)$:

Lemma 1: For a cell C and a strategy (x_1,x_2,x_3):

$\Delta((x_1,x_2,x_3),C) = prob_C(y_1 \in \{x_1,x_1+1\}$ and $(y_2 = x_2$ or $y_3 = x_3))/4+prob_C(y_1 \in \{x_1+1,x_1\}$ and either $y_2 > x_2$ and $x_3 > y_3$ or $y_2 < x_2$ and $x_3 < y_3)/2$.

Proof: See appendix.

A strategy is said to be an *edge* strategy if all of its positive components are on the edges of the categories Low and $High$, that is within the set $\{1, T/2, T/2+1, T\}$. Let $edge(C)$ be the set of edge strategies in C. Lemma 1 implies Lemma 2 which states that the maximization of the expected payoff over cell C_1 below, while playing against the uniform distribution on a cell C_2, has a solution on the edge of C_1.

Lemma 2: For any two cells C_1 and C_2, the maximization $max_{s\in C_1} W(s,C_2)$ has a solution in $edge(C_1)$.

Proof: Consider three values $t, t+1, t+2$ belonging to the same category, either Low or $High$. The events presented in the above table for $x_1 = t$ and $x_1 = t+1$ have the same probability. Thus, by Lemma 1, $\Delta(t+1,x_2,x_3),C) = \Delta((t,x_2,x_3),C)$. Therefore, the maximization $max_{s\in C_1} W(s,C_2)$ must have a solution that is an edge strategy.

The following three claims show that only $(2,H)$ and $(3,mix)$ are MD-equilibria:

Claim 1: **All the cells besides** $(2,H)$ **and** $(3,mix)$ **are not MD-equilibria**.

The following are intuitive considerations which show those cells unstable.

$PR_1(1,L) \neq 1$: The strategy $(1,1,1) \in (3,L)$ always wins against $(1,L)$ with almost no cost; in contrast a strategy in $(1,L)$ wins with probability of at most 7/12 against $(1,L)$.

$PR_1(1,H) \neq 1$ or $PR_2(1,H) \neq H$: There are two candidates for best strategy in $(1,H)$ when playing against $(1,H)$. The strategy $(T,0,0)$ wins with probability of about $7/12$ against $(1,H)$ but costs T and is inferior to $(T/2+1, T/2+1, T/2+1) \in (3,H)$ which costs only about $T/2$ more but adds $5/12$ to the chances of winning. The strategy $(T/2+1, 0, 0)$ is clearly inferior to $(1,0,0) \in (1,L)$ since using the lowest bid in H is a waste against $(1,H)$.

$PR_1(3,H) \neq 3$: With the expectation that the other player will choose some three high bids, a player believes that two maximal bids ($(2,H)$) are sufficient to almost ensure winning the two objects and saving the cost of one high bid.

$PR_1(3,L) \neq 3$ or $PR_2(3,L) \neq L$: If the best response to $(3,L)$ in $(3,L)$ involves only two assignments of $T/2$, then the third bid of 1 plays only a small role in determining the probability of winning and hence should be dropped (a strategy in $(2,L)$). If the best response to $(3,L)$ in $(3,L)$ happens to involve three assignments of $T/2$, then a strategy of the type $(1, T/2+1, T/2+1) \in (3, mix)$ guarantees winning with lower costs.

$PR_2(2,L) \neq L$: The assumption that $M > 2T$ makes $(T/2, T/2, 0)$ the best strategy within $(2,L)$ against the cell. However, in that case a strategy $(0, T/2+1, T/2+1) \in (2,H)$ involves only a small additional cost and guarantees winning.

$PR_1(2,mix) \neq 2$: any $(1, x_2, x_3) \in (3, mix)$ is superior to $(0, x_2, x_3) \in (2, mix)$ against $(2, mix)$ since it increases the probability of winning by at least $1/12$ and involves only a negligible additional cost.

Claim 2: **The cells $(2,H)$ and $(3,mix)$ are MD-equilibrium. They are non-global unless $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$.**

Proof: The formal proof for $(2,H)$ appears in the appendix. We present here only an intuitive proof for $(3,mix)$. All statements of the type "a strategy wins against $(3,mix)$ with probability α" should be read as "with probability of *approximately* α" and the symbol $\sim m$ stands for "m or $m+1$".

The strategy $(1,T,T)$ wins almost always against strategies in $(3,mix)$ and thus its expected payoff is about $M-2T$. We will show that the only edge strategy that does better than $(1,T,T)$ against $(3,mix)$ is $(0,T,T) \in (2,H)$ and hence $(3,mix)$ is a non-global MD-equilibrium.

Any edge strategy that costs $2T+T/2$ or more is inferior to $(1,T,T)$ against $(3,mix)$.

The edge strategies that cost $3T/2$ are $(\sim 0, \sim T/2, T)$ and $(\sim T/2, \sim T/2, \sim T/2)$, all of which win with probability $1/2$ against $(3,mix)$ and have an expected payoff of $M/2 - 3T/2$, which is smaller than $M-2T$.

The edge strategies that cost T are $(\sim 0, \sim T/2, \sim T/2)$, which win with probability $1/6$ (against all strategies in $High \times Low \times Low$), and $(\sim 0, \sim 0, T)$ which always lose. Thus, all these strategies yield an expected payoff of at most $M/6 - T < M-2T$.

Any edge strategy that costs $T/2$ or less always loses .

Finally, there are two types of edge strategies that cost $2T$: Any strategy of the type $(\sim T/2, \sim T/2, T)$ wins with probability $5/6$ and thus is inferior to $(1, T, T)$. The strategy $(0, T, T) \in (2, H)$ wins with probability 1 and its payoff evaluation requires a more precise calculation. It does (slightly) better than $(1, T, T)$ against $(3, mix)$ since

$\Delta((0, T, T), (3, mix)) = prob_{(3,mix)}(y_1 = 1$ and $(y_2 = T$ or $y_3 = T)]/4 = \frac{(2T-1)}{6(T/2)^3}\frac{1}{4}$

and $M\frac{2T-1}{24(T/2)^3} < 1$ for $M < T^2/2$.

Claim 3 states that $(2, H)$ is an MD-equilibrium and that it is not global. The result captures a natural process of strategic deliberation in which a player who expects his opponent to choose two high bids, believes that he will almost surely win if he makes two maximal bids. Reducing the number of bids to one will significantly undermine his chances of winning two objects. Making three high bids is wasteful since it will increase only marginally his chances of winning and involves a large additional cost. Making only two bids, with at least one of them low, dramatically reduces the chances of winning at least two objects.

Claim 3:

Proof: See Appendix.

Comparison to Nash equilibrium: The MD-equilibria $(2, H)$ and $(3, mix)$ are very different in nature from the mixed strategy Nash equilibrium characterized in Rosenthal and Szentes (2003) for the continuous case. The Nash equilibrium is a uniform distribution over all strategies that lie on the surface of a specific tetrahedron and its support includes strategies in all the partition cells except $(1, L)$ and $(1, H)$.

