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EQUILIBRIUM IN SUPERGAMES

by

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1. Introduction

There are significant differences between the situation of players undertaking to play a single game, and players who know that they will play the same game repeatedly in the future. Strategy in the first case is a single play; in the second, it is a sequence of rules, each one of which pertains to the outcomes preceding it. The preferences of the participants are determined partly by temporal considerations. The participants may adopt risky strategies, "protected" by threats of retribution in the future.*

Analysis of a finite sequence of identical games shows that this model is inadequate for examining the idea of repeated games. If the number of games is finite and known initially, the players will treat the last game as if it were a single game. Thus the threats implicit in the game before last are proven to be false threats. Therefore the game before last will be treated as a single game, and so on. Thus the situation we wish to describe is not expressed by such a sequence. (For a detailed analysis, see [9].)

In order to avoid "end-points" in the model, we define a supergame. A supergame is an infinite sequence of identical games, together with the players' evaluation relations. (That is their preference orders on utility

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* This paper was written as a Master's Thesis at the Hebrew University of Jerusalem, under the supervision of Professor B. Peleg. I wish to thank Professor Peleg for his advice and guidance.

sequences.) Obviously the assumption of an infinite planning horizon is unrealistic, but it is an approximation to the situation we wish to describe.

The literature deals mainly with comparison of equilibrium concepts in supergames and single games. (See Aumann [1] and [2]. The results are derived more simply in [5]; see also [3] and [4].) Other papers emphasize the uses of the concept of supergames in economics ([7], [8]).

In most of the papers, it was assumed that the participants evaluate the utility flows according to the criterion of the limit of the means of the flows. (But see [7].) The drawback of this evaluation relation is that it ignores any finite time interval.

One of the aims of this paper is to extend the discussion to supergames with different evaluation relations. In particular, to supergames with evaluation relations determined according to the "overtaking criterion". (The sequence $\{a_i\}_{i=1}^{\infty}$ is preferred to the sequence $\{b_i\}_{i=1}^{\infty}$ if $0 < \underline{\lim} \sum_{i=1}^{n} a_i - b_i$. See §6.)

The formal model, described in §2, is taken from Roth.[1/] The single game is given in strategic form (see [5]). This representation permits discussion or matrix games, when the set of strategies is the set of mixed strategies. In this treatment, the following simplifying assumptions are made:

i) Every participant has complete information about the others' plans.

ii) The evaluation of strategies in supergame is a function of the evaluation of the outcomes of the individual games.

iii) The players do not randomize on the strategies.

1/ I wish to thank Professor A.E. Roth for permitting me to use the model described in [10].

A (Nash) equilibrium is an n-tuple of supergame strategies (n players), such that no player may alone deviate profitably from his strategy. A stationary strategy is a supergame strategy which, if adhered to by all players, will produce identical outcomes for every game played. In §3, the stationary equilibrium points will be characterized by a "two-stage" finite game in which the time element is reduced to "present" and "future".

The "power" of the threats make possible many equilibrium points. We can add some other reasonable requirements. An equilibrium point will be called perfect if after any possible "history", the strategies planned are an equilibrium point. In other words no player ever has a motivation to change his strategy. This will be treated in §4.

A complete characterization is obtained for supergames with the limit of means evaluation relation. A partial characterization is obtained for supergames with evaluation according to the overtaking criterion. The requirement of perfection does not alter the outcomes of stationary equilibria in the first case, and only marginally in the second.

A strong equilibrium is an n-tuple of strategies where no coalition of players can alter their strategies to bring profit to all members of the coalition.

In §5, the concept of desirable payoff is defined for a single game.

The strong perfect equilibrium points are characterized in a supergame with evaluation relations determined by the limit of means criterion. An example is given of such a game with a strong perfect equilibrium, but in which, when the evaluation relations are determined by the overtaking criterion, none exist.

2. The Model

i) The single game G is a game in strategic form

$$G = \langle \{S_i\}_{i=1}^n , \{\pi_i\}_{i=1}^n \rangle \quad .$$

The set of players is $N = \{1,\ldots,n\}$. For each $i \in N$, the set of strategies of i is S_i ; S_i is assumed non-empty and compact. $S = \prod_{i=1}^n S_i$ is the set of outcomes. An element in S will be called an outcome of G . The preference relations of the player i are defined by utility function $\pi_i : S \to R$ (R - the reals), which are continuous in the product topology.

Given $\sigma \in S$, a payoff vector is the n-tuple $\pi(\sigma) = \langle \pi_1(\sigma),\ldots,\pi_n(\sigma) \rangle$. For convenience we will denote the n-1-tuple $\langle \sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n \rangle$ by σ^{-i} , and the n-tuple σ by $\langle \sigma^{-i},\sigma_i \rangle$. σ will be called a (Nash) equilibrium if for all i and for all $s_i \in S_i$,

$$\pi_i(\sigma^{-i},s_i) \leq \pi_i(\sigma) \quad .$$

If the set of strategies is finite, and it is possible to adopt mixed strategies we can identify S_i with the set of mixed strategies, and π_i with the expected payoff of i . Examples in a similar context may be found in [7] and [8].

ii) The supergame, G^∞ is $\langle G, \prec_1,\ldots,\prec_n \rangle$ where G is a single game and the $\prec_i$ are evaluation relations on real number sequences; more exactly, $\prec_i$ is a binary relation on $\pi_i(S)^N$ 2/ where $\pi_i(S)$ is the range of π_i on S . $\prec_i$ will be transitive, anti-symmetric, but not necessarily a total order.

1/ Let A be a set. A^N is the set of sequences of elements in A .

The set of outcomes at time t , $S(t)$, is S . A strategy for i in G^∞ is a set $\{f_i(t)\}_{t=1}^\infty$, where $f_i(1) \in S(1)$, and for $t \geq 2$, $f_i(t) : \prod_{j=1}^{t-1} \beta(j) \to S_i$. Thus a supergame strategy is a choice of strategies at every stage, possibly dependent on the outcomes preceding the choice. We assume all players know all the choices made by all the players in the past.

The set of strategies of i will be denoted by F_i . F is the set of n-tuples of strategies; $F = \prod_{i=1}^{n} F_i$.

Given $f \in F$, the outcome at time t will be denoted by $\sigma(f)(t)$, and is defined inductively by

$$\sigma(f)(1) = (f_1(1),\ldots,f_n(1))$$
$$\sigma(f)(t) = (\ldots,f_i(t)(\sigma(f)(1),\ldots,\sigma(f)(t-1)),\ldots) \quad .$$

We will define an evaluation relation $\bar{\prec}_i$ on F , induced by $\prec_i$, as follows:

For all $f,g \in F$, $f \bar{\prec}_i g$ if and only if

$$\{\pi_i(\sigma(f)(t))\}_{t=1}^\infty \prec_i \{\pi_i(\sigma(g)(t))\}_{t=1}^\infty \quad .$$

Given $f \in F$, we will denote $(f_1,\ldots,f_{i-1},f_{i+1},\ldots,f_n)$ by f^{-i} , and f by (f^{-i},f_i) . $f \in F$ is a (Nash) equilibrium in the supergame G^∞ if for all i , and for all $h_i \in F_i$, $f \not\bar{\prec}_i (f^{-i},h_i)$.

$f \in F$ is stationary if there exists $\sigma \in S$ such that for all t , $\sigma(f)(t) = \sigma$. If $f \in F$ is stationary we will denote the corresponding σ by $\hat{\sigma}(f)$. Note that, in contrast to accepted definitions, the stationary strategies produce constant outcomes (as in [10]).

Let us examine various evaluation relations. If A is a bounded set of real numbers, the following are evaluation relations on A^N .

i) Limit of means evaluation relation, defined by

$$x \prec y \quad \text{iff} \quad \underline{\lim} \frac{\sum_{t=1}^{n} x_t}{n} < \underline{\lim} \frac{\sum_{t=1}^{n} y_t}{n} .$$

ii) Overtaking criterion evaluation relation, defined by

$$x \prec y \quad \text{iff} \quad \underline{\lim} \sum_{i=1}^{n} y_i - x_i > 0 .$$

iii) The evaluation relation with discount parameter $0 < \delta < 1$, defined by

$$x \prec y \quad \text{iff} \quad \sum_{t=1}^{\infty} \delta^t x_t < \sum_{t=1}^{\infty} \delta^t y_t .$$

iv) The evaluation relation determined solely by the present:

$$x \prec y \quad \text{iff} \quad x_1 < y_1 .$$

We do not assume yet that the players are characterized by a single evaluation relation. We will merely assume that the evaluation relations are reasonable, that is, satisfy:

(A.1) If for all t, $y_t \equiv y_0$, and $x_t \equiv x_0$, then $x_0 < y_0$ implies $x \prec y$.

