**Coordinating with a "Problem Solver"**

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**Abstract**

A "problem solver" (PS) is an agent who when interacting with other agents does not "put himself in their shoes" but rather chooses a best response to a uniform distribution over all possible configurations consistent with the information he receives about the other agents' moves.

We demonstrate the special features of a PS by analyzing a modified coordination game. In the first stage, each of the other participants - who are treated as conventional players - chooses a location. The PS then receives some partial information about their moves and chooses his location. The PS wishes to coordinate with any one of the conventional players and they wish to coordinate with him but not with each other. Equilibria are characterized and shown to have different properties than those of Nash equilibria when the PS is treated as a conventional player.

The first author thanks the Henry Crown Institute of Business Research in Israel for financial support.

The second author acknowledges financial support from ERC grant 269143.

Our thanks go to Noga Alon, Ayala Arad, Kfir Eliaz, Gil Kalai and Rani Spiegler for their comments.

## 1. **Introduction**

Consider a gamelike situation in which all agents but one choose their actions in the first stage and the other agent chooses his action after receiving some information about the choices made in the first stage. According to the standard game-theoretic approach, the second-stage player "puts himself in the shoes" of the first-stage players, forms beliefs about their behavior, updates his beliefs according to the information he received and responds optimally to those beliefs. However, there are situations in which the second-stage player does not base his choice on the beliefs he forms about the other agents' moves but rather on the set of configurations of actions consistent with what he observes.

For example, imagine that someone has written a crossword puzzle and offers you a prize if you can solve it. It is unlikely that you will think much about his "strategy". It is more likely that you will use your knowledge and powers of logic to fill in the crossword based on the clues given.

Alternatively, imagine a situation in which someone writes the names of two countries (members of the UN), each on a different card, and places them face down in front of you. You are told that each name has four different letters and that they have three letters in common. Your task is to guess one of the two names. You will eventually come to the conclusion that the only possible configuration is {Iran, Iraq} and thus either of them will be a successful guess. However, what would be your guess if you are told that the two names share precisely two letters. In that case, there are three possible pairs: {Chad,Cuba}, {Iran, Mali} and {Iraq,Mali}. If you aren't thinking strategically about the motives of the person who chose the two countries, then your guess will probably be Mali, since it appears in two of the possible configurations whereas the others appear in only one.

Finally, imagine you are considering entering a particular market of several similar products (such as breakfast cereals) with an incumbent who produces all the products. You observe your competitor's cumulative profits but not their breakdown by product. You need to decide which of the products to produce if you enter the market. In such a situation, you are likely to first compute values for the unknowns that are consistent with what you observe and then assign probabilities to each consistent scenario, rather than forming beliefs about the competitor's moves and updating them according to what you observe.

In these examples, agents do not think strategically but rather reason as if they were solving a puzzle. Accordingly, we introduce a new type of economic agent, which we refer to as a problem solver (PS). The PS interacts with the other agents and receives only partial information about their moves; he does not deliberate about their motives. We assume that the PS calculates the set of possible configurations of the other players' moves that are consistent with what he observes and chooses a best response to the uniform distribution over that set. For the PS, finding all the configurations of moves that are consistent with what he observes is like solving a puzzle.

The platform we use to demonstrate the idea is a new version of the coordination game. The agents in the interaction are labelled $0, 1, .., n$. Each agent chooses an alternative from a set $X$. Agent $0$ is the PS and the other $n$ agents are treated as conventional players. The PS is interested in coordinating his choice with one of the other players. Each of the players $1, .., n$ would like to coordinate with the PS and to avoid coordinating with any other player. Players $1, .., n$ first make their choices simultaneously, following which the PS receives some information about their choices and then makes his own.

If the PS is treated as a conventional player, then the game would have many trivial pure equilibria which do not make much sense. In fact, regardless of the information player $0$ receives, every profile of $n$ distinct choices, one of which is chosen by player $0$ with probability 1, is consistent with a pure sequential equilibrium. The success of the coordination is due to player $0$'s knowledge of the equilibrium. The information player $0$ receives regarding the other $n$ players is superfluous.

We suggest a different approach according to which player $0$ is a problem solver. He identifies all the profiles of the players' choices that are consistent with the data he receives and treats them as equally likely. He then chooses an action in order to maximize the chance that his choice will match that of at least one of the other players.

