

Coordinating with a "Problem Solver"

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Abstract

We introduce a new type of agent called a "problem solver" (PS). When interacting with other agents the PS does not "put himself in their shoes" but rather chooses a best response to a uniform distribution over all possible configurations consistent with the information he receives about the other agents' moves.

We demonstrate the special features of a PS by analyzing a modified coordination game. In the first stage, each of the other participants, which are treated as conventional players, chooses a location. The PS then, receives some partial information about their moves and chooses his location. The PS wishes to coordinate with any of the conventional players and they wish to coordinate with the PS but not with each other. Equilibria are characterized and shown to have different properties than those of Nash equilibria when the PS is treated as a conventional player.

The second author acknowledges financial support from ERC grant 269143.

Our thanks to Noga Alon, Ayala Arad, Kfir Eliaz, Gil Kalai and Rani Spiegler for helpful conversations.

1. Introduction

Consider a gamelike situation in which some agents choose actions in the first stage and another agent chooses his action after observing some information about the choices made in the first stage. According to the standard game-theoretic approach, the second-stage player "puts himself in the shoes" of the first-stage players, forms beliefs about their behavior, updates his beliefs according to what he has observed and responds optimally to those beliefs. However, there are situations in which the second-stage player does not make his choice based on the beliefs he forms about the other agents' moves but rather bases his action solely on the set of configurations of actions consistent with what he observes.

For example, imagine that someone has written a crossword puzzle and offers you a prize if you can solve it. It is unlikely that you will think much about his "strategy". It is more likely that you will use your knowledge and powers of logic to fill in the crossword according to the clues given.

Alternatively, imagine a situation in which someone writes the names of two countries (members of the UN), each on a different card, and places them face down in front of you. You are told that each name has four different letters and that they have three letters in common. Your task is to guess one of the two names. You will eventually come to the conclusion that the only possible configuration is {Iran, Iraq} and thus either of these two names will be a successful guess. However, what would be your guess if you are told that the two names share precisely two letters. In that case, there are three possible pairs that fit this description: {Chad, Cuba}, {Iran, Mali} and {Iraq, Mali}. If you aren't thinking strategically about the motives of the person who chose the two countries, then your guess will probably be Mali, since it appears in two of the possible configurations whereas the others appear in only one.

Finally, imagine you are a new competitor entering a market and you observe some partial information about your competitor's behavior. You are likely to first look for values of the unknowns that are consistent with what you observe and then assign probabilities to the consistent scenarios rather than forming beliefs about the competitor's moves and updating them according to what you observe.

These are examples of situations in which agents do not think strategically but rather reason as if they were solving a puzzle. Accordingly, we introduce a new type of economic agent, which we refer to as a problem solver (PS). The PS interacts with the

other agents and receives only partial information about their moves; he does not deliberate about their motives. We assume that the PS calculates the set of possible configurations of the other players' moves that are consistent with what he observes and chooses a best response to the uniform distribution over that set. For the PS, finding all the configurations of moves that are consistent with what he observes is like solving a puzzle.

The platform we use to demonstrate the idea is a new version of the coordination game. The agents in the interaction are labelled $0, 1, \dots, n$. Each agent chooses an alternative from a set X . Agent 0 is the PS and the other n agents are treated as conventional players. The PS is interested in coordinating his choice with one of the other players. Each of the players $1, \dots, n$ would like to coordinate with the PS and would like to avoid coordinating with any other player. First, players $1, \dots, n$ make their choices simultaneously after which the PS receives some information about their choices and then makes his own.

If the PS is treated as a conventional player, then the game would have many trivial pure equilibria which do not make much sense. In fact, regardless of the information player 0 receives, every profile of n distinct choices, one of which is chosen by player 0 with probability 1, is consistent with a pure sequential equilibrium. The success of the coordination is due to player 0's knowledge of the equilibrium. The information player 0 receives regarding the other n players is superfluous.

We suggest a different approach according to which player 0 is a problem solver. He identifies all the profiles of the players' choices that are consistent with the data he receives and treats them as equally likely. He then chooses an action in order to maximize the chance that his choice will match that of at least one of the other players.

We analyze the equilibria of the modified game assuming that the set X is a large matrix and the PS observes only the number of players located in each row and in each column of the matrix. The equilibrium of the model in the presence of a problem solver differs significantly from that of the above coordination game in which the PS is treated as a conventional player. We show that in all equilibria, the PS coordinates with one of the players with certainty. However, an interesting phenomenon emerges: There exist equilibria in which the PS chooses a position in the matrix which he believes might be vacant even though it is occupied with certainty. Such a

phenomenon would not arise in the case that the PS is a conventional player.