(A.2) If $z \prec x$, and $x \leq y$, then $z \prec y$. [1/]

(A.3) If there exists $a \in A$ such that $(a,x_1,x_2,\ldots) \prec (a,y_1,y_2,\ldots)$, then $x \prec y$.

All four evaluation relations described satisfy these conditions.

1/ If $x,y \in A^N$, we will write " $x \leq y$ " for "for all $t \in N$, $x_t \leq y_t$ ".

3. Characterization of Stationary Equilibria

This section has two aims. The first, to characterize the stationary equilibria, using the equilibria of a "finite" two-stage game, derived from G^{∞}. We will denote this game G^2 ; it will be a twofold repetition of G. A strategy in G^2, contains decisions about the "present", the first game, and the "future", the second game. The latter decision depends on the outcome of the former.

The second aim is to examine the verisimilitude of the formal model in representing human behavior characterized by the expectation which the players have that after every game they will play further games. We will do this at least as far as characterization of stationary equilibria is concerned.

We will now define the derived game, G^2. A strategy for a player i in G^2 is a pair $\langle f_i(1), f_i(2) \rangle$ where $f_i(1) \in S_i$, and $f_i(2) : S(1) \to S_i$. We will denote the strategies of i by F_i^2, and write $F^2 = \prod_{i \in N} F_i^2$.

We define a partial order $\prec_i^2$ on $\pi_i(S) \times \pi_i(S)$ as follows:

$$(b_1, b_2) = b \prec_i^2 a = (a_1, a_2) \quad \text{iff}$$

1) $b_1 < a_1$

2) $(b_1, b_2, b_2, b_2, \ldots) \prec_i (a_1, a_2, a_2, a_2, \ldots)$,

where $\prec_i$ is the evaluation relation of a player i in G^{∞}.

The outcome of G^2, where players adopt strategies $f \in F^2$, we define inductively by:

$$\sigma(f)(1) = (f_1(1),\ldots,f_n(1))$$

$$\sigma(f)(2) = (f_1(\sigma(f)(1)),\ldots,f_n(\sigma(f)(1))) \ .$$

$f \in F^2$ will be called stationary if there exists $\sigma \in S$ such that $\sigma(f)(1) = \sigma(f)(2) = \sigma$. Such a σ will be denoted by $\hat{\sigma}(f)$.

$f \in F^2$ will be called an equilibrium in G^2 if there does not exist i and $g_i \in F_i^2$ such that

$$(\pi_i(\sigma(f)(t)))_{t=1}^2 \prec_i^2 (\pi_i(\sigma(f^{-i},g_i)(t)))_{t=1}^2$$

Examples:

1) If $\prec$ is the limit of means evaluation relation, the relation $\prec^2$ induced by $\prec$ in G^2 is:

$$b \prec^2 a \quad \text{iff} \quad b_1 < a_1 \quad \text{and} \quad \underline{\lim}\left(\frac{a_1 + (n-1)a_2}{n} - \frac{b_1 + (n-1)b_2}{n}\right) > 0$$

$$\text{iff} \quad b_1 < a_1 \quad \text{and} \quad b_2 < a_2 \quad (a,b \in R^2) \ .$$

2) If $\prec$ is the overtaking evaluation relation,

$$b \prec^2 a \quad \text{iff} \quad b_1 < a_1 \quad \text{and} \quad b_2 \le a_2 \ .$$

3) If $\prec$ is the evalution relation with discount parameter $0 < \delta < 1$, then

$$b \prec^2 a \quad \text{iff} \quad b_1 < a_1 \quad \text{and} \quad b_1 + \frac{\delta}{1-\delta} b_2 < a_1 + \frac{\delta}{1-\delta} a_2 \ .$$

Remark: For a supergame $G^\infty = \langle G, \prec_1, \ldots, \prec_n \rangle$ where for all $1 \le i \le n$ the evaluation relations are reasonable, and where there exists an equilibrium σ in G , define $f \in F$ by: "For all i and for all t ,

$f_i(t) = \sigma_i$." Clearly f is a stationary equilibrium in G^∞ (even perfect; see the definition in §4), satisfying $\hat{\sigma}(f) = \sigma$. Thus in a supergame where the individual game has an equilibrium, we are guaranteed the existence of an equilibrium.

Proposition 3.1: Let $G^\infty = \langle G, \prec_1 \ldots, \prec_n \rangle$ be a supergame where the $\prec_i$ are reasonable evaluation relations. If there exists $g \in F^2$, a stationary equilibrium in G^2 , the derived game, such that $\hat{\sigma}(g) = \sigma$ then there exists $f \in F$, f a stationary equilibrium in G^∞ , such that $\hat{\sigma}(f) = \sigma$.

Proof: Given $s \in S$, let $r_i(s) \in S_i$ satisfy

$$\pi_i(r_i(s), s^{-i}) = \max_{t_i \in S_i} \pi_i(t_i, s^{-i})$$

(S_i is compact and π_i is continuous).

We will first show that there exists a "punishing strategy" $\gamma^i \in S$ satisfying

$$(\pi_i(\sigma), \pi_i(\sigma)) \not\prec_i^2 (\pi_i(r_i(g(1)), g^{-i}(1)), \pi_i(r_i(\gamma^i), \gamma_i^{-i})) .$$

Let $h_i \in F_i^2$, satisfying

$$h_i(1) = r_i(g(1)) .$$

$$h_i(2)(g^{-i}(1), h_i(1)) = r_i(g(2)(g^{-i}(1), h_i(1))) .$$

Assuming there is no such γ^i we have

$$(\pi_i(\sigma(g^{-i}, h_i)(t)))_{t=1}^2 \succ_i^2 (\pi_i(\sigma), \pi_i(\sigma)) = (\pi_i(\sigma(g)(t)))_{t=1}^2 ,$$

contradicting the fact that g is a G^2 equilibrium. Let $\gamma^i \in S$ be "punishing strategies". We may assume that if $\pi_i(r_i(\sigma), \sigma^{-i}) \le \pi_i(\sigma)$, then $\gamma^i = \sigma$.

Define $f_i \in F_i$ for $i \in N$ as follows:

$$f_i(1) = \sigma_i$$

$$f_i(t)(s(1)\ldots s(t-1)) = \begin{cases} \gamma_i^j & \text{if there exists } T \leq t-1 \\ & \text{such that } s(1) = \ldots = s(T+1) = \sigma \\ & \text{and } s^{-j}(T) = \sigma^{-j} \\ & \text{and } s_j(T) \neq \sigma_j \\ \sigma_i & \text{otherwise .} \end{cases}$$

Then $\hat{\sigma}(f) = \sigma$. f is a G^∞ equilibrium, since for all $h_i \in F_i$

1) If $\pi_i(r_i(\sigma), \sigma^{-i}) \leq \pi_i(\sigma)$, then $\pi_i(\sigma) \geq \pi_i(\sigma(g^{-i}, h_i)(t))$ for all t , and thus according to (A.2) and the irreflexivity following from the asymmetry, $f \not\succ_i (f^{-i}, h_i)$

2) If $\pi_i(\sigma) < \pi_i(r_i(\sigma), \sigma^{-i})$, let t_o be the minimum satisfying $f_i(t_o)(\sigma,\sigma,\ldots,\sigma) \neq \sigma_i$ whenever [1/]

$$\{\pi_i(\sigma(f^{-i}, h_i)(t))\}_{t=1}^{\infty} \succ_i (\underset{\sim}{\pi_i(\sigma)})$$

Repeated application of (A.3) yields

$$\{\pi_i(\sigma(f^{-i}, h_i)(t))\}_{t=t_o}^{\infty} \succ_i (\underset{\sim}{\pi_i(\sigma)})$$

But $\pi_i(r_i(\sigma), \sigma^{-i}) \geq \pi_i(\sigma(f^{-i}, h_i)(t_o))$ and for $t > t_o$,

$\pi_i(r_i(\gamma^i), \gamma^{i^{-i}}) \geq \pi_i(\sigma(f^{-i}, h_i)(t))$.

[1/] We use the notation: if $a,b \in \pi(S)$, $\underset{\sim}{a} = (a,a,a\ldots)$, and $(a,\underset{\sim}{b}) = (a,b,b,b,\ldots)$

Thus applying (A.2)

$$(\pi_i(r_i(\sigma), \sigma^{-i}), \underset{\sim}{\pi_i(r_i(\gamma^i), \gamma^{i^{-i}}}))\succ_i (\underset{\sim}{\pi_i(\sigma)})$$

contradicting the choice of γ^i .