Discussion: We find that there are two MD-equilibria in this edited game. The first fits a norm of behavior according to which players bid high for two of the three objects. The other involves bidding on all three objects but to put in at least one low bid on one object and at least one high bids on another. The global best response to each of the MD-equilibria lies in the other equilibrium cell and the expected payoff for maximization against the equilibrium cell is almost identical, i.e. $M - 2T$. We find the $(2, H)$ MD-equilibrium particularly attractive since it is supported by more intuitive considerations, whereby eliminating one of the high bids or replacing one of the high bids with a low bid reduces total cost but does not justify the significantly diminished probability of winning.

5. The tennis coach game with costly players: The role of a designated location

The basic game in this example is an extension of Arad (2012)'s tennis coach problem. Two tennis teams, each managed by a coach, compete on three courts denoted 1,2 and 3. Each coach recruits his set of players and assigns a single tennis player to each court. Each tennis player has one of the skill levels $0, 1, 2, \dots, T$. Thus, a coach's strategy is a triple (x_1, x_2, x_3) where $x_j \in \{0, 1, .., T\}$. When a team $x = (x_1, x_2, x_3)$ plays against a team $y = (y_1, y_2, y_3)$ three matches are played: in court i the tennis player with skill level x_i confronts the player with skill level y_i. The team scores a point in each court i where its player is more skilled than his opponent $(x_i > y_i)$; it scores half a point if the two players are equally skilled $(x_i = y_i)$ and none if $x_i < y_i$. A tennis player of skill level x_i costs the coach cx_i (where $c > 0$). Each coach faces a trade-off between performance and the cost of the players. The trade-off is inserted into the utility function as follows: $u(x, y) = |\{i|x_i > y_i\}| + |\{i|x_i = y_i\}|/2 - c\sum x_i$. Coaches maximize their expected utility.

The game involves a typical contest in which the outcome depends on costly investments made by the competitors. From a public welfare perspective, the investment would be considered a waste if the contest is viewed as pointless. The investment might be considered worthwhile if the public enjoys the contest and their enjoyment increases with the level of investment.

People often view locations asymmetrically even if there is no payoff-relevant difference between them. In other words, there is often a salient location, the assignment to which is viewed differently than the assignment to other locations. The standard solution concepts (and particularly the mixed strategy Nash equilibrium) treat all locations symmetrically. Some researchers and in particular Bacharach (2006), have developed equilibrium concepts that take into account framing effects (for example, attraction to salience) in games where players make a decision involving locations (such as choosing a location or a subset of locations or deciding on the priority between locations). We construct an edited game in which players perceive one of the courts as distinct from the others. We formalize this by specifying the values in one of the dimensions to be whether the coach assigns the strongest tennis player on his team to the designated court. Evidence that some people consider whether to focus on the center or the side locations has been found in experiments of related games (see Arad (2012)). We will see that in an MD-equilibrium of the edited game, the strongest tennis player on the team will never be assigned to the designated court.

Formally, the edited game consists of two dimensions:

(i) The sum of the skill levels of the three tennis players. A value in this dimension can be any integer between 0 and $3T$.

(ii) Whether to assign the most skilled tennis player to the center court (namely, whether $x_2 = \max\{x_1, x_2, x_3\}$). This dimension can receive one of two values: "strong" (having the most skilled player on the center court) and "not strong".

To illustrate, the following matrix presents the partition of the strategy space according to the two dimensions for the case of $T = 2$:

	0	1	2	3	4	5	6
strong	000	010	020, 011, 110	120, 021, 111	022, 220, 121	122, 221	222
not strong		100, 001	002, 101, 200	012, 201, 102, 210	211, 202, 112	212	

To make the situation non-trivial, we limit the range of c and T as follows:

(i) $1/2 > c$: That is, the coach is interested in increasing his expenses by one unit if it guarantees scoring half a point more.

(ii) $Tc > 1/2$: That is, it is not beneficial for the coach to spend the maximal possible amount in order to earn half a point more.

The first and main part of proposition 3 states that a cell of the type $(Q, strong)$ is never an MD-equilibrium. The intuition is as follows: The skill level assigned in this cell to each of the side courts cannot exceed that assigned to the center court and thus cannot be more than half of the total skill level. A coach can find a strategy against the cell that is superior to any strategy in the cell by using a strategy in $(Q, not\ Strong)$ that abandons the center court and divides the total skill level as equally as possible between the two side courts. This strategy will score almost 2 points against $(Q, strong)$ while the maximal score obtained by a strategy in the cell when played against the cell is closer to 1.5 (though above).This result is independent of the parameters of the game.

Calculating the MD-equilibria among the cells of the form $(Q, not\ strong)$ depends on the values of c and T. For some parameters, no MD-equilibrium exists. The second part of proposition 3 states that for values of c around $1/3$ the cell $(3, not\ strong)$ is an MD-equilibrium for all T. In this range, $(3, not\ strong)$ is the "most efficient" MD-equilibrium. (The cell $(1, not\ strong)$ is never an

MD-equilibrium since $101 \in (2, not\ strong)$) outperform all strategies in the cell (that tie with all strategies in the cell). Similarly, the strategies in cell $(2, not\ strong)$) are outperformed by $110 \in (2, strong)$).

Proposition 3: **In the edited tennis coach game with parameters** T **and** c **satisfying** $1/2 > c$ **and** $(3T-1)c > 3/2$:

(**a**) **No cell** $(Q, Strong)$ **is an MD-equilibrium**.

(**b**) **If** $c \in (5/16, 3/8]$, **then the cell** $(3, not\ Strong)$ **is a non-global MD-equilibrium for all** T.

Proof: See Appendix.

Comparison to Nash equilibrium: Finally, we compare the MD-equilibrium to the standard mixed strategy Nash equilibrium. Note that the independence of the payoffs in each court and the additivity of the costs make it possible to calculate the Nash equilibria of the game using the equilibria of the game in each single court. Consider the case of $T = 2$. The strategies in the auxiliary game are $0, 1$ and 2 and the payoff matrix is given by:

	0	1	2
0	$1/2$	0	0
1	$1-c$	$1/2-c$	$-c$
2	$1-2c$	$1-2c$	$1/2-2c$

In the range of $1/2 \geq c \geq 1/4$, the auxiliary game has a unique mixed strategy Nash equilibrium $(1-2c, 4c-1, 1-2c)$ and the Nash equilibrium of the entire game will be any mixed strategy that induces this distribution of values in each of the three courts.