2. The model

Consider the following romantic matching story. There are n men, each of whom would like to be picked by a certain woman. Each man has to declare his favorite vacation country to vacation in (a member of set A), as well as his favorite cuisine (a member of set B). The declarations don't have to be truthful. Men would like to be unique, i.e., they do not want to be viewed as having the same pair of characteristics as other men. The woman gets to view the distributions of the country declarations and the cuisine declarations, after which she is to declare the type of man (a pair in $A \times B$) she is ready to be matched with. If her declaration does not match any of the men's declarations, then everyone is disappointed. If there is a man whose declaration matches her declaration, then they become a happy couple and the other men are miserable.

There are $n + 1$ agents. We refer to agent 0 as a PS and to agents $1, \dots, n$ as players. Each agent chooses a position, which is a pair of characteristics in the set $X = A \times B$ (where A and B are finite and disjoint sets). For simplicity, assume that both A and B contain at least n elements. Consistent with the convention in chess, we refer to the elements of A as columns and to the elements of B as rows. A product set of columns and rows is called a *box*.

The n players make their choices simultaneously. The PS observes only the number of occupied entries in each column and row. With that information he chooses his position.

The PS gets utility of 1 if his choice coincides with the choice of one of the players and 0 otherwise. Each of the players $1, \dots, n$ gets utility of 1 if his choice coincides only with the choice of the PS, utility of -1 if his choice coincides with at least one of the other players and 0 otherwise.

Note that in the last section we analyze two variants of the model. In the first, a player does not have disutility from making the same choice as another player and in the case that the PS picks an entry chosen by more than one player he is randomly matched with one of them. The second is a version of the hide-and-seek game: the

PS wants to pick a player and players want not to be picked by the PS.

Given these preferences we can confine ourselves to strategy profiles where no two players occupy the same position. An outcome of the n players' choices is a *matrix* $M = (M_{a,b})$, where $M_{a,b} \in \{0, 1\}$, with n 1's. The notation $M_{a,b} = 1$ signifies that the position (a, b) is occupied and $M_{a,b} = 0$ signifies that it is vacant. The PS observes only the *data vector* $d(M) = (d(M)(x))_{x \in A \cup B}$ where $d(M)(x)$ is the number of players occupying entries with a characteristic x in the matrix M (i.e., $d(M)(a) = \sum_i M_{ai}$ and $d(M)(b) = \sum_k M_{kb}$). A vector d is consistent if there is a matrix M such that $d = d(M)$. By definition, the PS observes a consistent data vector. We refer to $d(a) + d(b)$ as the *score* of the entry (a, b) .

The PS assumes that all matrices consistent with the data he observes are equally likely and randomly picks an entry with the highest probability of being occupied. Formally, a matrix M is said to be d -consistent if $d(M) = d$. Let $\lambda(d, x)$ be the proportion of d -consistent matrices, in which x is occupied. If $\lambda(d, x) = 1$, we say that x is *revealed to be occupied* by d . If $\lambda(d, x) = 0$, we say that x is *revealed to be vacant* by d (hereafter, we use the term "revealed" to mean "revealed to be occupied"). We say that the matrix M is *revealed* if it is the only matrix consistent with $d(M)$. Denote by $C(M) = \{x \mid \lambda(d(M), x) \text{ is maximal}\}$ the set of all entries with the highest probability of being occupied, given $d(M)$. The probability that the PS picks x is $\mu(M, x) = 1/|C(M)|$ for each $x \in C(M)$ and $\mu(M, x) = 0$ for each $x \notin C(M)$.

An *equilibrium* is a matrix M such that no player can increase his probability of being picked by moving to a vacant entry. Formally, let $M(x \rightarrow y)$ be the matrix derived from M after switching the values of entries x and y . The matrix M is an *equilibrium* if, for each occupied entry x , $\mu(M, x) \geq \mu((M(x \rightarrow y)), y)$ for any entry y that is not occupied in M .