Proposition 3.2: Let $G^\infty = \langle G, \prec_1, \ldots, \prec_n \rangle$ and $\prec_i$ reasonable evaluation relations. If $f \in F$ is a stationary equilibrium in G^∞ , and $\hat{\sigma}(f) = \sigma$, then there exists a G^2 stationary equilibrium, $g \in F^2$, such that $\hat{\sigma}(g) = \sigma$.

Proof: Suppose not. Then there is an i and $\tau_i \in S_i$ satisfying:

1) $\pi_i(\sigma^{-i}, \tau_i) > \pi_i(\sigma)$

2) For all $s \in S$, there are $t_i \in S_i$ such that

$$(\pi_i(\sigma^{-i}, \tau_i), \underset{\sim}{\pi_i(t_i, s^{-i})}) \succ_i (\underset{\sim}{\pi_i(\sigma)}) .$$

Thus, applying (A.2), $(\pi_i(\sigma^{-i}, \tau_i), \underset{\sim}{\min_{s^{-i}} \max_{t_i} \pi_i(t_i, s^{-i})}) \succ_i (\underset{\sim}{\pi_i(\sigma)})$.

Define h_i , a strategy in G^∞ , by

$$h_i(1) = s_i$$

$$h_i(t)\ (s(1)\ldots s(t-1)) = r_i(f(t)\ (s(1)\ldots s(t-1))).$$

Applying (A.2), we obtain

$$\{\pi_i(\sigma(f^{-i}, h_i)(t))\}_{t=1}^\infty \succ_i \{\underset{\sim}{\pi_i(\sigma)}\}$$

in contradiction to f being an equilibrium.

Definition: $s \in S$ is a weakly forced outcome in G if, for all i , there is $r \in S$ such that for all $t_i \in S_i$, $\pi_i(r^{-i}, t_i) \leq \pi_i(s)$.

Thus in a weakly forced outcome each player's payoff is at least as large as the amount the other players can force on him, that is at least

$\min_{r \in S} \max_{t_i \in S_i} \pi_i(r^{-i}, t_i)$.

<u>Definition</u>: $s \in S$ is a strongly forced outcome in G if for all i , and for all $t_i \in S_i$, $\pi_i(s^{-i}, t_i) \leq \pi_i(s)$, or there exists $r \in S$ such that for all $t_i \in S_i$, $\pi_i(r^{-i}, t_i) < \pi_i(s)$.

Thus a strongly forced outcome is one where any one player making a "profitable" deviation may have a loss forced on him by the other players; that is if $\max_{t_i \in S_i} \pi_i(t_i, s^{-i}) > \pi_i(s)$, then $\min_{r \in S} \max_{t_i \in S_i} \pi_i(r^{-i}, t_i) < \pi_i(s)$.

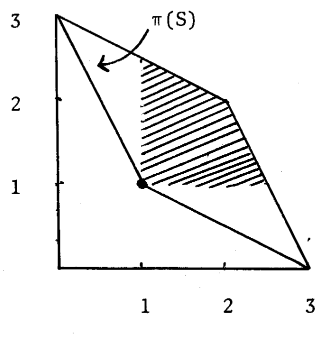
<u>Example</u>:

Let S_i be the set of mixed strategies of i , $i = \{1,2\}$. In a matrix game with payoff matrix

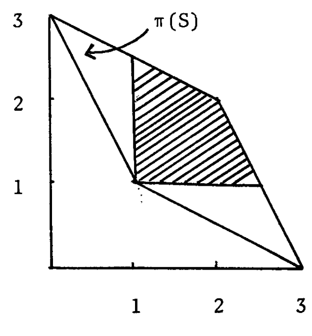
2,2	0,3
3,0	1,1

π_i is the expected payoff of i .

Strongly forced outcomes



Weakly forced outcomes



Proposition 3.3: 1) If $\prec_i$ is the limit of means evaluation relation in G^∞, the stationary outcomes of equilibria are the weakly forced outcomes.

2) If $\prec_i$ is the overtaking evaluation relation the stationary outcomes of equilibria are the strongly forced outcomes.

3) If $\prec_i$ is the evaluation relation according to the present, then the stationary outcomes of equilibria are the Nash equilibria.

Proof: 3) is immediate. 2) is similar to 1), which will now be proved.

There exists $g \in F$, a stationary G^∞ equilibrium, such that $\hat{\sigma}(g) = \sigma$, iff there exists $f \in F^2$ a stationary G^2 equilibrium such that $\hat{\sigma}(f) = \sigma$; and this holds iff for all $h_i \in F_i^2$,

$$(\sigma,\sigma) \not\prec_i^2 (\sigma(h_i, f^{-i})(t))_{t=1}^2$$

i.e. iff for all i, either $\pi_i(r_i(\sigma),\sigma^{-i}) \le \pi_i(\sigma)$ or

$$\pi_i(r_i(\gamma^i),\gamma^{i^{-i}}) = \underline{\lim} \frac{\pi_i(r_i(\sigma),\sigma^{-i}) + (n-1)\pi_i(r_i(\gamma^i),\gamma^i)}{n} \le \pi_i(\sigma) ;$$

and this will hold iff for all i, there exists $\gamma^i \in S$ such that $\pi_i(r_i(\gamma^i),r^{i^{-i}}) \le \pi_i(\sigma)$.

Proposition 3.4: Let $G^\infty = <G, \prec_1,\dots,\prec_n>$ be a supergame with reasonable evaluation relations. A necessary condition for $f \in F$ to be a stationary equilibrium is that $\hat{\sigma}(f)$ is a weakly forced outcome.

Proof: By proposition 3.2, there exists $g \in F$, a stationary equilibrium in G^2 such that $\hat{\sigma}(g) = \hat{\sigma}(f)$. If $\hat{\sigma}(f)$ is not weakly forced, there exists i such that for all $s \in S$,

$$\pi_i(r_i(s),s^{-i}) > \pi_i(\hat{\sigma}(f)) \ .$$

S is compact. Thus there exists $\varepsilon > 0$ such that

$$\pi_i(r_i(s),s^{-i}) > \pi_i(\hat{\sigma}(f)) + \varepsilon \ .$$

Define $h_i \in F_i^2$ by:

$$h_i(1) = r_i(\hat{\sigma}(f))$$

$$h_i(2) = r_i(g(1)(\hat{\sigma}^{-i}(f),r_i(\hat{\sigma}(f)))) \ .$$

Applying (A.1), we obtain:

$$(\pi_i(\sigma(g^{-i},h_i)(t))_{t=1}^2 \succ_i^2 (\pi_i(\hat{\sigma}(f)),\pi_i(\hat{\sigma}(f))) \ ,$$

in contradiction to g being a G^2 equilibrium.

Remark: Let us return to the question raised at the beginning of this section. In G^2, each player considers whether to deviate from the stationary norm, taking in account future plays. The decision is far simpler than in supergame, where the relevant factors range over an infinite future. For example, a player deviating at any one time only if he will derive certain profit in the forseable future, behaves according to the evaluation relation induced in G^2 by the limit of means relation. A player deviating only if he will not "lose" in the future, behaves according to the relation in G^2 induced by the overtaking criterion, and if there exists $0 < \varepsilon$ such that a player will deviate in the future iff the difference between his present profit and future loss exceeds ε, $\left(\frac{a_1-b_1}{b_2-a_2} > \varepsilon \Leftrightarrow a_1 + \varepsilon a_2 > b_1 + \varepsilon b_2\right)$ then then the evaluation relation in G^2 corresponding to this behavior is that induced by the evaluation relation with discount parameter $\delta = \frac{\varepsilon}{1+\varepsilon}$.

Thus at least as far as the equilibria are concerned, the conclusions arrived at in supergame and G^2 are identical. It appears that the relation induced in G^2 by the overtaking criterion evaluation relation is more acceptable intuitively than that induced by the limit of means, pointing to the superiority of the former.

Remark: As has previously been mentioned, by identifying mixed strategies and the expected payoff with S_i and π_i, matrix games are covered by the theory so far.

In this model, a strategy $f \in F$ is preferred to $g \in F$, if i 's sequence of utility contents "induced" by f is $\prec_i$ - better than that "induced" by g. Alternative definitions are possible, as in [1]. Thus a possible definition exending the concept of equilibrium is:

$f \in F$ is preferred to $g \in F$ as far as i is concerned if his payoff sequence resulting from adoption of f is prefereable with probability 1 to the expectation sequence resulting from adoption of g. (Compare with [5]).