The strategies in the MD-equilibrium cell $(3, not\ Strong)$ cannot span such a mixed strategy since none of the strategies in the cell use the skill level of 2 in the center court. Furthermore, for $1/3 > c$, the Nash equilibrium strategy uses extreme values more than does the MD-equilibrium: the strength 1 appears in Nash equilibrium with less than half the probability of the appearance of a skill level within $\{0, 2\}$, whereas in any mixture of strategies from the MD-equilibrium cell $(3, not\ strong)$, the skill level 1 appears with at least half the probability of $\{0, 2\}$.

Discussion: Once again, we feel that the MD-equilibrium captures an intuitive consideration that is missing from the standard Nash equilibrium analysis. Given a decision about the total budget spent on the team, a coach naturally considers whether or not to follow the convention by assigning the strongest player to the central court. The convention of assigning the strongest player to the designated court is not stable since a coach might think that if the other coach follows the norm, he can achieve almost 2 points without increasing the total cost of his team by breaking with convention and putting his two strongest player on the edge courts. A stable mode of behavior must therefore include a norm not to put the strongest tennis player on the designated court.

6. Variations and Extensions

We conclude with three extensions of the analysis: (a) analyzing the edited game when a strategy is a vector of K numbers and the dimensions are the K components of the vector; (b) applying the concept of MD-equilibrium to asymmetric games; and (c) proving the existence result for edited asymmetric two-player games with supermodular payoff functions.

a. Product edited games

The concept of MD-equilibrium includes two ingredients that differentiate it from standard Nash equilibrium:

(i) An MD-equilibrium is a set of strategies which share a specific value in each dimension.

(ii) In looking for a best response strategy to a cell, a player considers only strategies that differ from the cell in at most one dimension.

The second ingredient can be discussed independently of the first for the family of edited games that we call *product edited games*. A product edited game is a tuple $< S, u, (D_k)_{k=1,..,K} >$ for which the set of strategies S is a product set $S = \times_{k=1,..,K} S_k$, u is a payoff function and $D_k(s) = s_k$. For product edited games, each cell is a singleton. A symmetric MD-equilibrium for product games is a strategy such that any deviation of a player in only one dimension (component) is not profitable. Note that if the payoff function is concave and differentiable, then the lack of profitable deviations in any dimension implies that other types of possible deviations are not profitable either. However, this is not the case for games in which the payoff function is not differentiable.

Note both the similarities and differences between MD-equilibrium in product edited games and the concept of D-Nash equilibrium of Guney and Richter (2016). According to the latter, players bear a cost for any switch of strategies and that cost depends on both the strategy being abandoned by the player and the new strategy he is adopting. In an extreme version of their concept, players consider deviating to a limited subset of strategies, as in the case of MD-equilibrium in product edited games.

The following proposition states a simple condition that guarantees the existence of MD-equilibrium in product edited games:

Proposition 4: **If** $< S, u, (P_k)_{k=1,..,K} >$ **is a product edited game where** S_k **is a closed interval of real numbers and** $u(s_1,..,s_K)$ **is a continuous function that is concave in each of it components**, **then it has an MD-equilibrium**.

Proof: Consider the correspondence $T : S \to\to S$ defined by

$T(s) = \times_{k=1,..,K} \{x_k \in S_k \mid x_k = \arg\max_y u((s_1, s_2, \ldots s_{k-1}, y, s_{k+1}, .., s_K), s\}$.

All sets in the range of the correspondence are products of closed intervals. By a standard fixed point argument, the correspondence has a fixed point that is an MD-equilibrium. ■

To illustrate a product edited game, consider the following version of a two-dimensional Hotelling model: Two political candidates are competing for votes. The set of policies S is the unit square and each policy represents a stand on two public issues. Each voter has an ideal point $h \in S$ and holds a strictly convex preference relation represented by a continuous function $u(s,h)$. The ideal points are distributed according to F. Each candidate selects a point in S. Voters maximize their utility, while candidates wish to maximize the number of votes they receive. A candidate considers each issue separately. This strategic situation can therefore be analyzed as a product edited game in which the two dimensions are the two issues.

This example is not covered by Proposition 4 since the candidates' induced payoff functions are not continuous. However, the following argument, (due to Roemer (2001, ch 6)) proves that an MD-equilibrium exists. Consider a strategy (s_1, s_2). Each voter has a preferred point in $\{(s_1, y) | \ y \in [0,1]\}$. Let $m(s_1)$ be the median of those points and define $m(s_2)$ similarly. The function $M(s_1, s_2) = (m(s_2), m(s_1))$ has a fixed point which is an MD-equilibrium by our definition. In the special case where the marginal distribution of preferred positions on one issue is independent of the preferred position on the other, the MD-equilibrium is the pair of medians of the two marginal distributions. Roemer (2001) also shows that the fixed point of the function M is generically not a Nash equilibrium of the Hotelling game. This result can be viewed as a proof that an MD-equilibrium of this product game is generically not global.

b. Asymmetric Edited Games

The MD-equilibrium introduced in Section 2 for symmetric games can easily be extended to asymmetry between the players, whether due to differences in their sets of strategies or the payoff functions in the basic game or differences in the players' perceptions of the dimensions that partition the strategy space in the edited game.

An asymmetric edited game is a tuple $< S^i, u^i, (D^i_k)_{k=1,..,K_i} >_{i \in N}$ where N is the set of players, S^i is player i's set of strategies, u^i is i's payoff function and $(D^i_k)_{k=1,..,K_i}$ is the collection of player i's dimensional functions. A cell for player i is a set of all strategies $s \in S^i$ which share the K_i values $(D^i_k(s))_{1 \leq k \leq K_i}$, that is, each cell is characterized by the choice of a value in each of the K_i dimensions considered by i. An MD-equilibrium of the edited game is a profile of cells $(d^*(i))_{i \in N}$ such that for each i, a best response from among $\{s \in S^i \mid D^i_k(s) = d^*_k(i)$ for all k besides at most one dimension$\}$ to the uniform distribution over $\times_{j \neq i} (d^*(j))$ is in the cell $d^*(i)$.

As an example, consider a Blotto game with three battlefields and two players B_6 and B_4. Player B_6 has 6 troops and B_4 has 4. Players have in mind two dimensions:

i) The number of reinforced battlefields, which are battlefields where player B_T deploys at least $\lfloor T/3 \rfloor$ (that is, at least 2 troops for B_6 and at least 1 for B_4).

ii) The "order" dimension: whether or not the order of the player's three troop assignments is monotonic.

Each of the following matrices represents one player's partition of his strategy space:

B_6	1	2	3
monotonic	411, 510, 600	321, 330, 420	–
	006, 015, 114	024, 033, 123	–
no order	051, 060, 105	**042**, **132***, **204**, **213**, **231**	222
	141, 150, 501	**240**, **303**, **312**, **402**	

B_4	1	2	3
monotonic	400	$\mathbf{310}^*, \mathbf{220}$	211
	004	$\mathbf{022}, \mathbf{013}$	112
no order	040	$202, 031, 103, 130, 301$	121

One of the game's MD-equilibria is $((3, \textit{no order}), (1, \textit{monotonic})) = (\{222\}, \{400, 004\})$, which is not global because 411 is a better response to $\{400,\ 004\}$ for B_6.