We analyze the equilibria of the modified game assuming that the set $X$ is a large matrix and the PS observes only the number of players located in each row and in each column of the matrix. The equilibrium of the model in the presence of a problem solver differs significantly from that of the above coordination game in which the PS is treated as a conventional player. We show that in all equilibria, the PS coordinates with one of the players with certainty. However, an interesting phenomenon emerges: There exist equilibria in which the PS chooses a position in the matrix which he believes might be vacant even though it is occupied with certainty. Such a phenomenon would not arise in the case that the PS is a conventional player.

The concept of a PS in this paper is strongly linked to the idea of an Artificial Intelligence (AI) agent. An AI agent is one that receives percepts from the environment and performs actions in order to maximize its chance of success in achieving some goal (see, for example, Russell and Norwig (2009)). Our PS has exactly these features: The environment is the players' actions. The partial information received by the PS concerns the players' moves. The PS goal is to maximize the chances of choosing an occupied entry.

## 2. **The model**

Consider the following "romantic game". There are $n$ men, each of whom would like to be picked by a certain woman. Each man has to declare his favorite country to vacation in (from a set $A$), as well as his favorite cuisine (from a set $B$). The declarations don't have to be truthful. Men would like to be viewed as unique. In other words, they do not want to be viewed as having the same pair of characteristics as other men. The woman gets to view the distributions of the declared countries and the declared cuisines, after which she declares the type of man (i.e., a pair in $A \times B$) that she is willing to be matched with. If her declaration does not match any of the men's declarations, then everyone is disappointed. If there is a man whose declaration matches hers, they are happy and the other men feel miserable.

Formally, there are $n+1$ agents. We refer to agent $0$ as a PS and to agents $1,..,n$ as players. Each agent chooses a position, which is a pair of characteristics in the set $X = A \times B$ (where $A$ and $B$ are finite and disjoint sets). For simplicity, assume that both $A$ and $B$ contain at least $n$ elements. Consistent with the convention in chess, we refer to the elements of $A$ as columns and to the elements of $B$ as rows. A product set of columns and rows is called a *box*.

The $n$ players make their choices simultaneously. The PS observes only the number of occupied entries in each column and row. With that information he chooses his position.

The PS gains utility of $1$ if his choice coincides with that of one of the players and $0$ otherwise. Each of the players $1,\ldots,n$ gains utility of $1$ if his choice coincides only with that of the PS, $-1$ if his choice coincides with that of at least one of the other players and $0$ otherwise. (In the final section, we analyze two variants of the model. In the first, a player does not have disutiluty from making the same choice as another player and if the PS picks an entry chosen by more than one player he is randomly matched with one of them. The second is a version of the hide-and-seek game: the PS wants to pick a player and players want not to be picked by the PS.)

Given these preferences, we can confine ourselves to strategy profiles in which no two players occupy the same position. An outcome of the $n$ players' choices is a *matrix* $M = (M_{a,b})$, where $M_{a,b} \in \{0,1\}$, with $n$ 1's. The notation $M_{a,b} = 1$ signifies that the position $(a,b)$ is occupied and $M_{a,b} = 0$ signifies that it is vacant. The PS observes only the *data vector* $d(M) = (d(M)(x))_{x\in A\cup B}$ where $d(M)(x)$ is the number of players occupying entries with a characteristic $x$ in the matrix $M$ (i.e., $d(M)(a) = \Sigma_l M_{al}$ and $d(M)(b) = \Sigma_k M_{kb}$). A vector $d$ is consistent if there is a matrix $M$ such that $d = d(M)$. By definition, the PS observes a consistent data vector. We refer to $d(a) + d(b)$ as the *score* of the entry $(a,b)$.

The PS assumes that all matrices consistent with the data he observes are equally likely and randomly picks an entry with the highest probability of being occupied. Formally, a matrix $M$ is said to be $d$-consistent if $d(M) = d$. Let $\lambda(d,x)$ be the proportion of $d$-consistent matrices, in which $x$ is occupied. If $\lambda(d,x) = 1$, we say that $x$ is *revealed to be occupied* by $d$. If $\lambda(d,x) = 0$, we say that $x$ is *revealed to be vacant* by $d$. Hereafter, we use the term "revealed" to mean "revealed to be occupied". We say that the matrix $M$ is *revealed* if it is the only matrix consistent with $d(M)$. Denote by $C(M) = \{x \mid \lambda(d(M),x)$ is maximal$\}$ the set of all entries with the highest probability of being occupied, given $d(M)$. The probability that the PS picks $x$ is $\mu(M,x) = 1/|C(M)|$ for each $x \in C(M)$ and $\mu(M,x) = 0$ for each $x \notin C(M)$.

