A Note on Story Builders

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Abstract: An individual ("a story builder") seeks to construct a coherent story on the basis of evidence. A coherent story is a path from the origin to a terminal node in a directed graph. The discussion focuses on the property that if the story builder believes in a particular story, given a certain set of facts, then he believes in it given a broader set of facts consistent with the story. If the graph is a tree (but not necessarily otherwise), this property is equivalent to the maximization of an ordering on the set of stories that are consistent with the evidence.
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1. Introduction

It is common that an individual receives partial evidence about some chain of events and must complete the picture, in an attempt to obtain a full and coherent "story". In constructing the story, the individual is aware of the constraints on how a story can develop. We are interested in methods for constructing a full story from partial evidence. Given that people often fail to apply Bayesian reasoning even in very simple situations, the approach we take is non-Bayesian.

In our setting, a story is a sequence of events, i.e., a path along a finite directed graph (without cycles), which starts from the origin and continues along the graph until it reaches a terminal node. Each node in the graph represents a possible event. A story builder receives some evidence in the form of a subset of events consistent with at least one story and expands it into a full coherent story he believes in. Thus, the story builder holds a point belief rather than a probabilistic belief about the real story.

The formalization of the concept of "story builder" allows us to define and analyze a variety of procedures. The "rational" story builder is prominent in our analysis. He has in mind an ordering over the set of possible stories and chooses the story that maximizes the ordering given the evidence. The ordering might be a belief about the likelihood of the stories, in which case he chooses the most likely one. Alternatively, the ordering might reflect wishful thinking, and then he will select the best story that does not conflict with the evidence.

Whereas the rational story builder approaches the situation holistically, other types of story builders construct the story in steps, sequentially adding events according to some rule. For example, the story builder might have in mind a probability measure over the set of possible stories and will advance from the origin by selecting the most likely event at each stage, given the evidence and the path he has chosen so far.

Much of the analysis centers around what we call Story Consistency which states that if, after receiving a particular evidence set, the story builder believes in some story, then he continues to do so if he receives the same evidence set together with an additional piece of evidence consistent with the story.

If the graph is a tree, then Story Consistency is satisfied only by a story builder who can be described as rational. If not, then there are story builders that satisfy the property
but are not "rational". We also investigate the connection between Story Consistency and two other properties that relate to the case in which the story builder believes that the evidence is being presented by an interested party. Each of the properties requires that the story builder's method be consistent with the assumption that the speaker cannot do better by manipulating the evidences he chooses to present.

Finally, we present an example of a matching game in a large population consisting of various types of story builders. The story builders are matched into pairs and each needs to come up with a story identical to that of his partner, given a particular graph. We show that miscoordination may take place in equilibrium and that (unlike the case of mixed strategies equilibria in a standard game) this equilibrium is robust to small changes in the distribution of types in the population.

**Comment:** Sadler (2018) is the closest work to ours. In his model, an agent encounters information in terms of propositions that can be either true or false. A proposition is identified with a subset of states in which it is true. A belief is a set of propositions held by the individual. An individual is modeled as an updating function that determines belief as a function of previous beliefs and a new proposition. This formalization provides a language for specifying a number of updating rules that are not necessarily consistent with Bayesian reasoning.

In Bjorke (2019), the set of "states" is a product set. The individual receives information about some components of the vector. The individual possesses a "focal state" function (analogous to our story builder) that completes any subset of characteristics into a full vector. An individual chooses a "focal state" for every subset of values. Bjorke (2019) focuses on two functions ("most likely" and "most distinctive") and investigates an implementability question: What are the states that can be evoked given the true state?

Eliaz and Spiegler (2018) focus on a decision maker who interprets objective data about the realizations of variables in terms of a causal model (a Bayesian network). The decision maker in their framework receives statistical data and builds a model (a narrative) that organizes the data. Our story builder receives partial information about the realizations of some of the variables and wishes to develop it into a complete and coherent chain of events.
2. The model

Let $G = \langle A, \rightarrow \rangle$ be a finite directed graph. The set $A$ is a finite set of possible events, each of which may or may not have occurred. The existence of an arch $a \rightarrow b$ in the graph means that the event $b$ may succeed the event $a$. Denote $N(a) = \{b \mid a \rightarrow b\}$. The event $a$ is terminal if $N(a) = \emptyset$. A path in $G$ is a sequence of events $(a_0, a_1, \ldots, a_K)$ where $a_0 = O$ and $a_k \rightarrow a_{k+1}$ is an arch in $G$, for every $0 \leq k \leq K - 1$.

We restrict attention to graphs satisfying two properties:

(i) There is no cycle. That is, $G$ does not have a path of the form $(a_0, a_1, \ldots, a_K = a_0)$ with $K \geq 1$. It follows that a terminal event exists.

(ii) The graph contains an event, denoted by $O$ (the starting point), such that for every $a \in A$ there is at least one path from $O$ to $a$.