The only other MD-equilibrium (marked in bold) is $((2, \textit{no order}), (2, \textit{monotonic}))$, which is global. The strategy 132 is B_6's best response to $(2, \textit{monotonic})$ with a payoff of 2.128. The structure of $(2, \textit{monotonic})$ for B_4 allows B_6 to win all battles in the center by deploying 3 troops there, which leaves him enough troops to win half of the other battles on the edges. The strategy 310 is B_4's best response to $(2, \textit{no order})$ with a payoff of 1.2778.

Both MD-equilibria are consistent with the intuition that in a stable situation the weaker player focuses on a smaller number of fields than the stronger one in order to have a chance of winning at least in those fields despite his overall inferiority. Thus, in MD-equilibrium the weaker player uses an assignment of at least 2 troops in either one or two fields, whereas the stronger player assigns at least 2 troops to 2 fields according to all strategies in his MD-equilibrium cell.

The above feature of the MD-equilibrium concept is shared with the game's field-symmetric mixed strategy Nash equilibria. Consider the following auxiliary game where B_6's strategies are $[600], [511], [411],\ [420],\ [330],\ [321],\ [222]$ while those of B_4 are $[400],\ [310],\ [220],\ [211]$, where the meaning of $[abc]$ is that all its permutation will be played with equal probability.

	[400]	[310]	[220]	[211]
[600]	5/3	4/3	4/3	1
[510]	2	10/6	1.5	4/3
[411]	13/6	2	10/6	10/6
[420]	11/6	11/6	11/6	11/6
[330]	10/6	11/6	13/6	2
[321]	2	2	2	13/6
[222]	2	2	2	15/6

The value of the auxiliary game is 2 and in all MD-equilibria the stronger player uses either $[321]$ or $[222]$ and the weaker player never uses $[211]$. Thus, in Nash equilibrium, the stronger player spreads his forces over the three fields (and reinforces two of them) while the weaker one places his troops on either one or two fields.

c. Existence of MD-equilibrium in supermodular games

The existence of MD-equilibrium is obviously not guaranteed and our focus in this paper has not been on finding conditions for its existence. However, we conclude the paper with a simple example of an existence claim (suggested by Michael Richter) which relates to edited asymmetric two-player games with supermodular payoff functions, in which each player has a one-dimensional space of strategies partitioned into a finite set of intervals.

Proposition 5: **Let** $< S^i, u^i, D^i >_{i \in \{1,2\}}$ **be an edited game where** $S^i = [m^i, M^i] \subset \Re$, **and** $u^i(s^i, s^j)$ **is continuous and supermodular (in the sense of Topkins (1979)). Assume that for each** i **there is a sequence of points in** S^i, $m^i = a_0^i < a_1^i < \ldots < a_{L_i}^i = M^i$ **such that** $D^i(s) = l$ **if** $s \in P_l^i = [a_{l-1}^i, a_l^i]$. **Then, an asymmetric MD-equilibrium exists.**

Comment: Note that D^i is actually a correspondence in this case since a border point between two intervals receives two values. This is a straightforward extension of our model and solution concept.

Proof: By continuity, the best response correspondences BR^1 and BR^2 are well-defined. Define the proper response correspondence PR^1 (and similarly PR^2) as $l_1 \in PR^1(l_2)$ if there is $s \in P_{l_1}^1$ which is a best response to the uniform distribution over $P_{l_2}^2$. Define the maximal proper response function by $MPR^1(l_2) = l_1$ where l_1 is the highest index in $PR^1(l_2)$. We will see that MPR^1 (and similarly MPR^2) is a non-decreasing function. Suppose it is not. Then, there are pairs $l_1 > l_1'$ and $l_2 < l_2'$ such that $MPR^1(l_2) = l_1$ and $MPR^1(l_2') = l_2'$. Therefore, there are $a \in P_{l_1}^1$ and $b \in P_{l_1'}^1$ such that $u^1(a, Unif(P_{l_2}^2)) \;\; > u^1(b, Unif(P_{l_2}^2))$ and $u^1(b, Unif(P_{l_2'}^2)) \;\; > u^1(a, Unif(P_{l_2'}^2))$, violating the supermodularity of u^1. Thus, the maximal proper response function is non-decreasing in the values attached to the strategies. A well-known result (which is a special case of Tarski's fixed point theorem) guarantees the existence of (l_1, l_2) such that $PR^1(l_2) = l_1$ and $PR^2(l_1) = l_2$ and the cell $P_{l_1}^1 \times P_{l_2}^2$ is an MD-equilbrium. ■

7. Discussion

The paper introduces a concept of stability, in the spirit of Nash equilibrium, for strategic situations in which a player's strategy space is large and complex and players reason in terms of characteristics of strategies rather than the strategies themselves. A symmetric MD-equilibrium is a stable mode of behavior in the sense that a player who considers deviating from it by altering his choice in one of the dimensions (while keeping the others fixed) will not find a justification for doing so.

There are two natural directions for continuing our discussion. First, we adopted in the paper a particular formal definition of proper responsiveness. Replacing our definition with another definition would yield a different equilibrium concept. Second, rather than following a Nash equilibrium approach, one can use the framework to formulate non-equilibrium approaches like k-level reasoning.

The MD-equilibrium provides a prediction of a profile of strategies' characteristics rather than the traditional prediction of a distribution of strategies. We find this approach to be particularly realistic. When asked about their strategy in a complicated strategic interaction, people usually do not describe a specific pure or mixed strategy. Rather, they tend to make some qualitative statement that sums up their strategy, such as: "I always bid high", "I concentrate my attention on only two fields of study", "I cooperate with only a few players" or "I am playing aggressively".

In order to apply the MD-equilibrium concept one has to augment the description of the situation with a description of the way in which players characterize the strategies. But the additional information needed should not deter one from using the concept – how players characterize strategies will affect their behavior and hence we need to understand and integrate this within the analysis. Once this is done, we have a new tool for explaining the stability of certain modes of behavior in strategic situations.

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Appendix

Proposition 1: **In the edited Blotto game with 3 fields and N which is a multiple of 6**, $(2, other)$ **is the only MD-equilibrium and is global**.

Proof: Denote by $p(x,y)$ the number of points that a player using the strategy x scores against the strategy y. Given a cell C, let $p(x,C) = \sum_{y \in C} p(x,y)$. Obviously, comparing two strategies x and y played against the uniform distribution over C is equivalent to comparing $p(x,C)$ to $p(y,C)$.

Claim 1: **All cells besides** $(2, other)$ **are not MD-equilibria**.