An *equilibrium* is a matrix $M$ such that no player can increase his probability of being picked by moving to a vacant entry. Formally, let $M(x \to y)$ be the matrix derived from $M$ after switching the values of entries $x$ and $y$. The matrix $M$ is an *equilibrium* if, for each occupied entry $x$, $\mu(M,x) \geq \mu((M(x \to y),y)$ for any entry $y$ that is not occupied in $M$.

**Example 1**: Consider the following matrices with $n = 5$ (vacant rows and columns are not depicted).

$$M_1 = \begin{array}{|c|c|c|}\hline 1^* & 1 & 1 \\ \hline 1 & 0^{**} & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array} \qquad M_2 = \begin{array}{|c|c|c|}\hline 1^* & 1 & 1 \\ \hline 1 & 1 & 0 \\ \hline 0 & 0^{**} & 0 \\ \hline \end{array} \qquad M_3 = \begin{array}{|c|c|c|c|c|}\hline 1^* & 1 & 1 & 1 & 1 \\ \hline ** & *** & & & \\ \hline \end{array}$$

The matrix $M_1$ is revealed. Thus, the probability of each occupied entry being picked is $1/5$. However, $M_1$ is not an equilibrium. If the occupier of $*$ moves to $**$, the new data vector will be $((2,2,1),(2,2,1))$, which is consistent with 5 matrices, such that each of the four entries $(a,b)$ for which $(d(a),d(b)) = (2,2)$ is occupied in 4 of the 5 matrices. Each of the 4 entries $(a,b)$ for which $(d(a),d(b))$ is $(2,1)$ or $(1,2)$ is occupied in 2 of the 5 matrices. The unique entry $(a,b)$, for which $(d(a),d(b)) = (1,1)$, is occupied in only one of the $d$-consistent matrices. Thus, by moving from $*$ to $**$ the mover increases his probability of being picked from $1/5$ to $1/4$.

The matrix $M_2$ is revealed but is not an equilibrium since the player who occupies $*$ can increase his probability of being picked from $1/5$ to $1/3$ by moving to $**$.

The matrix $M_3$ is revealed and is an equilibrium. If the player at $*$ moves to an entry such as $**$ (which does not share any characteristics with the other four occupied entries), then the new data vector will be consistent with 5 matrices and in only one of them is $**$ occupied. Each of the other occupied entries is occupied in 4 of the 5 matrices consistent with the new data and thus the mover reduces his probability of being picked from $1/5$ to $0$. If the occupier of $*$ moves to an entry such as $***$ (which shares one characteristic with one other occupied entry), then the new matrix will also be revealed and the player gains nothing by moving.

*Comment*: Notice that the coordination problem which arises here is very different from that due to the lack of common language regarding the available alternatives, as discussed, for example, in Crawford and Haller (1990) and Bacharach (1993). In fact, if the PS in our model interacts with only one player then coordination is trivial since the PS receives full information about the location of the player. What makes coordination non-trivial in our model is the lack of strategic reasoning on the part of the

PS and the partial information he observes about the locations of the players, information that might not enable him to identify an occupied entry.

### 3. The Problem Solver's Behavior

In this section, we present some properties of the set of matrices that are consistent with a given data set. These properties determine the Problem Solver's "response function". In particular, we will show that either:

(1) The set of entries that are revealed to be occupied is nonempty (and consists of all entries with a score above a certain number) and the PS randomly picks one of the revealed entries;

or

(2) No entry is revealed and the PS randomly picks one of the entries with the maximal score.

**Claim 1**: Let $d$ be a consistent data vector such that $d(1) > d(2)$, where $1$ and $2$ are elements of $A$. Then, for all $b \in B$, $\lambda(d,(1,b)) \geq \lambda(d,(2,b))$. Furthermore, if there is a $d$-consistent matrix $M$ such that for some $b$, $M_{1,b} = 1$ and $M_{2,b} = 0$ , then $\lambda(d,(1,b)) > \lambda(d,(2,b))$.