Example 1: Consider the following matrices with $n = 5$ (when describing a matrix, vacant rows and columns are not depicted).

$$M_1 = \begin{array}{|c|c|c|} \hline 1^* & 1 & 1 \\ \hline 1 & 0^{**} & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array} \quad M_2 = \begin{array}{|c|c|c|} \hline 1^* & 1 & 1 \\ \hline 1 & 1 & 0 \\ \hline 0 & 0^{**} & 0 \\ \hline \end{array} \quad M_3 = \begin{array}{|c|c|c|c|c|} \hline 1^* & 1 & 1 & 1 & 1 \\ \hline * & * & * & * & * \\ \hline \end{array}$$

The matrix M_1 is revealed. Thus, the probability of each occupied entry being picked is $1/5$. However, M_1 is not an equilibrium. If the occupier of $*$ moves to $**$, the new

data vector will be: $((2, 2, 1), (2, 2, 1))$, which is consistent with 5 matrices, such that each of the four entries (a, b) for which $(d(a), d(b)) = (2, 2)$ is occupied in 4 of the 5 matrices. Each of the 4 entries (a, b) for which $(d(a), d(b))$ is $(2, 1)$ or $(1, 2)$ is occupied in 2 of the 5 matrices. The unique entry (a, b) , for which $(d(a), d(b)) = (1, 1)$, is occupied in only one of the d -consistent matrices. Thus, by moving from $*$ to $**$ the mover increases his probability of being picked from $1/5$ to $1/4$.

The matrix M_2 is revealed but is not an equilibrium since the player who occupies $*$ can increase his probability of being picked from $1/5$ to $1/3$ by moving to $**$.

The matrix M_3 is revealed and is an equilibrium. If the player at $*$ moves to an entry such as $**$ (which does not share any characteristics with the other four occupied entries), then the new data vector will be consistent with 5 matrices and in only one of them is $**$ occupied. Each of the other occupied entries is occupied in 4 of the 5 matrices consistent with the new data and thus the mover reduces his probability of being picked from $1/5$ to 0. If the occupier of $*$ moves to an entry such as $***$ (which shares one characteristic with one other occupied entry), then the new matrix will also be revealed and the player gains nothing by moving.

3. The Problem Solver's Behavior

In this section, we present some properties of the set of matrices that are consistent with a given data set. These properties determine the Problem Solver's "response function". In particular, we will show that either:

(1) The set of entries that are revealed to be occupied is nonempty and forms a step set, every other entry has a strictly lower score than at least one revealed entry and the PS randomly picks one of the revealed entries;

or

(2) No entry is revealed and the PS randomly picks one of the entries with the maximal score.

Claim 1: Let d be a consistent data vector such that $d(1) > d(2)$, where 1 and 2 are elements of A . Then, for all $b \in B$, $\lambda(d, (1, b)) \geq \lambda(d, (2, b))$. Furthermore, if there is a d -consistent matrix M such that for some b , $M_{1,b} = 1$ and $M_{2,b} = 0$, then

$$\lambda(d, (1, b)) > \lambda(d, (2, b)).$$

Proof: Let $b \in B$. Fix the values for all entries other than those in columns 1 and 2. Partition the class \mathbb{C} of all d -consistent matrices with these fixed values outside columns 1 and 2 into (up to) four cells, denoted by $\mathbf{M}(\alpha, \beta)$, $\alpha \in \{0, 1\}$, $\beta \in \{0, 1\}$, such that $\mathbf{M}(\alpha, \beta)$ is the cell in this partition that consists of the matrices for which $M_{1,b} = \alpha$ and $M_{2,b} = \beta$. We will show that $|\mathbf{M}(1, 0)| \geq |\mathbf{M}(0, 1)|$ and if $\mathbf{M}(1, 0)$ is not empty, then the inequality is strict. This is sufficient since if there is a d -consistent matrix M such that $M_{1,b} = 1$ and $M_{2,b} = 0$, then for at least one set of entries in columns $B - \{1, 2\}$ we have $\mathbf{M}(1, 0) \neq \emptyset$.

We first show that if $\mathbf{M}(1, 0) = \emptyset$, then $\mathbf{M}(0, 1) = \emptyset$. If $\mathbf{M}(0, 1)$ is not empty, then there is $M \in \mathbb{C}$ with $M_{1,b} = 0$ and $M_{2,b} = 1$. By $d(1) > d(2)$, there is a row b' where $M_{1,b'} = 1$ and $M_{2,b'} = 0$. Switching all values in $\{1, 2\} \times \{b, b'\}$ (i.e., changing all 0's to 1's and vice versa), we obtain another matrix in \mathbb{C} which is in $\mathbf{M}(1, 0)$. Therefore, if $\mathbf{M}(1, 0)$ is empty, then the number of matrices in \mathbb{C} in which $M_{1,b} = 1$ is the same as the number of matrices in \mathbb{C} in which $M_{2,b} = 1$.