In the following example, both players adopt an evaluation relation according to the overtaking criterion. The payoffs matrix is:

1,0	1,0
2,0	0,2
0,2	2,0

We will see that there is an equilibrium point in the single game which is a stationary outcome of an equilibrium point according to our definition, but not according to the second definition. The strategy (0,1/2,1/2) for the row player and (1/2,1/2) for the column player is an equilibrium, and the outcome is strongly forced (neither player can profitably deviate). Thus it is a stationary outcome of an equilibrium in supergame, according to our definition of equilibrium.

However, if for every $f \in F$ satisfying $\hat{\sigma}(f) = ((0,1/2,1/2),(1,0))$, the row player deviates by the following rule:

i) He plays $(0,1/2,1/2)$ until time T when the total payoff he has accumulated is $T + 1$.

ii) He then deviates and plays $(1,0,0)$.

The probability of such a T occuring, and hence the probability of deviating, is 1. Since the row player's evaluation is according to the overtaking criterion, his payoff sequence will be preferable with probability 1 to that obtained had he not deviated.

4. Perfect Equilibria

The definition of equilibrium as in §2 was shown to be too general in §3. The set of equilibria is too large, and it is natural to introduce further reasonable restrictions to obtain a stronger characterization.

One possible requirement is that a deviation will prove unprofitable to a player at all stages of the game, and not only at the beginning; thus no circumstances will induce him to change his original strategy.

<u>Definition</u>: $f \in F$ is a perfect equilibrium point if for all $r(1),\ldots,r(T) \in S$,

$\bar{f}_i(t)(s(1),\ldots,s(t-1)) = f_i(t+T)(r(1),\ldots,r(T),s(1),\ldots,s(t-1))$

satisfies " $\bar{f}$ is an equilibrium in supergame".

Not only is it unprofitable for one player to alter his strategy, but also no players can perform manipulative manoeuvres since after every "history", all players prefer not to deviate.

The following proposition characterizes the stationary perfect equilibrium in a supergame with evaluation relations determined by the limit of means criterion. A similar result was discovered independently, by Aumann and Shapley.

Proposition 4.1: If $\sigma \in S$, there is a perfect stationary equilibrium $f \in F$ such that $\hat{\sigma}(f) = \sigma$, iff σ is a weakly forced outcome.

Proof: Necessity follows from 3.3

Let $\sigma \in S$ be weakly forced outcome.

Let γ^i be strategies satisfying

$$\max_{t_i \in S_i} \pi_i(\gamma_i^{-i}, t_i) \leq \pi_i(\sigma)$$

(γ^i will be the strategy for punishing player i). Define $f_i(t+1)(s(1),\ldots,s(t))$ and $P(s(1),\ldots,s(t))$ inductively as follows:

($P(s(1),\ldots,s(t))$ will denote the players - in practice a single player - deserving punishment after $s(1),\ldots,s(t)$).

$P(\emptyset) = \emptyset$.

$f_i(1) = \sigma_i$.

$$P(s(1),\ldots,s(t)) = \begin{cases} \{i\} & \text{if } s^{-i}(t) = \gamma^{i^{-i}},\ P(s(1),\ldots,s(t-1)) = \{i\} \\ & \text{and } \dfrac{\sum_{k=1}^{t} \pi_i(s(k))}{t} \geq \pi_i(\sigma) + \dfrac{1}{\sqrt{t}} \\ \{i\} & \text{if } P(s(1),\ldots,s(t-1)) = \emptyset,\ s^{-i}(t) = \sigma^{-i} \text{ but} \\ & s_i(t) \neq \sigma_i, \text{ and } \dfrac{\sum_{k=1}^{t} \pi_i(s(k))}{t} \geq \pi_i(\sigma) + \dfrac{1}{\sqrt{t}} \\ \emptyset & \text{otherwise} \end{cases}$$

$$f_i(t+1)(s(1),\ldots,s(t)) = \begin{cases} \gamma_i^j & \text{if } j \neq i\text{ , and } P(s(1),\ldots,s(t)) = \{j\} \\ \sigma_i & \text{otherwise.} \end{cases}$$

Let $r(1),\ldots,r(T) \in S$; denote

$$\bar{f}_i(1) = f_i(T+1)(r(1),\ldots,r(T))$$

$$\bar{f}_i(t)(s(1),\ldots,s(t-1)) = f_i(t+T)(r(1),\ldots,r(T),s(1),\ldots,s(t-1))$$

We will show that $\bar{f}$ is an equilibrium.

Lemma 1: There exists T_1 such that for all $t \geq T_1$, $\sigma(\bar{f})(t) = \sigma$.

Proof: If $P(r(1),\ldots,r(T)) = \emptyset$, then $\sigma(\bar{f})(t) = \sigma$ for all $t \geq 1$. If $P(r(1),\ldots,r(T)) = \{j\}$, then j 's mean gain is

$$\frac{\sum_{k=1}^{T} \pi_j(r(k)) + t\pi_j(\gamma^j)}{T+t} \leq \frac{\sum_{k=1}^{T} \pi_j(r(k)) + t\pi_j(\sigma)}{T+t}$$

$$\leq \frac{\sum_{k=1}^{T} \pi_j(r(k))}{T+t} + \pi_j(\sigma)$$

$$< \frac{1}{\sqrt{T+t}} + \pi_j(\sigma)$$

for sufficiently large t .

Let $h_i \in F_i$.

Lemma 2: For all t_0 there is $t \geq t_0$ such that

$$\frac{\sum_{k=1}^{t} \pi_i(\sigma(f^{-i},h_i)(k))}{t} < \pi_i(\sigma) + \frac{1}{\sqrt{t}} .$$

__Proof__: It is sufficient to consider the case $t_0 \geq T_1$, where T_1 is given by lemma 1; note also that $t \geq T_1$ implies $P(s(1),\ldots,s(t)) \subseteq \{i\}$. If for all $t \geq T_0$, i does not deserve punishment after $\{\sigma(\bar{f}^{-i},h_i)(k)\}_{k=1}^{t}$, then $\sigma(\bar{f}^{-i},h_i) = \sigma$ for all $t_0 < k$, and

$$\sum_{k=1}^{t} \frac{\pi_i(\sigma(\bar{f}^{-i},h_0)(k))}{t} = \sum_{k=1}^{t_0} \frac{\pi_i(\sigma(\bar{f}^{-i},h_i)(k))}{t} + \frac{(t-t_0)\pi_i(\sigma)}{t} \leq \pi_i(\sigma) + \frac{1}{\sqrt{t}}$$

for large enough t.

If there exists $t_0 \leq t_1$, such that i deserves punishment after $\{\sigma(\bar{f}^{-i},h_i)(k)\}_{k=1}^{t_1}$, then

$$\sum_{k=1}^{t} \frac{\pi_i(\sigma(\bar{f}^{-i},h_i)(k))}{t} \leq \sum_{k=1}^{t_1} \frac{\pi_i(\sigma(\bar{f}^{-i},h_i)(k))}{t} + \frac{(t-t_1)\pi_i(\sigma)}{t} \leq \pi_i(\sigma) + \frac{1}{\sqrt{t}}$$

for large enough t.

Using lemma 2, we have $\bar{f} \not\prec_i (f^{-i},h_i)$.

Example:

Not every strongly forced outcome is the outcome of a perfect stationary equilibrium in $\langle G, \prec_1, \ldots, \prec_n \rangle$ where all players have evaluation relations according to the overtaking criterion.

Consider the following matrix game:

	B_1	B_2	B_3
A_1	(1,1)	(0,0)	(0,0)
A_2	(2,0)	(0,0)	(2,1)

$S_1 = \{A_1,A_2\}$ $\qquad$ $S_2 = \{B_1,B_2,B_3\}$

(A_1,B_1) is a strongly forced outcome, but is not an outcome of a perfect stationary equilibrium; for if $(f_1,f_2) \in F$ is a stationary perfect equilibrium such that $\hat{\sigma}(f_1,f_2) = (A_1,B_1)$, then for all $s(1),\ldots,s(t) \in S$,

$$f_2(t+1)(s(1),\ldots,s(t)) = \begin{cases} B_1 & \text{if } f_1(t+1)(s(1),\ldots,s(t)) = A_1 \\ B_3 & \text{if } f_1(t+1)(s(1),\ldots,s(t)) = A_2 \end{cases}$$

But then the column player can profitably alter his strategy by $\hat{f}_1 \equiv A_2$. with a utility flow $(1,1,\ldots) \prec_2 (2,2,\ldots)$.

<u>Proposition 4.2</u>: In the supergame $\langle G, \prec_1, \ldots, \prec_n \rangle$ where $\prec_i$ are evaluation relations according to the overtaking criterion, and $s \in S$ satisfies "there exist $\gamma^i \in S$ such that $\max_{t_i \in S_i} \pi_i({\gamma^i}^{-i}, t_i) < \pi_i(s)$." then there exists $f \in F$, a stationary perfect equilibrium such that $\hat{\sigma}(f) = s$.