**Proof**: Let $b \in B$. Fix the values for all entries other than those in columns $1$ and $2$. Partition the class $\mathbb{C}$ of all $d$-consistent matrices with these fixed values outside columns $1$ and $2$ into four cells, denoted by $\mathbf{M}(\alpha,\beta)$, $\alpha \in \{0,1\}$ , $\beta \in \{0,1\}$, such that $\mathbf{M}(\alpha,\beta)$ is the cell in this partition that consists of the matrices for which $M_{1,b} = \alpha$ and $M_{2,b} = \beta$. We will show that $|\mathbf{M}(1,0)| \geq |\mathbf{M}(0,1)|$ and if $\mathbf{M}(1,0)$ is not empty, then the inequality is strict. This is sufficient since if there is a $d$-consistent matrix $M$ such that $M_{1,b} = 1$ and $M_{2,b} = 0$, then for at least one set of entries in columns $B - \{1,2\}$ we have $\mathbf{M}(1,0) \neq \emptyset$.

We first show that if $\mathbf{M}(1,0) = \emptyset$, then $\mathbf{M}(0,1) = \emptyset$. If $\mathbf{M}(0,1)$ is not empty, then there is $M \in \mathbb{C}$ with $M_{1,b} = 0$ and $M_{2,b} = 1$. By $d(1) > d(2)$, there is a row $b'$ where $M_{1,b'} = 1$ and $M_{2,b'} = 0$. Switching all values in $\{1,2\} \times \{b,b'\}$ (i.e., changing all $0$'s to $1$'s and vice versa), we obtain another matrix in $\mathbb{C}$ which is in $\mathbf{M}(1,0)$. Therefore, if $\mathbf{M}(1,0)$ is empty, then the number of matrices in $\mathbb{C}$ in which $M_{1,b} = 1$ is the same as the number of matrices in $\mathbb{C}$ in which $M_{2,b} = 1$.

We next show that if $\mathbf{M}(1,0) \neq \emptyset$, then the number of elements in $\mathbf{M}(0,1)$ is strictly smaller than that in $\mathbf{M}(1,0)$. Define $L_{1,1}$ to be the set of rows in which the data regarding the rows implies that the missing values in columns $1$ and $2$ are $(1,1)$, and define $L_{0,0}$ in a similar manner. For the rows in $B - L_{11} - L_{00}$, the data dictates that the missing values in columns $1$ and $2$ be either $(0,1)$ or $(1,0)$. It must be that in any $\delta(1) = d(1) - |L_{1,1}|$ of these rows the values in the two columns are $(1,0)$ and in the other $\delta(2) = d(2) - |L_{1,1}|$ rows the values must be $(0,1)$. Thus, $|\mathbf{M}(1,0)| = C(\delta(1) - 1, \delta(1) + \delta(2) - 1) > |\mathbf{M}(0,1)| = C(\delta(1), \delta(1) + \delta(2) - 1)$ where $C(k,l)$ is the number of sets of size $k$ in a set of size $l$. The strict inequality follows from the fact that $\delta(1) > \delta(2)$. ■

**Claim 2**: Let $d$ be a consistent data set. Assume that $(a^*,b^*)$ maximizes the score over $A \times B$ and is not revealed. Then, for any cell $(a,b)$ that does not maximize the score $\lambda(d,(a^*,b^*)) > \lambda(d,(a,b))$.

**Proof**: By Claim 1, without loss of generality, it is sufficient to show that $\lambda(d,(a^*,b^*)) > \lambda(d,(a,b^*))$ for any $a$ for which $d(a^*) > d(a)$. Also by claim 1, it is sufficient to show that there exists a matrix $M$ consistent with $d$, such that $M_{a^*,b^*} = 1$ and $M_{a,b^*} = 0$.

Since $1 > \lambda(d,(a^*,b^*))$, there exists a $d$-consistent matrix $M$ with $M_{a^*,b^*} = 0$. Since $d(a^*) > d(a)$, there must be some $b \in B$ with $M_{a^*,b} = 1$ and $M_{a,b} = 0$.

If $M_{a,b^*} = 1$, then by switching the values in the four entries $\{a^*,a\} \times \{b^*,b\}$, we obtain a $d$-consistent $M'$ with $M'_{a^*,b^*} = 1$ and $M'_{a,b^*} = 0$.