Define the relation $a$ appears (weakly) before $b$ by $a = b$ or there is a path from $O$ to $b$ that contains $a$. This relation is anti-symmetric (but incomplete).

A story in $G$, denoted by $x = (x_0, \ldots, x_{l(x)})$, is a path where $x_0 = O$ and $x_{l(x)}$ is terminal. The integer $l(x)$ is the length of $x$. The set of all stories is denoted by $S$. We have in mind that one and only one of the stories is the true story. For any two stories $x$ and $y$ denote by $j(x, y)$ the maximal $j$ for which $(x_0, \ldots, x_j) = (y_0, \ldots, y_j)$. That is, $j(x, y)$ is the length of the head of the stories that is joint to $x$ and $y$. Denote $a_{xy} = x_{j(x,y)} = y_{j(x,y)}$, the event where the two stories split.

An individual gets to see evidence in the form of a set of events that have occurred. We do not allow for the individual to receive direct information about an event that has not occurred, though he might infer the information from the graph and the evidence he receives. Formally, an evidence set is a subset $E \subseteq A$ such that there is at least one story $s \in S$ that is consistent with $E$, in the sense that it passes through all the events in $E$. The model does not specify what the source of the evidence is; it may be, for example, the individual himself that simply information comes his way, or an interested party.

The individual is familiar with the graph $G$, observes an evidence set and builds a story that is consistent with the evidence. A story builder is a method of developing any
evidence set into a story. Let \( \mathcal{E} \) be the set of all evidence sets. For every \( E \in \mathcal{E} \) let \( S_E \) be the (non-empty) set of stories consistent with \( E \). A story builder (for the graph \( G \)) is a function that assigns a unique story in \( S_E \) to every \( E \in \mathcal{E} \).

The "rational" story builder. A prime example of a story builder is based on a strict ordering \( \succeq \) on the set \( S \). For each evidence set \( E \), the story builder \( F_{\succeq} \) selects the story in \( S_E \) that is \( \succeq \)-maximal according to the ordering. The ordering \( \succeq \) can represent a variety of psychological phenomena. In particular, it might express a likelihood relation between the stories. Under this interpretation, the rational story builder can be thought of as a Bayesian agent who uses only point-wise beliefs. Other interpretations include wishful thinking (what the story builder wishes the truth to be) or a simplicity measure (the story builder seeks a simple story).

We say that a story builder \( F \) is rationalizable if there is an ordering \( \succeq \) such that \( F = F_{\succeq} \).

3. The story builder as a choice function

A story builder who observes an evidence set \( E \) chooses a story in \( S_E \), the set of stories consistent with \( E \). Our notion of a story builder allows for the individual to believe in two different stories after receiving two different evidence sets that share the same set of consistent stories. In other words, if we think about the graph as a description of the logical constraints on a story, then the concept of a story builder allows for two evidence sets to logically lead to the same conclusion (i.e. the same set of consistent stories) but nevertheless the story builder will believe in two different stories.

**Invariance:** A story builder \( F \) satisfies the invariance property if \( F(E) = F(E') \) whenever \( S_E = S_{E'} \).

When a story builder satisfies the invariance property it can be thought of as a choice function with the (restricted) domain \( D = \{ S_E \mid E \text{ is an evidence set} \} \). The graph \( G \) imposes restrictions on the story builder’s domain. For example, if \( G \) is a tree then an evidence set \( E \) must be a set of events along some path and \( S_E \) a set of all stories that include the event in \( E \) that is the furthest away from \( O \). Thus, the number of elements in the domain \( D \) is at most \( |A| \).
The choice from the sets of size 2 is often a basic ingredient of a choice function. However, in the current model, there is only a limited set of subsets of size 2 in \( D \).

**Claim 1:** For no graph \( G \) are there three stories \( p, q, r \in S \) and three evidence sets \( E_{pq}, E_{qr} \) and \( E_{pr} \), such that \( S_{E_{pq}} = \{p, q\} \), \( S_{E_{qr}} = \{q, r\} \) and \( S_{E_{pr}} = \{p, r\} \).

**Proof:** Assume to the contrary that such a graph \( G \) exists.

Note first that since \( G \) does not contain a cycle, the relation "the event \( a \) appears before the event \( b \) in some story" is anti-symmetric.