$(1,\searrow)$ (and similarly $(1,\nearrow)$): The strategy $(0, 2N/3+1, N/3-1) \in (1, other)$ wins against all strategies in $(1,\searrow)$ except for $(N/3+2, N/3-1, N/3-1)$ with which it ties. Any strategy $(x_1,x_2,x_3) \in (1,\searrow)$ ties with itself and with at least one other strategy in $(1,\searrow)$: either (x_1+1, x_2, x_3-1) (if $x_3 \geq 1$), or $(x_1+1, x_2-1, 0)$ (if $x_2 \geq 1$ and $x_3 = 0$) or $(N-1, 1, 0)$ (if $x_2 = x_3 = 0$).

$(2,\searrow)$ (and similarly $(2,\nearrow)$): The strategy $(0, 2N/3, N/3) \in (2, other)$ wins against all strategies in $(2,\searrow)$, whereas any strategy in $(2,\searrow)$ ties with itself.

$(1, other)$: The strategy $(N/3, N/3, N/3) \in (3, other)$ wins against all the strategies in $(1, other)$.

$(3, other)$: The strategy $(N/2, 0, N/2) \in (2, other)$ wins against $(N/3, N/3, N/3)$, the only strategy in $(3, other)$.

Claim 2: **The cell** $(2, other)$ **is a (global) MD-equilibrium**.

To illustrate, the following table presents the distribution of assignments in the center field and in an edge field for the case of $N = 18$ (where $(2, other)$ contains 51 strategies). The cell corresponding to assignment n and field i contains the number of strategies in $(2, other)$ for which the assignment in field i is n.

assignment	*field* 2	*category*	*field* 3	*category*
12	2	H_2	1	H_3
11	4	H_2	2	H_3
10	6	H_2	3	H_3
9	6	M_2	4	H_3
8	4	M_2	$5+2$	M_3
7	2	M_2	$6+4$	M_3
6	0	M_2	$6+6$	M_3
5	2	L_2	1	L_3
4	3	L_2	1	L_3
3	4	L_2	2	L_3
2	5	L_2	2	L_3
1	6	L_2	3	L_3
0	7	L_2	3	L_3

The pattern of the distributions is generalized in the following table (the explanations refer to six categories $H_2, M_2, L_2, H_3, M_3, L_3$ of pairs (n, i)).

	$x_2 = n \in$	# strategies in $(2, other)$	Explanation
H_2	$(N/2, 2N/3]$	$2(2N/3 - n + 1)$	x_1 or x_3 is in $[N/3, N - n]$
M_2	$[N/3, N/2]$	$2(n - N/3)$	x_1 or x_3 is in $[N/3, n - 1]$
L_2	$[0, N/3)$	$N/3 - n + 1$	edges are reinforced; $x_1 \in [N/3, 2N/3 - n]$

	$x_3 = n \in$	# strategies in $(2, other)$	Explanation
H_3	$[N/2, 2N/3]$	$2N/3 - n + 1$	$x_1 \in [N/3, N - n]$
M_3	$[N/3, N/2 - 1]$	$N - 2n + min\{2N/3 - n + 1, N/3\}$	$x_2 \in [n + 1, N - n]$ or $x_2 \in [0, N/3 - 1] \cap [0, 2N/3 - n]$
L_3	$[0, N/3)$	$\lceil N/6 - n/2 \rceil$	$N/3 \leq x_1 < x_2$ and $x_1 \in [N/3, (N - n)/2)$

To see that $(2, other)$ is a global MD-equilibrium, we need to prove that for each strategy $x \notin (2, other)$, there is a strategy $y \in (2, other)$ which does at least as well as x against $(2, other)$.

Case 1: x has only one reinforced field.

If x has a peak in the center, it will lose against half of the strategies in $(2, Other)$, which are "sinks", and thus its expected score is no more than 1.5. On the other hand, any set of strategies contains a strategy that achieves an expected score of at least 1.5 when playing against the set. Thus, there must be a strategy in $(2, other)$ that is weakly better than x. Otherwise, and without loss of generality, field 3 is reinforced. When playing against all strategies in $(2, other)$, x scores as follows: in field 3, at most $|(2, other)|$; in field 2, less than the sum of entries in category L_2; and in field 1, less than the sum of entries in category L_3. The sum of the entries in L_3 equals the sum of the entries in H_2:

$\sum_{n<N/3} \lceil N/6 - n/2 \rceil = \sum_{N/2<n\leq 2N/3} 2(2N/3 - n + 1)$. The sum of entries in M_2 is $2(\sum_{i=1,..,N/6} i)$. Therefore,

$p(x, (2, other)) \ < 2|(2, other)| - 2(\sum_{i=1,..,N/6} i)$.

We will show that $p((N/2, 0, N/2), (2, other)) \ > p(x, (2, other))$. In each edge field, $(N/2, 0, N/2)$ wins a point against any assignment except those in H_3, where it loses to any assignment in $(N/2 + 1, 2N/3]$ (there are $\sum_{i=N/2+1,..,2N/3} (2N/3 - i + 1) = \sum_{i=1,..,N/6} i$ strategies with such assignments) and ties with the assignment $N/2$ (there are $N/6 + 1$ such strategies in $(2, other)$). Furthermore, it scores half a point in the center when playing against an assignment 0 (there are $N/3 + 1$ strategies in $(2, other)$ with 0 in the center). Thus,

$p((N/2, 0, N/2), (2, other)) \ \geq 2[|(2, other)| - \sum_{i=1,..,N/6} i - (N/6 + 1)/2] + (N/3 + 1)/2 =$

$2|(2, other)| - 2(\sum_{i=1,..,N/6} i) - 1/2$.

Case 2: x has at least two reinforced fields.

We show that x is inferior to $(N/2, 0, N/2) \in (2, other)$.

Since $x \notin (2, other)$, it follows that the center must be reinforced and w.l.o.g. field 3 must also be, which implies that $x_2 \leq x_3$.

Consider a strategy x with $x_3 < N/2$ and therefore $x_1 > 0$. The strategy $(x_1 - 1, x_2, x_3 + 1)$ is superior to x against $(2, other)$ since the minimal marginal gain from adding 1 to field 3 (category M_3) is larger than the maximal marginal loss from subtracting 1 from field 1 (category L_3). Thus, x is inferior to a strategy of the type $(a, N/2 - a, N/2) \in (2, \nearrow)$.

Consider $x = (a, N/2 - a, N/2)$ where $a \leq N/6$. The strategy is not superior to $(0, N/2, N/2)$ because the sequence of marginal gains from increasing the assignment in field 1 $(N/6, N/6, N/6 - 1, N/6 - 1, N/6 - 2, N/6 - 2. \ldots)$ (category L_3) is dominated by the sequence of marginal losses from decreasing the assignment in the center $(N/3, N/3-2, N/3-4, \ldots)$ (category M_2).