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{b}^*$ | 0 | | 1 | | |
| | | | | | |
| $\mathbf{b}$ | 1 | | 0 | | |
| | $\mathbf{a}^*$ | | $\mathbf{a}$ | | |

If $M_{a,b^*} = 0$, then since $b^*$ maximizes $d$ over $B$, there also exists an $a'$ such that $M_{a',b^*} = 1$ and $M_{a',b} = 0$. By switching the values in $\{a^*,a'\} \times \{b^*,b\}$, we obtain a $d$-consistent matrix $M'$ with $M'_{a^*,b^*} = 1$ and $M'_{a,b^*} = 0$.

| | | | | | |
|---|---|---|---|---|---|
| $\mathbf{b}^*$ | 0 | | 0 | | 1 |
| | | | | | |
| $\mathbf{b}$ | 1 | | 0 | | 0 |
| | $\mathbf{a}^*$ | | $\mathbf{a}$ | | $\mathbf{a}'$ |

■

**Claim 3**: Let $M^*$ be a matrix such that the box $C \times R$ is occupied and the "dual" box $C^c \times R^c$ is vacant. Then, each element in $C \times R$ is revealed to be occupied and each element in the dual box is revealed to be vacant.

| | | | | | |
|---|---|---|---|---|---|
| **R** | 1 | 1 | 1 | ? | ? |
| **R** | 1 | 1 | 1 | ? | ? |
| | ? | ? | ? | 0 | 0 |
| | ? | ? | ? | 0 | 0 |
| | ? | ? | ? | 0 | 0 |
| | **C** | **C** | **C** | | |

**Proof**: Given a data vector $d$ and a set $E$ of rows or columns, let $d(E) = \sum_{e \in E} d(e)$. It must be that $d(M^*)(C^c) + d(M^*)(R^c) + |C|\times|R| = n$. On the other hand, for every matrix $M$ such that $M_{a,b} = 0$ for some $(a,b) \in C \times R$, it must be that $d(M)(C^c) + d(M)(R^c)$ $+|C|\times|R| > n$ and thus $M$ is inconsistent with $d(M^*)$. An analogous argument can be used to show that the positions in the dual box are revealed to be vacant. ■

Let $d$ be a consistent data vector. Order the elements of $A$ so that $d(a_1) \geq d(a_2) \geq \ldots \geq d(a_{|A|})$ and order the elements of $B$ so that $d(b_1) \geq d(b_2) \geq \ldots \geq d(b_{|B|})$. By Claim 1, if $(a_i, b_j)$ is revealed to be occupied, then so are all the entries in the box $\{a_1,..,a_i\} \times \{b_1,..,b_j\}$. Thus, the set of revealed entries is a "step set". That is, there is a sequence of disjoint sets of columns $A(1), A(2), \ldots, A(I)$ and a sequence of disjoint sets of rows $B(1), B(2), \ldots, B(I)$ such that (i) the set of revealed (to be occupied) entries is $\cup_{i+j \leq I+1} (A(j) \times B(i))$, (ii) the $d$-value is constant over each $A(i)$ (or $B(i)$) and (iii) the $d$-value of any entry in $A(i)$ (or $B(i)$) is larger than the $d$-value of any entry in $A(j)$ (or $B(j)$) where $j > i$. (In the illustration below the sequences are $\{a_1,a_2\},\{a_3\},\{a_4,a_5\},\{a_6\}$ and $\{b_1\},\{b_2,b_3\},\{b_4\},\{b_5\}$)).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\mathbf{b}_1$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | X |
| $\mathbf{b}_2$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | X | $Z$ |
| $\mathbf{b}_3$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | X | $Z$ |
| $\mathbf{b}_4$ | $Y$ | $Y$ | $Y$ | X | X | $Z$ | $Z$ |
| $\mathbf{b}_5$ | $Y$ | $Y$ | X | $Z$ | $Z$ | $Z$ | $Z$ |
| $\mathbf{b}_6$ | X | X | $Z$ | $Z$ | $Z$ | $Z$ | $Z$ |
| $\mathbf{b}_7$ | X | X | $Z$ | $Z$ | $Z$ | $Z$ | $Z$ |
| | $\mathbf{a}_1$ | $\mathbf{a}_2$ | $\mathbf{a}_3$ | $\mathbf{a}_4$ | $\mathbf{a}_5$ | $\mathbf{a}_6$ | $\mathbf{a}_7$ |

In what follows, we denote the step set of revealed entries by $Y = \cup_{i+j \leq I+1} A(i) \times B(j)$, the dual step set by $Z = \cup_{i+j \geq I+3} A(i) \times B(j)$ and the union of the boxes between $Y$ and $Z$ by $X = \cup_{i+j=I+2} A(i) \times B(j)$.