<u>Proof</u>: The idea is to construct $f \in F$ such that a player deviating from the stationary position, or the punishing strategy of another player, will be punished sufficiently to eliminate his "profit". After punishment, the players return to the stationary position.

By assumption, there exist γ^i such that $\max_{t_i \in S_i} \pi_i({\gamma^i}^{-i}, t_i) = \pi_i(\gamma^i) < \pi_i(s)$ (γ^i is the strategy punishing i; the i 'th component of γ^i is i 's optimal defense strategy). We write $L_i = \pi_i(s) - \max_{t_i \in S_i} \pi_i({\gamma^i}^{-i}, t_i) > 0$. L_i is the punishment i will receive every time the punishing strategy γ^i is employed against him. We will write

$R_i = \max_{r \in \{s,\gamma^1,\ldots,\gamma^n\}} \{\max_{t_i \in S_i} \pi_i(t_i, r^{-i}) - \pi_i(r), \pi_i(s) - \pi_i(r)\} \geq 0$. R_i is the maximal relative profit a player i can gain by deviating from one of the $n+1$ single game strategies deployed in f, and by bringing to an end the punishment of another player.

We will now define $m_i(s(1),\ldots,s(t))$ and $f_i(t+1)(s(1),\ldots,s(t))$ inductively as follows:

($m_i(s(1),\ldots,s(t))$ is the length of time a player will be punished for participating in $s(1),\ldots,s(t)$.)

$m_i(\emptyset) = 0$

$f_i(1) = s_i$.

$$m_i(s(1),\ldots,s(t+1)) = \begin{cases} \left[\dfrac{R_i}{L_i}\right] + 1 & \text{if for all } j,\ m_j(s(1),\ldots,s(t)) = 0 \quad (1) \\ & s_i(t+1) \neq \sigma_i \text{ and } s^{-i}(t+1) = \sigma^{-i}. \\ \left[\dfrac{m_j(s(1),\ldots,s(t))\cdot R_i}{L_i}\right] + 1 & \text{if there exists } j \neq i \quad (2) \\ & \text{such that} \\ & m_j(s(1),\ldots,s(t)) > 0 \\ & s_i(t+1) \neq \gamma_i^j \text{ and} \\ & s^{-i}(t+1) = \gamma^{j^{-i}}. \\ m_i(s(1),\ldots,s(t)) - 1 & \text{if } m_i(s(1),\ldots,s(t)) > 0 \quad (3) \\ & \text{and } s^{-i}(t+1) = \gamma^{i^{-i}}. \\ 0 & \text{otherwise}. \quad (4) \end{cases}$$

Lemma: For all $s(1),\ldots,s(t) \in S$, the number of players i for whom $m_i(s(1),\ldots,s(t)) > 0$ is at most 1.

Proof: By induction on t. When $t = 0$, the number is 0. Suppose lemma is true for t; we will prove it for $t+1$. If for all i, $m_i(s(1),\ldots,s(t)) = 0$, then $m_j(s(1),\ldots,s(t+1)) > 0$ only if condition (1) in the definition of m_i is satisfied, and this can happen for one player only. For if $m_{i_0}(s(1),\ldots,s(t)) > 0$, and if all other players are "disciplined", i.e., $s^{-i_0}(t) = \gamma^{-i_0}$, then only for i_0 will $m_{i_0}(s(1),\ldots,s(t+1)) > 0$ (the case (2)). If only one player $j \neq i_0$ will be undisciplined, i.e., $s_j(t) \neq \gamma_j^{i_0}$, then only $m_j(s(1),\ldots,s(t+1))$ will be greater than zero. In all other cases, no player deserves punishment.

Thus we can define

$$f_i(t+1)(s(1),\ldots,s(t)) = \begin{cases} \gamma_i^j & \text{if} \quad m_j(s(1),\ldots,s(t)) > 0 \\ s_i & \text{otherwise} . \end{cases}$$

Clearly f is stationary and $\hat{\sigma}(f) = s$.

Let $r(1),\ldots,r(T) \in S$, and define $\bar{f}_i(t)(s(1),\ldots,s(t-1)) = f_i(t+T)(r(1),\ldots,r(T),s(1),\ldots,s(t-1))$. We will show that $\bar{f}$ is an equilibrium.

Let $h_i \in F_i$; we will prove that $\bar{f} \not\prec_i (\bar{f}^{-i},h_i)$. It is sufficient to show that if there exists t_0 such that $\pi_i(\sigma(\bar{f})(t_0)) < \pi_i(\sigma(\bar{f}^{-i},h_i(t_0))$, then there exists $t_0 < t_1$ such that $\sum_{t=t_0}^{t_1} \pi_i(\sigma(\bar{f})(t)) \geq \sum_{t=t_0}^{t_1} \pi_i(\sigma(\bar{f}^{-i},h_i)(t))$.

We will denote $m_i(\{\sigma(\bar{f}^{-i},h_i)(t)\}_{t=1}^{t_0-1})$ by m_i. If $m_i > 0$, i cannot profitably deviate, since

$$\pi_i(\gamma^i) = \max_{t_i \in S_i} \pi_i(\gamma^{i^{-i}}, t_i) \quad .$$

If $m_j = 0$ for all j , we will define $t_1 = t_0 + \left[\frac{R_i}{L_i}\right] + 1$; for all $t_0 < t \leq t_1$, $\sigma^{-i}(\bar{f}^{-i}, h_i) = \gamma^{i^{-i}}$, thus

$$\sum_{t=t_0}^{t_1} [\pi_i(\sigma(\bar{f}^{-i}, h_i)(t)) - \pi_i(\sigma(\bar{f}(t))] \leq R_i - (\left[\frac{R_i}{L_i}\right] + 1)L_i \leq 0 \ .$$

If $m_j > 0$, $j \neq i$, we define $t_1 = t_0 + \left[\frac{m_j \cdot R_i}{L_i}\right] + 1$; for all $t_0 < t \leq t_1$, $\sigma^{-i}(\bar{f}^{-i}, h_i) = \gamma^{i^{-i}}$, and thus

$$\sum_{t=t_0}^{t_1} \pi_i(s) - \pi_i(\sigma(\bar{f}(t))) + \pi_i(\sigma(\bar{f}^{-i}, h_i)(t)) - \pi_i(s) \leq m_j R_i - (\left[\frac{m_j \cdot R_i}{L_i}\right] + 1)L_i \leq 0 \ .$$

5. Strong Equilibrium

We have so far considered stationary equilibria in supergame, where equilibrium was defined as a strategy n-tuple where no deviation by a single player would be worthwhile for him. In this section, we will consider strong equilibria, that is strategy n-tuples with the property that no coalition exists which enables the members of the coalition to deviate en masse, and give every coalition member a better utility flow than he would have otherwise obtained.

We will no longer use the concept of stationarity, since the central element in the behavior of coalitions in supergame will be the co-ordination between the players at different times. Thus in the following 2-player game, where $S_1 = \{a_1, a_2\}$, $S_2 = \{b_1, b_2\}$, and the payoff function is given by the matrix

	b_1	b_2
a_1	(1,0)	(0,0)
a_2	(0,0)	(0,1)

we should expect a non-stationary equilibrium, for example:

"At time 2k+1 the players play (a_1,b_1) , and
at time 2k the players play (a_2,b_2) . "

<u>Definitions</u>: $f \in F$ is a strong equilibrium if there does not exist $\emptyset \neq B \subseteq N$, and $g_i \in F_i$ such that for all $i \in B$, $(f^{N-B},g^B) \succsim_i f$. $f \in F$ is a strong perfect equilibrium if for all $r(1),\ldots,r(T) \in S$, $\bar{f} \in F$, given by

$$\bar{f}(1) = f(T+1)(r(1),\ldots,r(T))$$

$$\bar{f}(t)(s(1),\ldots,s(t)) = f(T+t+1)(r(1),\ldots,r(T),s(1),\ldots,s(t))$$

is a strong equilibrium.

We will denote $S^B = \prod_{i \in B} S_i$.