We next show that if $\mathbf{M}(1, 0) \neq \emptyset$, then the number of elements in $\mathbf{M}(0, 1)$ is strictly smaller than that in $\mathbf{M}(1, 0)$. Define $L_{1,1}$ to be the set of rows in which the data regarding the rows implies that the missing values in columns 1 and 2 are (1, 1), and define $L_{0,0}$ in a similar manner. For the rows in $B - L_{1,1} - L_{0,0}$, the data dictates that the missing values in columns 1 and 2 be either (0, 1) or (1, 0). It must be that in any $\delta(1) = d(1) - |L_{1,1}|$ of these rows the values in the two columns are (1, 0) and in the other $\delta(2) = d(2) - |L_{1,1}|$ rows the values must be (0, 1). Thus, $|\mathbf{M}(1, 0)| = C(\delta(1) - 1, \delta(1) + \delta(2) - 1) > |\mathbf{M}(0, 1)| = C(\delta(1), \delta(1) + \delta(2) - 1)$ where $C(k, l)$ is the number of sets of size k in a set of size l . The strict inequality follows from the fact that $\delta(1) > \delta(2)$. ■

Claim 2: Let d be a consistent data set. Assume that (a^*, b^*) maximizes $d(a) + d(b)$ over $A \times B$ and is not revealed. If, for some a , $d(a^*) > d(a)$, then $\lambda(d, (a^*, b^*)) > \lambda(d, (a, b^*))$.

Proof: By claim 1, it is sufficient to show that there exists a matrix M consistent with d , such that $M_{a^*, b^*} = 1$ and $M_{a, b^*} = 0$.

Since $1 > \lambda(d, (a^*, b^*))$, there exists a d -consistent matrix M with $M_{a^*, b^*} = 0$.

If $M_{a,b^*} = 1$, then since $d(a^*) > d(a)$, there must be some $b \in B$ with $M_{a^*,b} = 1$ and $M_{a,b} = 0$. By switching the values in the four entries $\{a^*, a\} \times \{b^*, b\}$, we obtain a d -consistent M' with $M'_{a^*,b^*} = 1$ and $M'_{a,b^*} = 0$.

b*	0		1		
b	1		0		
	a*		a		

If $M_{a,b^*} = 0$, then by $d(a^*) > d(a)$ there exists a row b with $M_{a^*,b} = 1$ and $M_{a,b} = 0$. However, since b^* maximizes d over B , there also exists an a' such that $M_{a',b^*} = 1$ and $M_{a',b} = 0$. By switching the values in $\{a^*, a'\} \times \{b^*, b\}$, we obtain a d -consistent matrix M' with $M'_{a^*,b^*} = 1$ and $M'_{a,b^*} = 0$.

b*	0		0		1
b	1		0		0
	a*		a		a'



Claim 3: Let M^* be a matrix such that the box $C \times R$ is occupied and the "dual" box $C^c \times R^c$ is vacant. Then, each element in $C \times R$ is revealed to be occupied and each element in the dual box is revealed to be vacant.

R	1	1	1	?	?
R	1	1	1	?	?
	?	?	?	0	0
	?	?	?	0	0
	?	?	?	0	0
	C	C	C		

Proof: Given a data vector d and a set E of rows or columns, let $d(E) = \sum_{e \in E} d(e)$. It must be that $d(M^*)(C^c) + d(M^*)(R^c) + |C| \times |R| = n$. On the other hand, for every matrix M such that $M_{a,b} = 0$ for some $(a,b) \in C \times R$, it must be that $d(M)(C^c) + d(M)(R^c) + |C| \times |R| > n$ and thus M is inconsistent with $d(M^*)$. An analogous argument can be used