**Claim 4**: Let $d$ be a consistent data vector. Assume that the set of revealed elements $Y$ is not empty. Then, every element in the dual step set $Z$ is revealed to be vacant.

**Proof**: Given a consistent data vector $d$, the Gale-Ryser algorithm (see Gale (1957), Ryser (1957) and Krause (1996)) ends with a $d$-consistent matrix. The algorithm is sequential and starts with a certain initial matrix. In each step of the algorithm, a permissible pair of entries that are positioned in the same row – one occupied and the other vacant – is selected and their values are swapped. To be precise, order the elements in each of the sets $A$ and $B$ according to their $d$-values $a_1, \ldots, a_{|A|}$ and $b_1, \ldots, b_{|B|}$. The algorithm starts with a matrix $M_0$ in which, for any row $b$, $1$'s are assigned to the entries in columns $a_1, \ldots, a_{d(b)}$ of this row. For $M_0$, the number of $1$'s in column $a_k$ is $z(k) = |\{b \mid \text{the number } d(b) \geq k\}|$. Obviously, $\sum_{i=1,..,k} z(i) \geq \sum_{i=1,..,k} d(a_i)$ for all $k$. In each step of the algorithm, a "$1$" in the lowest index column for which the number of $1'$s is strictly greater than $d(a)$ is moved to the first column $a'$ in which the number of $1'$s is strictly less than $d(a')$.

In his proof that the algorithm ends with a $d$-consistent matrix, Krause (1996) shows that the algorithm works by starting with any matrix having the following two properties: (i) The number of 1's in each $b \in B$ is $d(b)$ and (ii) for each $k$, the sum of the 1's in the first $k$ columns is at least as large as $\sum_{i=1,..,k} d(a_i)$.

Note that if $\sum_{i=1,..,j^*} z(j) = \sum_{i=1,..,i^*} d(a_i)$, then for every $d$-consistent matrix the entries in $\{a_1,..,a_{i^*}\} \times \{b_1,..,b_{j^*}\}$ must be occupied and the entries in $(A - \{a_1,..,a_{i^*}\}) \times (B - \{b_1,..,b_{j^*}\})$ must be vacant.

Thus, to prove Claim 4, it is sufficient to show that, for all $l$, $\sum_{i=1,..,|A(1)|+...+|A(l)|} z(i) = \sum_{i\in A(1)\cup...\cup A(l)} d(a_i)$. Assume not. Then, for some $l$ we have $\sum_{i=1,..,|A(1)|+...+|A(l)|} z(i) > \sum_{i\in A(1)\cup...\cup A(l)} d(a_i)$. Let $k^*$ be the lowest $k > |A(1)|+\ldots+|A(l)|$ for which $\sum_{i=1,..,k} z(i) = \sum_{i=1,..,k} d(a_i)$. It must be that $d(b_{|B(1)|+...+|B(I+1-l)|}) < k^*$. Otherwise, $z(b_{|B(1)|+...+|B(I+1-l)|}) \geq k^*$ and in any $d$-consistent matrix all entries in $\{a_1,..,a_{k^*}\} \times \{b_1,..,b_{|B(1)|+...+|B(I+1-l)|}\}$ are occupied, in contradiction to the definition of $|A(l)|$. Now start the Gale-Ryser algorithm from a matrix that modifies $M_0$ by moving a single "1" from the entry $(a_{|A(1)|+...+|A(l)|}, b_{|B(1)|+...+|B(I+1-l)|})$ to the empty entry $(a_{k^*}, b_{|B(1)|+...+|B(I+1-l)|})$. The matrix satisfies properties (i) and (ii) and thus the algorithm starting with the modified matrix leads to a $d$-consistent matrix in which one of the revealed entries is 0, a contradiction. ■

## 4. Equilibrium

In this section, we show that in all equilibria the PS picks an occupied entry with probability 1 (Proposition 1) and we classify the structure of all equilibria (Proposition 2). Even though in every equilibrium the PS picks an occupied entry for certain, there are equilibria in which the PS assigns a strictly positive probability to the possibility that the position he is picking is vacant (Proposition 3).

**Proposition 1**: In equilibrium, the PS picks an occupied entry with probability 1.

**Proof**: Let $M$ be an equilibrium. If there are revealed positions, then the PS obviously chooses one of them. If no entry is revealed, then, by Claim 2, $C(M)$ is the box of all the entries that maximize the $d(M)$ score. It is left to show that all entries in $C(M)$ are occupied.