Consider $x = (0, N/2, N/2)$. The strategy $(N/2, 0, N/2)$ is superior to x and its advantage is $p((N/2,0,N/2),(2,other)) - p((0,N/2,N/2),(2,other)) = \sum_{i=1,..,N/6} i + 1/2$. The number of assignments in $(2, other)$ that are greater than $N/2$ in field 2 is twice the number in field 3, which is $\sum_{i=1,..,N/6} i$. The relative advantage of $(N/2, 0, N/2)$ over $(0, N/2, N/2)$, since it ties with the 0 assignment (the former strategy ties in the center and the latter ties in the edge), is larger than the disadvantage that it ties with the $N/2$ assignment. More formally, the total points scored by $(N/2, 0, N/2)$ in fields 1 and 2 when playing against all strategies in $(2, other)$ is $[|(2,other)| - \sum_{i=1,..,N/6} i - (N/6+1)/2] + [(N/3+1)/2]$, while the analogous number of points for $(0, N/2, N/2)$ is $[(N/6)/2] + [|(2,other)| - 2\sum_{i=1,..,N/6} i - (N/3)/2]$.

Finally, consider a strategy of the type $x = (a, N/2-a-b, N/2+b)$ where $a+b \leq N/6$ ($a \geq 0$, $b > 0$). The gain in field 3 from the extra b in the edge, compared to $(a, N/2-a, N/2)$, is $[(N/6+1) + 2(N/6) + \ldots + 2(N/6-b) + (N/6-b+1)]/2$ (category H_3) while the loss in the center from decreasing the assignment to $N/2-a$ is at least b. Thus, for all b we have $[(N/6+1) + 2(N/6) + \ldots + 2(N/6-b) + (N/6-b+1)]/2 - b = (N/6+1)/2 - b - (N/6-b+1)/2 + \sum_{i=N/6-b+1}^{N/6}$ $= -b/2 + \sum_{i=N/6-b+1}^{N/6} i < \sum_{i=1,..,N/6} i$. Therefore, x is inferior to $(N/2, 0, N/2)$. ■

Proposition 2: **For the edited 3-object two-bidder simultaneous all-pay auction with** $2T < M < T^2/2$, **the cells** $(3, mix)$ **and** $(2, H)$ **are the only MD-equilibria**. The cells MD-equilibrium $(2, H)$ is not global unless $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$ (that is, with the exception of the case in which M/T is around 3)

Lemma 1: For a cell C and a strategy (x_1, x_2, x_3): $\Delta((x_1, x_2, x_3), C) =$ $prob_C(y_1 \in \{x_1, x_1+1\}$ and ($y_2 = x_2$ or $y_3 = x_3$))/4+$prob_C(y_1 \in \{x_1+1, x_1\}$ and either $y_2 > x_2$ and $x_3 > y_3$ or $y_2 < x_2$ and $x_3 < y_3)/2$.

Proof: The Lemma follows from the following table which classifies all strategies y for which moving from (x_1, x_2, x_3) to (x_1+1, x_2, x_3) changes the probability of winning M when playing against y.

The event	prob. of winning M increases				the increase
$y_1 \in \{x_1, x_1+1\}$ and...	$y_1 = x_1$		$y_1 = x_1 + 1$		
	from	to	from	to	
either $y_2 > x_2$ and $x_3 > y_3$ or $y_2 < x_2$ and $x_3 < y_3$	1/2	1	0	1/2	1/2
either $y_2 > x_2$ and $y_3 = x_3$ or $y_2 = x_2$ and $y_3 > x_3$	1/4	1/2	0	1/4	1/4
both $y_2 = x_2$ and $y_3 = x_3$	1/2	3/4	1/4	1/2	1/4
either $y_2 < x_2$ and $y_3 = x_3$ or $y_2 = x_2$ and $y_3 < x_3$	3/4	1	1/2	3/4	1/4

Lemma 2: For any two cells C_1 and C_2, the maximization $max_{s \in C_1} W(s, C_2)$ has a solution in $edge(C_1)$.

Proof: Consider three values $t, t+1, t+2$ belonging to the same category, either Low or $High$. The events presented in the above table for $x_1 = t$ and $x_1 = t+1$ have the same probability. Thus, by Lemma 1, $\Delta(t+1, x_2, x_3), C) = \Delta((t, x_2, x_3), C)$. Therefore, the maximization $max_{s \in C_1} W(s, C_2)$ must have a solution that is an edge strategy.

Notice that given the constraints $2T < M < T^2/2$, the inequality $\Delta((x_1, x_2, x_3), C) \geq \frac{1}{2T}$ guarantees that increasing x_1 by one unit is strictly beneficial when playing against C while the inequality $\Delta((x_1, x_2, x_3), C) \leq \frac{2}{T^2}$ implies that increasing x_1 by one unit is strictly harmful.

Claim 1: **All cells besides** $(2, H)$ **and** $(3, mix)$ **are not MD-equilibria**.

Proof:

(**1**,**H**): Within $(1, H)$ the optimal strategy against the cell must be either $(T, 0, 0)$ or $(T/2+1, 0, 0)$ and they are inferior to strategies in $(2, H)$ and $(1, L)$, respectively:

$u((T/2+1, T/2+1, T/2+1), (1, H)) = M - \frac{3T}{2} - 3 > u((T, 0, 0), (1, H)) = M(\frac{7}{12} - \frac{1}{6T}) - T$ and

$u((1, 0, 0), (1, H)) = \frac{5}{12}M - 1 > u((T/2+1, 0, 0), (1, H)) = (\frac{5}{12} + \frac{1}{6T})M - \frac{T}{2} - 1.$

(**1**,**L**): The strategy $(1, 1, 1) \in (3, L)$ always wins against $(1, L)$ and its payoff of $M - 3$ is higher than that of the two edge strategies in $(1, L)$: $u((1, 0, 0), (1, L)) = (\frac{5}{12} + \frac{1}{6T})M - 1$ and $u((T/2, 0, 0), (1, L)) = (7/12)M - T/2.$

(**3**,**H**): The strategy $(0, T, T) \in (2, H)$ is superior to $(T/2+1, T/2+1, T/2+1)$ because the expected payoffs of the former are approximately $M - 2T$ and of the latter $-\frac{3T}{2}$. Furthermore,

$u((0,T,T),(3,H)) = \frac{T^3-2T^2+T}{T^3}M - 2T > u((T,T,T),(3,H)) = \frac{T^3-3T+2}{T^3}M - 3T$ since $T^2/2 > M$.

It is straightforward to verify that $\Delta((T/2+1, T/2+1, T),(3,H)) > \frac{1}{2T}$ (the strategy $(T/2+1, T/2+1, T)$ almost always loses in the second bid and wins in the third and hence the marginal expected gain is on the scale of $\frac{4}{T}\frac{1}{2}$) and thus $(T, T/2+1, T)$ is a better strategy against $(3,H)$.