to show that the positions in the dual box are revealed to be vacant. ■

Let d be a consistent data vector. Order the elements of A so that $d(a_1) \geq d(a_2) \geq \dots \geq d(a_{|A|})$ and order the elements of B so that $d(b_1) \geq d(b_2) \geq \dots \geq d(b_{|B|})$. By Claim 1, if (a_i, b_j) is revealed to be occupied, then so are all the entries in the box $\{a_1, \dots, a_i\} \times \{b_1, \dots, b_j\}$. Thus, the set of revealed entries is a "step set". That is, there are two sequences of sets $B(1) \supset B(2) \supset \dots \supset B(I)$ and $A(1) \subset A(2) \subset \dots \subset A(I)$ such that the set of revealed (to be occupied) entries is $\cup(A(i) \times B(i))$ (in the illustration below $A(1) \times B(1) = \{a_1, a_2\} \times \{b_1, b_2, b_3, b_4, b_5\}$, $A(2) \times B(2) = \{a_1, a_2, a_3\} \times \{b_1, b_2, b_3, b_4\}$, etc.) and the d -value of any element in $A(i)$ ($B(i)$) is larger than the d -value of any element in $A(j)$ ($B(j)$) where $j > i$.

b₁	Y	Y	Y	Y	Y	Y	X
b₂	Y	Y	Y	Y	Y	X	Z
b₃	Y	Y	Y	Y	Y	X	Z
b₄	Y	Y	Y	X	X	Z	Z
b₅	Y	Y	X	Z	Z	Z	Z
b₆	X	X	Z	Z	Z	Z	Z
b₇	X	X	Z	Z	Z	Z	Z
	a₁	a₂	a₃	a₄	a₅	a₆	a₇

In what follows, we denote the step set of revealed entries by $Y = \cup A(i) \times B(i)$, the dual step set by $Z = \cup A(i)^c \times B(i)^c$ and the union of the boxes between Y and Z by $X = \cup(A_i - A_{i-1}) \times (B_{i-1} - B_i)$.

Claim 4: Let d be a consistent data vector. Assume that the set of revealed elements $Y = \cup A(i) \times B(i)$ is not empty. Then, every element in the dual set $Z = \cup A(i)^c \times B(i)^c$ is revealed to be vacant.

Proof: Given a consistent data vector d , the Gale-Ryser algorithm (see Ryser (1963) and Krause (1996)) ends with a d -consistent matrix. The algorithm is sequential and starts with a certain initial matrix. In each step of the algorithm, a permissible pair of entries that are positioned in the same row – one occupied (the one with the higher d) and the other vacant – is selected and their values are swapped. An important property of the algorithm is that any choice of a chain of permissible pairs leads to a

d -consistent matrix.

In order to describe the algorithm precisely, order the elements in the set A according to their d -values $a_1, \dots, a_{|A|}$ and order the elements in B according to their d -values $b_1, \dots, b_{|B|}$. The algorithm starts with a matrix M_0 in which, for any row b , 1's are assigned to the $d(b)$ entries in this row in the columns $a_1, \dots, a_{d(b)}$. For M_0 , the number of 1's in column a_k is $z(k) = |\{b \mid \text{the number } d(b) \geq k\}|$. Obviously, (a) $\sum_{i=1, \dots, k} z(i) \geq \sum_{i=1, \dots, k} d(a_i)$ for all k and (b) if $\sum_{i=1, \dots, k} z(i) = \sum_{i=1, \dots, k} d(a_i)$, then in any d -consistent matrix all entries in $\{a_1, \dots, a_k\} \times \{b_1, \dots, b_{z(k)}\}$ are occupied and all entries in $\{a_{k+1}, \dots, a_{|A|}\} \times \{b_{z(k)+1}, \dots, b_{|B|}\}$ are vacant. In each step of the algorithm, a "1" in the lowest index column for which the number of 1's is strictly greater than $d(a)$ is moved to the first column a' in which the number of 1's is strictly less than $d(a')$.

In his proof that the algorithm ends with a d -consistent matrix, Krause (1996) shows that the algorithm works by starting with any matrix having the following two properties: (i) The number of 1's in each $b \in B$ is $d(b)$ and (ii) for each k , the sum of the 1's in the first k columns is at least as large as $\sum_{i=1, \dots, k} d(a_i)$.