If an entry $(a,b)$ outside $C(M)$ is occupied and $(a^*,b^*) \in C(M)$ is vacant, then the move from $(a,b)$ to $(a^*,b^*)$ is beneficial: the score of $(a^*,b^*)$ in $M((a,b) \to (a^*,b^*))$ increases by at least 1 relative to the score in $M$ and the score of any other entry increases by at most 1. Thus, $(a^*,b^*)$ maximizes the score after the move, and by

Claim 1 it will be picked with a positive probability.

If all the occupied entries are in $C(M)$ and it contains a vacant entry, then any vertical move from an occupied entry $(a^*,b)$ to a vacant entry $(a^*,b^*)$ is beneficial since: (i) $(a^*,b^*)$ maximizes the score after the move, (ii) $C(M((a^*,b) \to (a^*,b^*))) \subseteq C(M)$ and (iii) the entry $(a^*,b)$ is obviously not revealed and, by Claim 2, it is excluded from $C(M((a^*,b) \to (a^*,b^*)))$. ■

**Proposition 2**: If $M$ is an equilibrium, it must have one of the following three structures:

(1) The matrix $M$ is revealed and forms a "step set". The PS picks one of the occupied positions.

(2) There is a "step set" of entries that are revealed as occupied and the PS picks one of them. The "dual step set" is revealed to be vacant. Each box lying between these two sets contains at least three occupied entries that are not picked by the PS.

(3) None of the positions are revealed to be occupied. All positions with maximal score are occupied and the PS picks one of them.

**Proof**: The fact that the set of revealed entries forms a step set follows from Claim 1. By Claim 4, the dual step set is revealed to be vacant.

Assume that $M$ is an equilibrium in which some, but not all, occupied entries are revealed. We will show that any box $A' \times B'$ in area $X$ (i.e., the area consisting of all the positions between $Y$, the set of entries that are revealed as occupied, and $Z$, the set of entries that are revealed to be vacant) contains at least three occupied entries that do not share any characteristic (namely, they are positioned in three different rows and three different columns). If $A' \times B'$ is entirely vacant, then a move from an unrevealed occupied entry into this empty box will be beneficial to the mover since it will reveal this entry to be occupied (by Claim 3). It is impossible that all occupied entries in this box lie in the same row or the same column since (again, by Claim 3) they would then be revealed. It is also impossible that in equilibrium all occupied entries in this box lie in exactly two rows (or two columns) since if $(a_1,b_1)$ and $(a_2,b_2)$ in that box are occupied then a move from $(a_1,b_1)$ to $(a_1,b_2)$ is beneficial since the deviator will be revealed (once again, by Claim 3).

Finally, if none of the positions are revealed as occupied, then, by claim 2, $C(M)$

contains all of the positions with the highest score and, by the proof of Proposition 1, all of the entries in $C(M)$ must be occupied. ■

An interesting feature of the model is that although in equilibrium the PS always picks an occupied position, there are equilibria in which he does not know that the position is occupied. Such a phenomenon could not occur under the conventional equilibrium assumption but may emerge in our setup.

**Proposition 3**: For $n \geq 10$, there exists an equilibrium in which the PS picks an occupied entry although it is not revealed to be occupied.

**Proof**: The matrix $M_4$ demonstrates such an equilibrium for $n = 10$. The example can be modified for any $n > 10$ by "extending" the "arms of the top-left L".

$$M_4 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

No entry is revealed by $d(M_4)$ and the PS chooses the top-left position $x^*$ which achieves the maximum score of $8$, whereas any other entry achieves a score of at most 5. (The PS assigns a probability of 2400/2850 $\sim 0.85$ to $x^*$ being occupied.) For any occupied entry $x \neq x^*$ and unoccupied entry $y$, no entry is revealed in $M_4(x \to y)$ and $x^*$ remains the entry with the highest score in $M_4(x \to y)$ (being equal to $7$ or $8$ whereas the score of $y$ would not be higher than $6$). By Claim 2, $x^*$ would still be chosen by the PS given the data $d(M_4(x \to y))$. Thus, no deviation is profitable. ■

**Comment**: Were the PS to take into account that the $n$ players' profile of choices is consistent with Nash equilibrium, he would conclude that the entry he is choosing is occupied. However, the whole point of modeling an agent as a problem solver is that he does not think strategically. Rather, he treats the problem as a puzzle to be solved, using the information he possesses, without making any assumptions about the various solutions to the puzzle.