The characterization of strong equilibria will be obtained using the following definitions: Let A be a set. Then

$$c(A) = \{ c \mid c : A \to [0,1] \text{ , } c(a) > 0 \text{ for a finite number of } a \in A \text{ , and } \sum_{a \in A} c(a) = 1 \} .$$

$a \in E^n$ will be called a desired payoff if $a \in \text{conv}\ \pi(S)$ and satisfies "for all $\emptyset \neq B \subseteq N$ there exists $s^{N-B} \in S^{N-B}$ such that for no $b \in \text{conv}\ \{\pi(s^B,s^{N-B}) \mid s^B \in S^B\}$ $a^B << b^B$. " 1/ A desired payoff has the

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1/ If $x,y \in \pi(S)^B$, we will write $x << y$ if for all $i \in B$, $x_i < y_i$.

property that for all $\emptyset \neq B \subseteq N$, there is a correpsonding punishment inflicted by $N - B$ such that even if the players in B could randomly co-ordinate their strategies in a single game, they couldn't guarantee themselves more than the desired payoff offers. For every $c_B \in C(S^B)$ and $\gamma^{N-B} \in S^{N-B}$, we will write:

$$\pi^B(c^B, \gamma^{N-B}) = \sum_{s^B \in S^B} c^B(s^B) \cdot \pi^B(s^B, \gamma^{N-B}) .$$

The following lemma, taken from [2] (lemma 5.2 there) will be used in the main propositions of this section.

Lemma 5.1: Let Z be a finite set, and let $y \in C(Z)$. For every map $\psi : N \to Z$, and for every natural number k , given $z \in Z$ we will write

$$\rho_\psi(k,z) = |\{ j \mid \psi(j) = z , \; j \leq k \}|$$

(the number of times ψ takes the value z in the interval $[1,k]$) .

Then there exists $\psi : N \to Z$ such that for all $z \in Z$,

$$\lim_{k \to \infty} \frac{\rho_\psi(k,z)}{k} = y(z) .$$

The following proposition gives a sufficient condition for $f \in F$ to be a strong equilibrium in a supergame with evaluation relations containing the limit of means evaluation relation. 1/

Proposition 5.2: Let $G^\infty = \langle G, \prec_1, \ldots, \prec_n \rangle$, where for all $\prec_i$ is an evaluation relation contained in the relation determined by the

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1/ Let R, S be binary relations on a set A . S contains R if for all $a, b \in A$, $aRb \Rightarrow aSb$.

limit of means. If $f \in F$ is a strong equilibrium in G^∞, then for

$$a_i = \underline{\lim}_{T\to\infty} \frac{\sum_{t=1}^{T} \pi_i(\sigma(f)(t))}{T} \quad , \quad i = 1,\dots,n \ ,$$

a is a desired payoff.

<u>Proof</u>: conv $\pi(S)$ is closed, hence $a \in$ conv $\pi(S)$. Suppose a is not desired. Then there exists $B \neq \emptyset$ such that for all $s^{N-B} \in S^{N-B}$ there exists $b \in \{\pi(c^B, s^{N-B}) \mid c^B \in C(S^B)\}$ such that $a^B << b^B$. Let
$0 < \varepsilon < \min_{s^{N-B} \in S^{N-B}} \max_{c^B \in C(S^B)} \min_{i \in B} \{\pi_i(c^B, s^{N-B}) - a_i\}$.

Since π_i , $i = 1,\dots,n$, is uniformly continuous, there is a finite open cover $U_1,\dots,U_k$ such that for all $s,t \in U_j$, $|\pi_i(s) - \pi_i(t)| < \frac{\varepsilon}{4}$. Since S^{N-B} , S^B are compact, there is an open cover of S^{N-B} $\{O_1,\dots,O_L\}$, such that for all O_j and for all $s^B \in S^B$ there exists m such that $U_m \supseteq O_j \times \{s^B\}$. Let us chose $r^{N-B}(j) \in O_j$, and let $c^B(j)$ satisfy

$$\pi_i(c^B(j), r^{N-B}(j)) - a_i > \varepsilon \qquad \text{for all } i \ .$$

Let $\psi_i : N \to Z$ be maps satisfying, for all $s^B \in S^B$,

$$\lim_{k\to\infty} \frac{\rho_{\psi_i}(k, s^B)}{k} = c^B(s^B) \ .$$

Their existence is guaranteed by lemma 5.1.

We will now define $g_i \in F_i$ for all $i \in B$ inductively, together with $m_j(t)$, the number of times B use $c^B(j)$ up to time t .

Let j_1 satisfy $f^{N-B}(1) \in O_{j_1}$.

Then for all $i \in B$, $1 \leq j \leq L$

$$g_i(1) = (\psi_{j_1}(1))_i$$

$$m_j(1) = \begin{cases} 1 & j = j_1 \\ 0 & \text{otherwise} \end{cases}$$

Proceeding by induction, let j_{t+1} satisfy

$$f^{N-B}(t+1)(\{\sigma(f^{N-B},g^B)(h)\}_{h=1}^{t}) \in O_{j_{t+1}} .$$

Then $g_i(t+1)(\{\sigma(f^{N-B},g^B)(h)\}_{h=1}^{t}) = \psi_{j_{t+1}}(m_{j_{t+1}}(t) + 1)$ and $g_i(t+1)$ is defined arbitrarily at other points in its domain.

$$m_j(t+1) = \begin{cases} m_j(t) + 1 & j = j_{t+1} \\ m_j(t) & \text{otherwise.} \end{cases}$$

Let $j(t)$ satisfy $\sigma^{N-B}(f,g)(t) \in O_{j(t)}$. Then $\underline{\lim} \frac{\sum_{t=1}^{T} \pi_i(\sigma(f^{N-B},g^B)(t))}{T} \geq$

$$\geq \underline{\lim} \frac{\sum_{t=1}^{T} \pi_i(\sigma^B(f^{N-B},g^B)(t),\gamma_{j(t)}^{N-B})}{T} - \frac{\varepsilon}{2} =$$

$$= \underline{\lim} \frac{\sum_{j=1}^{L} T_j(T) \sum_{s^B \in S^B} \frac{\rho_{\psi_j}(T_j(T),s^B)}{T_j(T)} \cdot \pi_i(s^B,r^{N-B}(j(t)))}{T} - \frac{\varepsilon}{2}$$

where $T_j(T) = |\{ t \mid j(t) = j , \quad 1 \leq t \leq T \}|$.

For j satisfying $T_j(T) \underset{T}{\to} \infty$,

$$\lim_{T} \sum_{s^B \in S^B} \frac{\rho_{\psi_j}(T_j(T), s^B)}{T_j(T)} \cdot \pi_i(s^B, r^{N-B}(j(t))) \geq a_i + \varepsilon \ .$$

Thus

$$\underline{\lim} \frac{\sum_{t=1}^{T} \pi_i(\sigma(f^{N-B}, g^B)(t))}{T} \geq a_i + \frac{\varepsilon}{2} \ .$$

Therefore for all $i \in B$, $(g^B, f^{N-B}) \succsim_i f$.

The following proposition will give a sufficient condition for payoffs in $\pi(S)$ to be the average payoff of a strong perfect equilibrium of a supergame in which the evaluation relations are according to the limit of the means.

Proposition 5.3: Let $G^\infty = \langle G, \prec_1, \ldots, \prec_n \rangle$ be a supergame where for all $1 \leq i \leq n$, $\prec_i$ is an evaluation relation according to the limit of the means. If $a \in R^n$ is a desired payoff, then there exists $f \in F$, a strong perfect equilibrium such that for all $i = 1, \ldots, n$,

$$a_i = \lim_{T \to \infty} \frac{\sum_{t=1}^{T} \pi_i(\sigma(f)(t))}{T} \ .$$

Proof: For every $\emptyset \neq B \subsetneq N$ we will denote by γ^{N-B} a strategy in S^{N-B} which guarantees that for all $b \in \{\pi(c^B, \gamma^{N-B}) \mid c^B \in C(S^B)\}$, $a^B \not< b^B$. The existence of such a γ^{N-B} follows from the definition of desirability.

Now, $a \in \text{conv}\, \pi(S)$, thus there exists $c \in C(S)$ such that $\pi(c) = a$. Let ψ be a map from the natural numbers into S, satisfying

$$\lim_{k\to\infty} \frac{\rho_\psi(k,s)}{k} = c(s) .$$

By Lemma 5.1, such a map exists.

For every $s(1),\ldots,s(T) \in S$, we define the set of deviants after $s(1),\ldots,s(T)$, $B(s(1),\ldots,s(T))$ in parallel with the definition of $f \in F$, by induction on T.