To prove Claim 4, it is sufficient to show that, for all l , $\sum_{i=1, \dots, |A(l)|} z(i) = \sum_{i=1, \dots, |A(l)|} d(a_i)$. Assume not. Then, for some l we have $\sum_{i=1, \dots, |A(l)|} z(i) > \sum_{i=1, \dots, |A(l)|} d(a_i)$. Let k^* be the lowest $k > |A(l)|$ for which $\sum_{i=1, \dots, k} z(i) = \sum_{i=1, \dots, k} d(a_i)$. It must be that $d(b_{|B(l)|}) < k^*$; otherwise, by (b) above, the set $A(l)$ would be larger. Now start the Gale-Ryser algorithm from a matrix that modifies M_0 by moving a single "1" from the entry $(a_{|A(l)|}, b_{|B(l)|})$ to the entry in the same row in column k^* . The matrix satisfies properties (i) and (ii) above and the algorithm leads to a d -consistent matrix in which one of the first $|A(l)|$ entries in row $b_{|B(l)|}$ is 0, violating the assumption that this entry is revealed to be occupied.

4. Equilibrium

In this section, we show that in all equilibria the PS picks an occupied entry with probability 1 (Proposition 1). However, in some equilibria, the PS assigns a strictly positive probability to the possibility that the occupied position he is picking is vacant. We also classify the structure of all equilibria (Proposition 2).

Proposition 1: In equilibrium the PS picks an occupied entry with probability 1.

Proof: Let M be an equilibrium. If there are revealed positions, then the PS obviously chooses one of them. If no entry is revealed, then, by Claim 2, $C(M)$ is the box of all entries that maximize the $d(M)$ score. It is left to show that all entries in $C(M)$ are occupied.

If an entry (a, b) outside $C(M)$ is occupied and $(a^*, b^*) \in C(M)$ is vacant, then the move from (a, b) to (a^*, b^*) is beneficial: the score of (a^*, b^*) in $M((a, b) \rightarrow (a^*, b^*))$ increases by at least 1 relative to the score in M and the score of any other entry increases by at most 1. Thus, (a^*, b^*) maximizes the score after the move, and hence it will be picked with a positive probability.

If all the occupied entries are in $C(M)$ and $C(M)$ contains a vacant entry, then any vertical move from an occupied entry (a^*, b) to a vacant entry (a^*, b^*) is beneficial since (i) (a^*, b^*) maximizes the score after the move, (ii) $C(M((a^*, b) \rightarrow (a^*, b^*))) \subseteq C(M)$ and (iii) the entry (a^*, b) is excluded from $C(M((a^*, b) \rightarrow (a^*, b^*)))$. ■

Proposition 2: If M is an equilibrium, it must have one of the following three structures:

(1) The matrix M is revealed and forms a "step set". The PS picks one of the occupied positions.

(2) There is a "step set" of entries that are revealed as occupied and the PS picks one of them. The "dual step set" is revealed to be vacant. Each box lying between these two sets contains at least three occupied entries that are not picked by the PS.

(3) None of the positions are revealed to be occupied. All positions with maximal score are occupied and the PS picks one of them.

Proof: The fact that the set of revealed entries forms a step set follows from Claim 1. By Claim 4, the dual step set is revealed to be vacant.

Assume that M is an equilibrium in which some, but not all, occupied entries are revealed. We will show that any box $A' \times B'$ in area X (i.e., the area consisting of all the positions between Y , the set of entries that are revealed as occupied, and Z , the set of entries that are revealed to be vacant) contains at least three occupied entries that do not share any characteristic (namely, they are positioned in three different rows and three different columns). If $A' \times B'$ is entirely vacant, then a move from an unrevealed

occupied entry into this empty box will be beneficial to the mover since it will reveal this entry to be occupied (by Claim 3). It is impossible that all occupied entries in this box lie in the same row or the same column since (again, by Claim 3) they would then be revealed. It is also impossible that in equilibrium all occupied entries in this box lie in exactly two rows (or two columns) since if (a_1, b_1) and (a_2, b_2) are occupied then a move from (a_1, b_1) to (a_1, b_2) is beneficial since the deviator will be revealed (once again, by Claim 3).

Finally, if none of the positions are revealed as occupied, then, by claim 2, $C(M)$ contains all of the positions with the highest score and, by the proof of Proposition 1, all of the entries in $C(M)$ must be occupied. ■

An interesting feature of the model is that although in equilibrium the PS always picks an occupied position, there are equilibria in which he does not know that the position is occupied. Such a phenomenon could not occur under the conventional equilibrium assumption but may emerge in our setup.

Proposition 3: For $n \geq 10$, there exists an equilibrium in which the PS picks an occupied entry although the probability he assigns to it being occupied is strictly less than one.