## 5. Two variants of the model

### (a) **Players can share entries**

Thus far we have assumed that none of the players want to share an entry with another player regardless of whether or not the entry is picked by the PS. Assume now that players do not have disutility from sharing an entry with another player. Once the PS has chosen an occupied cell he randomly picks one of the occupiers and each player is interested in increasing the probability of being picked. Formally, a player's utility is $1/k$ if he shares the PS's chosen cell with $k-1$ other players and $0$ otherwise. A matrix is an assignment of non-negative integers (not necessarily zeros or ones) to all entries, such that the sum of the numbers in all entries is $n$. For each row (column), the PS receives information about the total number of *players* occupying that *row (column)*. The PS identifies the matrices that are consistent with the data and then randomly picks an entry with the largest number of consistent matrices in which this entry is occupied (by at least one player). For example, the data $d(a_1) = d(b_1) = 3$ and $d(a_2) = d(b_2) = 1$ is consistent with two matrices:

$\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 0 \\ \hline \end{array}$ and $\begin{array}{|c|c|} \hline 3 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$ .

Only the top-left entry is occupied in both matrices and therefore it is picked by the PS.

An equilibrium is a matrix such that a player who occupies one entry cannot increase his utility by moving to another. Formally, in equilibrium there is no occupied entry $x$ and (not necessarily unoccupied) entry $y$ such that $\frac{\mu((M(x\to y),y)}{M(y)+1} > \frac{\mu(M,x)}{M(x)}$ where $M(x \to y)$ is the matrix in which $x$ is decreased by $1$ and $y$ is increased by $1$.

Obviously, any matrix in which the $n$ players occupy $n$ entries in one row or in one column is an equilibrium (since the move of a player to another occupied entry reduces his utility from $1/n$ to $1/(2n-2)$ and the move of a player to an unoccupied entry cannot increase the probability of being picked by the PS).

We are able to prove that any other matrix is not an equilibrium (with the exception of $n = 2$ with the two players occupying the same entry). The proof is not presented here but can be obtained from the authors.

(b) **Players don't want to be picked by the PS**

Suppose that none of the $n$ players wishes to be "detected" by the PS. More precisely, each player receives a utility of $1$ if he discoordinates with the PS, $0$ if his choice matches that of the PS and negative utility if he chooses the same entry as one of the other players.

Any diagonal matrix $M$, in which each player chooses a distinct row and a distinct column, is an equilibrium. Given $d(M)$, the PS will select each of the $n^2$ entries with the score $2$ and thus a player's probability of being detected by the PS is $1/n^2$.

Consider the matrix $M(x \to y)$ that results from the move of a player from an occupied entry $x$ to an unoccupied entry $y$, where $y$ is an entry that shares a row or a column with one of the occupied entries in $M$. The score of $y$ in $M(x \to y)$ is maximal and thus it is one of at most $n(n-1)$ entries in $C(M(x \to y))$. Thus, the move from $x$ to $y$ increases a player's probability of being detected.

In any other matrix $M$, the minimal box that contains all occupied entries is of size $kl$, where $k \leq n$ and $l \leq n$, with at least one of the inequalities being strict. Then, there is an occupied entry in $M$ with a score strictly greater than $2$. Any player who occupies an entry $x$ with a maximal score will benefit by moving to an entry $y$, which does not share a row or column with any other player, since such a move reduces his chances of being picked by the PS from $\frac{1}{kl}$ to at most $max\{\frac{1}{(k+1)l}, \frac{1}{k(l+1)}\}$.

**References**

Bachrach, Michael J. (1993). "Variable Universe Games" in K. Binmore, A. Kirman and P. Tani (eds.), *Frontiers of Game Theory*. Cambridge, Mass: MIT Press.

Gale, David. (1957). "A Theorem on Flows in Networks". *Pacific Journal of Mathematics, 7*, 1073–1082.

Krause, Manfred. (1996). "A Simple Proof of the Gale-Ryser Theorem". *The American Mathematical Monthly*, *103*, 335-337.

Russell, Stuart J. and Peter Norwig. (2009). *Artificial Intelligence: A Modern Approach*. Englewood Cliffs: Prentice Hall.

Ryser, Herbert. J. (1957). "Combinatorial properties of matrices of zeros and ones". *Canadian Journal of Mathematics, 9*, 371–377.