$B(\emptyset) = \emptyset$

$f_i(1) = \psi(1)_i$.

$$B(s(1),\ldots,s(T)) = \begin{cases} A & \text{if } A = B(s(1),\ldots,s(T-1)) \cup \\ & \cup \{ i \mid s_i(T) \neq f_i(T)(s(1),\ldots,s(T-1)) \} \neq \emptyset \\ & \text{and if for all } i \in A \\ & \sum_{t=1}^{T} \frac{\pi_i(s(t))}{T} \geq a_i + \frac{1}{\sqrt{T}} \\ \emptyset & \text{otherwise} \end{cases}$$

$$f_i(T+1)(s(1),\ldots,s(T)) = \begin{cases} \gamma_i^{N-B(s(1),\ldots,s(T))} & \text{if } B(s(1),\ldots,s(T)) \neq \emptyset \\ & \text{and } i \notin B(s(1),\ldots,s(T)) \\ \text{arbitrary} & \text{if } B(s(1),\ldots,s(T)) \neq \emptyset \text{ and} \\ & i \in B(s(1),\ldots,s(T)) \\ \psi_i(T-k) & \text{if } B(s(1),\ldots,s(T)) = \emptyset \text{ and} \\ & k = \max \{ t \mid B(s(1),\ldots,s(t)) \neq \emptyset \vee t=0 \} \end{cases}$$

Let $r(1),\ldots,r(T) \in S$. Define

$$\bar{f}_i(t)(s(1),\ldots,s(t-1)) = f_i(t+T)(r(1),\ldots,r(T),s(1),\ldots,s(t-1))$$

We will show that $\bar{f}$ is a strong equilibrium, using the following four lemmas, on of which shows that the players account for the mean payoff a , and another that there does not exist a collective deviation profitable to all those deviating.

Lemma 1: If $f \in F$ has the property that for all $t > T_0$, $\sigma^B(f)(t) \equiv \gamma^B$, then there is $i \in N-B$ and $s > T_0$ such that

$$\sum_{k=1}^{s} \pi_i(\sigma(f)(k)) \le a_i + \frac{1}{\sqrt{s}} \quad .$$

Proof: $$\sum_{k=1}^{s} \frac{\pi_i(\sigma(f)(k))}{s} = \frac{\sum_{k=1}^{T_0} \pi_i(\sigma(f)(k)) + \sum_{k=T_0+1}^{s} \pi_i(\sigma(f)(k))}{s}$$

$$\le \frac{c + (s-T_0)a_i}{s} \quad \text{for some } i$$

$$\le a_i + \frac{1}{\sqrt{s}} \quad \text{for } s \text{ sufficiently large.}$$

Lemma 2: There exists T_1 such that for all $T_1 \le t$,

$$B(r(1),\ldots,r(T), \sigma(\bar{f})(1),\ldots,\sigma(\bar{f})(t)) = \emptyset \ .$$

Proof: If $B(r(1),\ldots,r(T)) = \emptyset$, then $T_1 = T$ will do. Otherwise, by Lemma 1, there exists $i \in B(r(1),\ldots,r(T))$ and T_1 such that

$$\sum_{k=1}^{T+T_1} \frac{\pi_i(\sigma(\bar{f})(k))}{T + T_1} \leq a_i + \frac{1}{\sqrt{T+T_1}} \quad .$$

Therefore, for every $T_1 \leq t$, $B(r(1),\ldots,r(T),\sigma(\bar{f})(1),\ldots,\sigma(\bar{f})(t)) = \emptyset$.

Lemma 3: Let $h^B \in F^B$. Then there exists T_2 such that for $C(t)$, defined by

$$C(t) = B(r(1),\ldots,r(T),\sigma(h^B,\bar{f}^{N-B})(1),\ldots,\sigma(h^B,\bar{f}^{N-B})(t)) \ ,$$

we have $C(T_2) = \emptyset$.

Proof: Suppose not. By the definition of the set of deviants, $C(t) \subseteq C(t+1)$ for all t . Thus there exists T' such that for all $T' \leq t$, $\sigma^{N-C(T')}(h^B,\bar{f}^{N-B})(t) = \gamma^{N-C(T')}$ and $C(t) = C(T')$. Thus by lemma 1, there exists $i \in C(T)$ and T'' such that

$$\frac{\sum_{k=1}^{T} \pi_i(r(k)) + \sum_{k=1}^{T''} \pi_i(\sigma(h^B,\bar{f}^{N-B})(k))}{T + T''} \leq a_i + \frac{1}{\sqrt{T+T''}}$$

contradicting $i \in C(T''+1)$.

Lemma 4: Let $h^B \in F^B$. Then for every t_0 , there exists $t \geq t_0$ such that

$$\frac{\sum_{k=1}^{T} \pi_i(r(k)) + \sum_{k=1}^{T+t} \pi_i(\sigma(h^B,\bar{f}^{N-B})(k))}{T + t} \leq a_i + \frac{1}{\sqrt{T+t}}$$

for some i .

Proof: By lemma 2, we may assume that $T_2 \leq t_0$, and thus the set of deviants is a subset of B . Define

$$D(t) = B(r(1),\ldots,r(T),\sigma(h^B,\bar{f}^{N-B})(1),\ldots,\sigma(h^B,\bar{f}^{N-B})(t)) .$$

If for every $t \geq t_0$, $D(t) = \emptyset$, then the lemma follows from lemma 1.

Suppose $D(t_0) \neq \emptyset$. If $D(s) \neq \emptyset$, and $D(s+1) = \emptyset$, then there exists $i \in D(s)$ satisfying

$$\frac{\sum_{k=1}^{T} \pi_i(r(k)) + \sum_{k=1}^{T+s} \pi_i(\sigma(h^B,\bar{f}^{N-B})(k))}{T+s} \leq a_i + \frac{1}{\sqrt{T+s}}$$

If the lemma is not true, then $D(s_2) \subseteq D(s_1)$ for all $t_0 \leq s_2 \leq s_1$. Thus there exists T_3 such that for every $T_3 \leq t$ $D(t) = D(T_3)$. By lemma 1, there exist $i \in D(T_3) \subseteq B$ and $T_3 \leq t$ such that

$$\frac{\sum_{k=1}^{T} \pi_i(r(k)) + \sum_{k=1}^{t} \pi_i(\sigma(h^B,\bar{f}^{N-B})(k))}{T+t} \leq a_i + \frac{1}{\sqrt{T+t}}$$

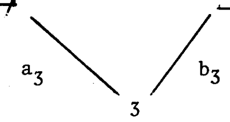
a contradiction.

Example and Comment

In matrix games, the desired payoffs are contained in the β-core, but the converse is not necessarily true. Let us look at the following 3-person game, where each player has two pure strategies, a_i and b_i . The payoff matrix is represented by two matrices, where player 1 is the row player, player 2 is the column player, and player 3 chooses the matrix.

	a_2	b_2
a_1	(2,2,2)	(1,1,0)
b_1	(1,1,3)	(0,0,4)

	a_2	b_2
a_1	(0,0,4)	(1,1,3)
b_1	(1,1,0)	(2,2,2)



For every $1 \leq i \leq 3$, S_i is the set of mixed strategies naturally identified with the interval [0,1], where the choice $p \in [0,1]$ corresponds to the strategy $p \cdot a_i + (1-p) \cdot b_i$.[1/] The payoff functions are the expected utilities. (2,2,2) is a payoff in the β-core, since the only coalition with a possible profitable strategy deviating from (a_1, a_2, a_3) is {3} ; but {1,2} have a punishing strategy $\frac{1}{2} \cdot (a_1\ b_2) + \frac{1}{2} \cdot (a_2, b_1)$, which reduces 3's expected payoff, whatever his strategy, to $1.5 < 2$.

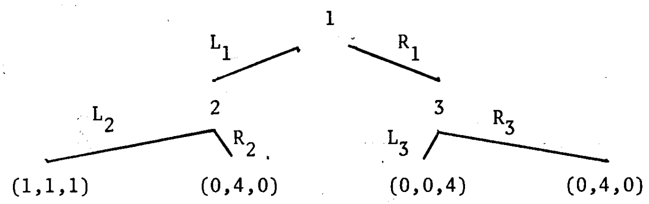
However, (2,2,2) is not a desired payoff since for $p \in S_1$, $q \in S_2$, player 3 may obtain

$$\max \{2pq + 3(1-p)q + 4(1-p)(1-q),\ 4pq + 3p(1-q) + 2(1-p)(1-q)\}$$ which is strictly greater than 2.

<u>Example</u>

Let us examine the following 3-person game presented in extended form. For every player, $S_i = \{R_i, L_i\}$, with evaluation relations according to the overtaking criterion.

1/ $p \cdot a_i + (1-p) \cdot b_i$ denotes the strategy "play a_i with probability p, and b_i with probability $(1-p)$."



(1,1,1) is a stationary payoff of a strong equilibrium in the supergame, despite the fact that for every $S \subseteq \{1,2,3\}$, either no deviation exists profitable to every player in the supergame, or $\{1,2,3\} - S$ can "retaliate", punishing at least one of the players in S .