Proof: The matrix M_4 demonstrates such an equilibrium for $n = 10$. The example can be adjusted to any $n > 10$ by "extending" the "arms of the top-left L".

$$M_4 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

No entry is revealed by $d(M_4)$ and the PS chooses the top-left position x^* which gets the maximum score of 8, whereas any other entry gets a score of at most 5. (The PS assigns a probability of $2400/2850 \sim 0.85$ to x^* being occupied.) For any occupied

entry $x \neq x^*$ and unoccupied entry y , no entry is revealed in $M_4(x \rightarrow y)$ and x^* remains the entry with the highest score in $M_4(x \rightarrow y)$ (being equal to 8 or 7 whereas the score of y would not be higher than 6). By Claim 2, x^* would still be chosen by the PS given the data $d(M_4(x \rightarrow y))$. Thus, no deviation is profitable. ■

Comment: Were the PS to take into account that the n players' profile of choices is consistent with Nash equilibrium, he would conclude that the entry he is choosing is occupied. However, the whole point of modeling an agent as a problem solver is that he does not think strategically. Rather, he treats the problem as a puzzle to be solved, using the information he possesses, without making any assumptions about the various solutions to the puzzle.

5. Two variants of the model

(a) Players can share entries

So far we have assumed that none of the players want to share an entry with another player regardless of whether or not the entry is picked by the PS. Assume now that players do not have disutility from sharing an entry with another player. Once the PS has chosen an occupied cell he randomly picks one of the occupiers and they are interested in increasing the probability of being picked. Formally, a player's utility is $1/k$ if he shares the PS' chosen cell with $k - 1$ other players and 0 otherwise. A matrix is an assignment of non-negative integers (not necessarily zeros or ones) to all entries, such that the sum of the numbers in all entries is n . For each row (column), the PS receives information about the total number of *players* occupying that *row* (*column*). The PS identifies the matrices that are consistent with the data and then randomly picks an entry with the largest number of consistent matrices in which this entry is occupied (by at least one player). For example, the data $d(a_1) = d(b_1) = 3$ and $d(a_2) = d(b_2) = 1$ is consistent with two matrices:

$$\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \text{ and } \begin{array}{|c|c|} \hline 3 & 0 \\ \hline 0 & 1 \\ \hline \end{array} .$$

Only the entry (a_1, b_1) is occupied in both matrices and therefore it is picked by the PS.

An equilibrium is a matrix such that a player who occupies one entry cannot increase his utility by moving to another. Formally, there is no occupied entry x and (not necessarily unoccupied) entry y such that $\frac{\mu(M(x \rightarrow y), y)}{M(y)+1} > \frac{\mu(M, x)}{M(x)}$ where $M(x \rightarrow y)$ is the matrix where x is decreased by 1 and y is increased by 1.

Obviously, any matrix in which the n players occupy n entries in one row or in one column is an equilibrium (since the move of a player to another occupied entry reduces his utility from $1/n$ to $1/(2n - 2)$ and the move of a player to an unoccupied entry cannot increase the probability that he is picked by the PS).

We are able to prove that any other matrix is not an equilibrium (with the exception of $n = 2$ and the two players occupying the same entry). The proof is not presented here but can be obtained from the authors.

(b) **Players don't want to be picked by the PS**

Suppose that none of the n players wishes to be "detected" by the PS. More precisely, each player gets a utility of 1 if he discoordinates with the PS, 0 if his choice matches that of the PS and negative utility if he chooses the same entry as one of the other players.

Any diagonal matrix M , in which each player chooses a distinct row and a distinct column, is an equilibrium. Given $d(M)$, the PS will select each of the n^2 entries with the score 2 and thus a player's probability of being detected by the PS is $1/n^2$.

Consider the matrix $M(x \rightarrow y)$ that results from the move of a player from an occupied entry x to an unoccupied entry y , where y is an entry that shares a row or a column with one of the occupied entries in M . The score of y in $M(x \rightarrow y)$ is maximal and thus it is one of at most $n(n - 1)$ entries in $C(M(x \rightarrow y))$. Thus, the move from x to y increases a player's probability of being detected.

In any other matrix M , the minimal box that contains all occupied entries is of size kl , where $k \leq n$ and $l \leq n$, with at least one of the inequalities being strict. Then, there is an occupied entry in M with a score strictly greater than 2. Any player who occupies an entry x with a maximal score will benefit by moving to an entry y , which does not share a row or column with any other player, since such a move reduces his chances of being picked by the PS from $\frac{1}{kl}$ to at most $\frac{1}{(k+1)l}$.

References

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