The strategy $(T/2, T, T) \in (3, mix)$ is superior to $(T/2+1, T, T)$ against $(3,H)$ since $M\Delta((T/2,T,T),(3,H)) = M(\frac{T-1}{2T(T/2)^2}) < 1$.

(**3**,**L**): Parallel arguments to those in the previous case for strategies consisting of bids in the category $High$ apply to strategies consisting of bids in the category Low: $(1,1,1)$ and $(1,1,T/2)$ are not best responses to $(3,L)$ and the strategy $(0,T/2,T/2) \in (2,L)$ is superior to $(1,T/2,T/2)$. The other edge strategy $(T/2,T/2,T/2)$ is inferior to $(T/2+1, T/2+1, 1) \in (3, mix)$, which wins M with certainty and costs less.

(**2**,**L**): As in step 1 of Claim 1, $(T/2,T/2,0)$ is the best strategy within $(2,L)$ against the cell.
The strategy $(T/2+1, T/2, 0) \in (2, mix)$ does better since $\Delta((T/2,T/2,0),(2,L)) = \frac{1}{2T}$.

(**2**,**mix**): We will show that for any $l \in Low$ and $h \in High$ the strategy $(1,l,h) \in (3,mix)$ is superior to $(0,l,h)$ against $(2,mix)$. The strategy $(1,l,h)$ increases the probability of winning M by at least $1/12$ (since it wins with probability 1 against any strategy in $(2,mix)$ of the form $(0,y_2,y_3)$ where $y_2 \in High$ and $y_3 \in Low$ whereas $(0,l,h)$ wins with probability 1/2 against those strategies). Thus, its expected improvement is at least $M/12 - 1$ which is positive for $T \geq 6$. ■

Claim 2: **The cell** $(2,H)$ **is an MD-equilibrium**. **It is not global unless** $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$.

Proof:

Step 1: The strategy $(T,T,0)$ is an optimal strategy within $(2,H)$ against the cell itself and achieves an expected payoff of $(1-\frac{1}{T})M - 2T$. (Note that the payoff is positive whenever $M \geq 2T+3$).

For $(x_1,x_2,0) \in (2,H)$, we have $\Delta((x_1,x_2,0),(2,H)) =$
$prob_{(2,H)}(y_1 \in \{x_1, x_1+1\}$and $y_3 = 0)]/4 + prob_{(2,H)}(y_1 \in \{x_1+1, x_1\}$ and $y_2 = 0)/2 = \frac{1}{4}\frac{4}{T}\frac{1}{3} + \frac{1}{2}\frac{4}{T}\frac{1}{3} = \frac{1}{T}$. Since this marginal is greater than $\frac{1}{2T}$, the strategy $(T,T,0)$ is optimal within the cell against $(2,H)$.

The strategy wins against all strategies in the cell (which contains $3T^2/4$ strategies), with the exception of: (i) $T+1$ strategies of the type $(T,0,y_3)$, $(0,T,y_3)$ and $(T,T,0)$, which it wins against with probability 1/2, and (ii) $T-2$ strategies in $(2,H)$ of the type $(T,y_2 < T,0)$ and $(y_1 < T,T,0)$, which it wins against with probability $3/4$. That is, it wins M with probability $1 - \frac{(T+1)/2+(T-2)/4}{3(\frac{T}{2})^2} = (1-\frac{1}{T})$ and its expected payoff against $(2,H)$ is $(1-\frac{1}{T})M - 2T$.

Step 2: The cell $(2,H)$ is an MD-equilibrium.

We wish to verify that no strategy in $(1,H),(3,H),(2,mix)$ or $(2,L)$ achieves an expected payoff against $(2,H)$ larger than that of $(T,T,0)$.

$(\mathbf{1},\mathbf{H})$: For $(x_1,0,0) \in (1,H)$, we have $\Delta((x_1,0,0),(2,H)) = prob_{(2,H)}(y_1 \in \{x_1+1,x_1\}$ and ($y_2 = 0$ or $y_3 = 0)]/4 = \frac{1}{4}\frac{2}{3}\frac{4}{T} = \frac{2}{3T}$. Thus, the strategy $(T,0,0)$ is optimal in $(1,H)$ against $(2,H)$. The strategy wins M with probability $\frac{2}{3}\frac{1}{2}(1-\frac{1}{2}\frac{2}{T}) = \frac{T-1}{3T}$. Given the constraints, which imply also that $T \geq 6$, the strategy is inferior to $(T,T,0)$ ($\frac{T-1}{T}M - 2T \geq \frac{T-1}{3T}M - T$).

$(\mathbf{3},\mathbf{H})$: By Lemma 2, it is sufficient to show that no strategy in $edge(3,H)$ does better than $(0,T,T)$ against $(2,H)$.

Any strategy that involves a cost of at least $2T + T/2$ is inferior to $(0,T,T)$ since at most it adds $\frac{M}{T} - \frac{T}{2} < 0$ to the expected payoff.

The other edge strategies in $(3,H)$ are $(T/2+1, T/2+1, T)$, which wins only with probability of about $2/3$ but which costs the same as $(T,T,0)$, and $(T/2+1,T/2+1,T/2+1)$, which wins with probability $\frac{3(2(\frac{T}{2}-1)\frac{1}{2}+\frac{3}{4})}{3(\frac{T}{2})^2} \leq \frac{2}{T}$ and $M\frac{2}{T} - \frac{3T}{2} - 3 < (1-\frac{1}{T})M - 2T$.

$(\mathbf{2},\mathbf{L})$: No strategy in $(2,L)$ wins M when playing against any strategy in $(2,H)$ and $(1-\frac{1}{T})M - 2T \geq -2$ given $M > 2T$.

$(\mathbf{2},\mathbf{mix})$: Placing a bid in Low that is higher than 1 is not optimal when playing against $(2,H)$. Therefore, an optimal strategy in $(2,mix)$ against $(2,H)$ is of the form $(x_1,0,1)$. Now,
$\Delta((x_1,0,1),(2,H)) = prob_{(2,H)}(y_1 \in \{x_1+1,x_1\}$ and
$y_2 = 0)]/4 + prob_{(2,H)}(y_1 \in \{x_1+1,x_1\},\ y_3 = 0)/2 =$
$\frac{1}{4}\frac{1}{3}\frac{4}{T} + \frac{1}{2}\frac{1}{3}\frac{4}{T} = \frac{1}{T}$ and thus $(0,1,T)$ is an optimal strategy in $(2,mix)$ against $(2,H)$.