In this example, we see the possibility of deviation by stages. An equilibrium strategy must punish player 2 for his deviation R_2 , as follows: every time 2 gains 4, 2 and 3 will play (R_1,L_3) at least three times; however, already after the first punishing game, both 2 and 3 will have averaged more than 1 . Thus the coalition {2,3} can plan the following ruse: 2 deviates; after being punished once, 3 plays alternatively L_3,R_3 , while 2 plays L_2 . This strategy is preferable according to the overtaking criterion to a strategy with a constant flow of 1.

On the other hand, the following proposition may be proven in a similar way to the last one.

<u>Proposition</u>: Let $a \in \text{conv}\ \pi(S)$ be such that for all B there is $\gamma^{N-B} \in S^{N-B}$, such that for all $c^B \in C(S^B)$, there exists $i \in B$ such that $\pi_i(c^B,\gamma^{N-B}) < a_i$; assume further, that $\prec_i$ is according to the overtaking criterion. Then there is a strong equilibrium f such that

$$\lim_{T\to\infty} \frac{\sum_{t=1}^{T} \pi_i(\sigma(f)(t))}{T} = a_i \quad , \quad \text{for all} \quad i \in N \ .$$

The next example shows among other things that the conditions of the proposition just given are not sufficient for the existence of a strong perfect equilibrium.

Example

Let us look at the following 2-person game:

	b_1	b_2
a_1	(2,2)	(1,1)
a_2	(4,0)	(1,1)

, $S_1 = \{a_1, a_2\}$, $S_2 = \{b_1, b_2\}$.

π_1 , π_2 are represented by the above matrix.

The β-core payoff is $\{(2,2)\}$, which is also the set of strongly forced payoff. But in supergame, with evaluation relations according to the overtaking criterion, there is no strong perfect equilibrium. For, suppose that $f \in F$ is a strong perfect equilibrium. If there exist $r(1),\ldots,r(T)$ such that $f_2(T+1)(r(1),\ldots,r(T)) = b_2$, then, writing

$$\bar{f}_i(t)(s(1),\ldots,s(t-1)) = f_i(t+T)(r(1),\ldots,r(T),s(1),\ldots,s(t-1))$$

$$g_1(1) = a_1 \ ,$$
$$g_1(t)(s(1),\ldots,s(t-1)) = \bar{f}_1(t)(\sigma(\bar{f})(1),s(2),\ldots,s(t-1))$$

$$g_2(1) = b_1 \ ,$$
$$g_2(t)(s(1),\ldots,s(t-1)) = \bar{f}_2(t)(\sigma(\bar{f})(1),s(2),\ldots,s(t-1))$$

we have $\sigma(g)(1) = (a_1,b_1)$ $\pi(\sigma(g)(1)) = (2,2) >> (1,1) = \pi(\sigma(\bar{f})(t))$, and for $t \geq 2$,

$$\sigma(g)(t) = \sigma(f)(t) , \quad \pi(\sigma(g)(t)) = \pi(\sigma(\bar{f})(t)) .$$

Thus for $i \in \{1,2\}$, $g \succsim_i f$. It follows that $f_2(t) \equiv b_1$ for all t . Clearly $f_1(t) \equiv a_2$ for all t , otherwise we obtain a contradiction to perfection, but now the deviation of player 2, given by $f_2(t) \equiv b_2$ guarantees him a utility flow $1 > 0$, contradicting f being an equilibrium.

Intuitively, the situation is as follows: players 1 and 2 agree to play (a_1,b_1) , but 1 deviates and plays a_2 . Player 2 threatened from the start to punish him by playing b_2 . If the players were unable to cooperate in supergame, an equilibrium would exist, by 1 threatening to punish 2 if 2 does not punish him, or, by participating in his own punishment by playing a_2 immediately after deviating.

However, assuming that the players can cooperate, after the first game they agree to "forgive" 1, since this is a common interest. Of course, 1 deviates again (why shouldn't he?) and he is again forgiven, and so on.

This example deals with the situation where punishment causes a loss to the punishers, who thus occasionally prefer to forgive for reasons similar to 2's here. It also, in my opinion, exposes the limitations of the description of supergame and the concepts of solution given in this paper. There is lacking a notion of "precedent"; if we do not ignore 2's expectations, we have to introduce into his calculations the possible consequences of not punishing 1.

In contrast to this example, the situation described as the "Prisoner's Dilemma" has a strong perfect equilibrium. This game is the same as the previous one, but with the following payoff matrix.

	b_1	b_2
a_1	(2,2)	(0,3)
a_2	(3,0)	(1,1)

We define $f_i(t)$ and $m_i(t)$ by induction on t .

$m_i(1) = 0$

$f_i(1) = 0$

$$m_i(t)(s(1),\dots,s(t-1)) = \begin{cases} 1 & \text{if } s_i(t-1) \neq a_i \text{ and } \\ & m_j(t-1)(s(1),\dots,s(t-2)) = 0 \quad \text{for all } j . \\ m_i(t-1)(s(1),\dots,s(t-2)) + 1 & \text{if } s_i(t) \neq a_i \text{ and } \\ & m_i(t-1)(s(1),\dots,s(t-2)) > 0. \\ 0 & \text{otherwise} \end{cases}$$

$$f_i(t)(s(1),\dots,s(t-1)) = \begin{cases} b_i & \text{if } j \neq i \Rightarrow m_j(t)(s(1),\dots,s(t-1)) \neq 0 \\ a_i & \text{otherwise.} \end{cases}$$

The players are planning to play (a_1,b_1) unless one of them deviates. If 2 (1) deviates, 1 (2) punishes him, by forcing (a_2,b_1) $[(a_1,b_2)]$.

If 2 (1) does not co-operate in his own punishment, player 1 (2) will increase the period of punishment. Clearly, in contrast to the previous example, the punishing player profits from the punishing arrangement and he has no motivation to forgive the deviant. It is easily verified that f is indeed a strong perfect equilibrium.

6. Some Comments on the Overtaking Criterion

(1) An axiomatic characterization is given in Brock [6].

(2) N is the natural numbers, R^N the set of real number sequences. Let F_0 be a filter of N. Define $\prec$ on R^N by: $x \prec y$ iff there exists $\varepsilon > 0$ sucht that $\{ n \mid S_n(y) - S_n(x) \geq \varepsilon \} \in F_0$ where $S_n(x) = \sum_{i=1}^{n} x_i$.

$\prec$ is an ordering relation, since if $x \prec y$ and $y \prec z$, then there are $\varepsilon_1, \varepsilon_2 > 0$ such that

$$A_1 = \{ n \mid S_n(z) - S_n(y) \geq \varepsilon_1 \} \in F_0$$

$$A_2 = \{ n \mid S_n(y) - S_n(x) \geq \varepsilon_2 \} \in F_0$$

Thus $\{ n \mid S_n(z) - S_n(x) \geq \varepsilon_1 + \varepsilon_2 \} \supseteq A_1 \cap A_2 \in F_0$.

Thus $\{ n \mid S_n(z) - S_n(x) \geq \varepsilon_1 + \varepsilon_2 \} \in F_0$ and $x \prec z$.

Also $x \prec y \Rightarrow$ There is an $0 < \varepsilon$ such that $\{n \mid S_n(y) - S_n(x) \geq \varepsilon\} \in F_0$

$\Rightarrow$ There is an $0 < \varepsilon$ such that $\{n \mid S_n(y) - S_n(x) < \varepsilon\} \notin F_0$

$\Rightarrow \{n \mid S_n(x) - S_n(y) \geq \delta\} \notin F_0$ for all $\delta > 0$

$\Rightarrow y \not\prec x$.

(3) In particular, if we choose $F_0 = \{ A \subset N \mid N - A \text{ is a finite set}\}$,

then the filter F_0 induces an order relation identical to the evaluation relation according to the overtaking criterion.

Denote by $\prec_c$ the overtaking relation, and by $\prec_0$ the relation induced by F_0. Then

$$x \prec_c y \quad \text{iff } \underline{\lim}\, S_n(y) - S_n(x) > 0 .$$

$$\text{iff there exists } \varepsilon > 0 \text{ such that } S_n(y) - S_n(x) > \varepsilon \text{ for all but finite } n .$$

$$\text{iff } x \prec_0 y .$$

(4) There does not exist utility function representing the overtaking cirterion, that is, no function $u: R^N \to R$ satisfies $u(x) < u(y) \Leftrightarrow x \prec_c y$ for all x,y which are $\prec_c$ related.

For, for every $a_0 \in R$, $(a_0,a_0,\ldots) \prec_c (a_0+1,a_0,\ldots)$ and for every $a_0 < a_1$, $(a_0+1,a_0,\ldots) \prec_c (a_1,a_1,\ldots)$. Thus R^N has $\aleph$ $\prec_c$ segments which are disjoint and non-empty, while $(R, <)$ has less.

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