This strategy is inferior to $(0,T,T)$ since saving T does not justify the loss of M with probability of approximately $1/2$. Formally, $W((0,1,T),(2,H)) = \frac{1}{2} - \frac{1}{2T}$ (i.e. it wins against any strategy in $(2,H)$

of the types $(0, y_2, T)$, $(0, y_2, y_3 < T)$, $(y_1, 0, y_3 < T)$, $(y_1, 0, T)$ and $(y_1, y_2, 0)$ with probabilities $1/4$, $1/2$, 1, $1/2$, and 0, respectively) and $M\frac{T-1}{2T} - T - 1 \leq \frac{T-1}{T}M - 2T$ since $2T < M$.

Step 3: The MD-equilibrium $(2, H)$ is not global unless $\frac{3(T-2)}{T-1} > \frac{M}{T} > \frac{3T}{T-1}$.

The strategy $(1, 1, T) \in (3, mix)$ wins M with probability $2/3$ against $(2, H)$ and its expected payoff is about $(2/3)M - T$, which is greater than $M - 2T$ if $3T > M$. The strategy $(T, T, 1) \in (3, mix)$ costs one unit more than $(T, T, 0)$ and increases the probability of winning against any strategy of the form $(T, y_2, 0)$ or $(y_1, T, 0)$ by $1/4$. The frequency of such strategies in $(2, H)$ is approximately $\frac{1}{3}\frac{4}{T}$. Thus, the approximate expected improvement is $\frac{1}{4}\frac{4M}{3T} - 1$ which is positive if $M/T > 3$. Formally, if $\frac{3(T-2)}{T-1} > \frac{M}{T}$, then

$$u((1, 1, T), (2, H)) = (1 - \tfrac{1}{T})M - T - 2 > (1 - \tfrac{1}{T})M - 2T = u((0, T, T), (2, H)).$$

Proposition 3: **In the edited tennis coach game with parameters T and c satisfying $1/2 > c$ and $(3T - 1)c > 3/2$:**

(**a**) **No cell $(Q, Strong)$ is an MD-equilibrium**.

(**b**) **If $c \in (5/16, 3/8]$, then the cell $(3, not\ Strong)$ is a non-global MD-equilibrium for all T.**

Proof of Proposition 3:(a) The upper bound on the range of c guarantees the claim for $Q = 0, 1,\ 3T - 1$ and the lower bound guarantees the claim for $3T$.

Let $2 \leq Q \leq 3T - 2$. Note that the total score in any match between two teams with the same total skill level is either $1, 1.5$ or 2.

Let $q = \min\{\lfloor Q/2 \rfloor, T\}$. We first show that either $a = (q, Q - 2q, q)$ or $b = (q, 0, q + 1)$, both of which are strategies in $(Q, not\ Strong)$, scores 2 points against all strategies in $(Q, Strong)$ except for two strategies against which they score 1.5 points.

For even Q or $3T - 2 \geq Q > 2T$, the strategy a scores 2 points against all strategies in $(Q, Strong)$ that do not assign q to a side court (since none of the strategies in this cell assign more than q to a side court). The strategy a scores 1.5 points against the only two strategies in $(Q, Strong)$ that assign q to one of the side courts, i.e. $(q, q, Q - 2q)$ and $(Q - 2q, q, q)$.

For odd $Q < 2T$, the strategy b always wins in the third court since there is no strategy $z \in (Q, Strong)$ with $z_3 \geq q + 1$. There are only two strategies in $(Q, Strong)$ in which the tennis player in court 1 has a skill level q: $(q, q + 1, 0)$ and $(q, q, 1)$, and b scores 1.5 points against these two strategies and 2 points against all others in $(Q, Strong)$.

It remains to show that every $x \in (Q, Strong)$ does worse than one of the strategies $a, b \in (Q, no\ Strong)$ since it ties against at least two strategies in $(Q, Strong)$ besides x. The following table specifies pairs of two strategies for every $x \in (Q, Strong)$ for which $x_1 < x_3$ (the case of $x_3 > x_1$ is symmetric). Note that the strategies identified in the third and fifth rows in the table are in $(Q, no\ strong)$ because $Q \leq 3T - 2$.

$x \in (Q, Strong)$	strategies in (Q, No_Strong) which tie with x	
$x_1 = x_3 = 0$	$(1, x_2 - 1, 0)$	$(0, x_2 - 1, 1)$
$x_2 > x_1 = x_3 > 0$	$(x_1 - 1, x_2, x_3 + 1)$	$(x_1 + 1, x_2, x_3 - 1)$
$x_1 = x_2 = x_3$	$(x_1 - 1, x_2 + 1, x_3)$	$(x_1, x_2 + 1, x_3 - 1)$
$T > x_2 = x_1 > x_3$	$(x_1 - 1, x_2, x_3 + 1)$	$(x_1 - 1, x_2 + 1, x_3)$
$T = x_2 = x_1 > x_3$	$(x_1 - 1, x_2, x_3 + 1)$	$(x_1 - 2, x_2, x_3 + 2)$
$x_2 > x_1 > x_3 \geq 0$	$(x_1 - 1, x_2, x_3 + 1)$	$(x_1, x_2 - 1, x_3 + 1)$

(b) Let $m_i(j) = u((j+1, x_{-i}), (3, not\ Strong)) - u((j, x_{-i}), (3, not\ Strong))$. Note that $m_i(j)$ is well-defined since it is independent of x_{-i}. Note also that $m_1(j) = m_3(j)$. Let $m_i = (m_i(0), m_i(1), \ldots.)$ be the vector of marginals for the i'th court.

For $T = 2$, the best strategy in $(3, not\ Strong)$ is 210 which achieves an average score of 1.625. Any strategy in $(3, Strong)$ scores only 1.5 points against the cell. It is straightforward to verify that (in this range of c) no strategy in (Q, not_strong) does better than 210 against $(3, not\ Strong)$. The MD-equilibrium is not global in this range since the strategy $010 \in (1, Strong)$ scores 1 point against all strategies in the cell and $1 - c > 1.625 - 3c$.

For $T \geq 3$, the cell $(3, not\ Strong)$ is identical for for every $T \geq 3$. The vectors of marginals are $m_2 = (6/12, 2/12, 0, 0, 0\ldots)$ and $m_1 = m_3 = (3/12, 3/12, 3/12, 1/12, 0, \ldots)$ which are decreasing sequences. Thus, the best strategy for $Q = 3$ against the cell is 210 which achieves an average score of $10/6$. In order to verify that there is no better strategy in any cell $(Q, not\ Strong)$, note that $c \geq 1/4$ guarantees that adding skill to the side courts is not beneficial. Furthermore, $c \leq 3/8$ guarantees that 100 is no better than 210 (since it reduces the average score by $3/4$, which is more than the saving of $2c$). Thus, $(3, not\ Strong)$ is an MD-equilibrium. It is not global since the strategy 010 is superior to 210 against the equilibrium cell (since the saving of $2c$ is larger than the loss of $5/8$ on the first court). ■