If the stories \( x \) and \( y \) share an event after \( a_{xy} \) (the last event in the joint heads of the stories \( x \) and \( y \)) then denote by \( b_{xy} \) the first event after \( a_{xy} \) that is common to both \( x \) and \( y \). At the earliest event where the three stories \( p, q \) and \( r \) do not coincide, one or them, say \( p \), is split from the other two, which might or might not split as well (see Figure 1). Thus, \( a_{pq} = a_{pr} \) and \( a_{pq} \) appears (weakly) before \( a_{qr} \). An event \( b_{pq} \) exists and \( E_{pq} \) contains and event which comes not later than \( b_{pq} \) since otherwise \( r \in E_{pq} \). Also \( b_{pr} \) exists and \( E_{pr} \) contains \( b_{pr} \) or an event that appears (weakly) before \( b_{pr} \). Without loss of generality, \( b_{pq} \) appears (weakly) before \( b_{pr} \) in \( p \). But then \( q \) is consistent with \( E_{pr} \), a contradiction. ■

Obviously, given a graph \( G \), any choice function with the restricted domain \( D \) induces a story builder. If the choice function does not satisfy Sen's property \( \alpha \), then the story builder is not rationalizable. Following is an example:

The story builder is a judge. The stories in \( S \) are the possible chains of events in a particular case. The set \( S \) is partitioned by the judge into two subsets - \( G \) ("guilty") and \( N \) ("not guilty"). The judge has in mind a prior \( p \) over \( S \). Given an evidence set \( E \), he compares \( p(G \cap S_E) \) to \( p(N \cap S_E) \). If the former is larger, he finds the defendant guilty and justifies the decision based on his belief in the most likely story in \( G \cap S_E \); otherwise, he finds the defendant not guilty and justifies the verdict based on his belief in the most likely story in \( N \cap S_E \). That is, the judge first makes up his mind whether the defendant is guilty or innocent, given the evidence that has been brought before him, and only then, does he adopt the most likely story given the evidence and given his verdict.
Such a judge may not be rationalizable. For example, given the tree and the probability measure on $S$ presented in Figure 2, the judge believes $(O, d, b)$ if he does not see any evidence and $(O, d, c)$ if he sees the evidence set $\{d\}$ although $d$ is consistent with $(O, d, b)$. A similar example can be easily constructed for the case in which the judge finds the defendant guilty only if he believes it "beyond a reasonable doubt" (i.e. above a certain probability).

Figure 2

4. Story Consistency

We focus on the property of a story builder which states that if, after receiving a particular evidence set, the story builder believes a particular story, then he does not change his belief if the evidence set contains one additional event consistent with the story.

**Story Consistency.** A story builder $F$ satisfies Story Consistency if $F(E \cup \{a\}) = F(E)$ for every evidence set $E$ and any event $a$ in the story $F(E)$.

**Claim 2:** If $F$ satisfies Property 1, then $F$ satisfies the invariance property.

**Proof:** Assume that $S_E = S_{E'}$. By definition $F(E) \in S_E$ and thus $F(E) \in S_{E'}$ and therefore the events in $E' - E$ appear in $F(E)$. Applying Story Consistency successively to the events in $E' - E$ yields $F(E \cup E') = F(E)$. Similarly, $F(E \cup E') = F(E')$. Thus, $F(E) = F(E')$.

**Comment:** Story Consistency seems intuitive, but in some contexts it conflicts with some principles of Pragmatics, as demonstrated in the following examples:

(i) A boss is about to enter the company’s compound. He believes that employees A and B are already there. Just before entering the building he meets an aide who coincidentally tells him that A is already at work. Although this information does not contradict the boss’ initial beliefs, he may conclude that B has not arrived yet. In contrast, if the boss does not meet his aide and coincidentally passes by A’s office and sees him there, then he probably will not change his mind as to whether B is already at work. Thus,
if the evidence appears coincidentally, the boss’ reasoning satisfies Story Consistency. But if the evidence is given in the course of conversation, it is possible that he attributes importance not only to the evidence but also to the evidence that he does not have, violating Story Consistency.

(ii) A student has taken four exams (1, 2, 3 and 4) and already knows his grades on them. A professor who knows the student but has not seen his results, believes that the student got an A on all four exams. The student coincidentally meets the professor in the corridor and tells him that he got an A on exams 1 and 3. Although the information does not contradict his initial beliefs, it is likely that the professor will conclude that the student did not get an A on exams 2 and 4. Given that the student bothered to report his grades only on two of the exams, the professor is likely to conclude that he has avoided mentioning the other two intentionally and therefore will revise his belief about the student’s grades, violating Story Consistency.

The story builder in our model receives evidence only once and then forms his belief. The model does not specify the order in which the evidences arrive. Consider a story builder with a function $F$ who first receives $E_1$ and then $E_2$. He may ignore the order they arrived in and adopt the belief $F(E_1 \cup E_2)$. Or he may first form the belief $F(E_1)$ and then stick with it if $E_2$ is consistent with $F(E_1)$. If $F$ satisfies Story Consistency, then the story builder reaches the same belief under both possibilities. If $F$ does not satisfy Story Consistency and if the story builder uses the second approach, then a speaker who possesses the evidence set $E_1 \cup E_2$ and is obliged to present all the evidence, might manipulate the order in which the evidence is presented in order to manipulate the story builder’s conclusion.

Obviously, a rational story builder satisfies Story Consistency. We now show (Claim 3) that if the graph is a tree then Story Consistency implies that the story builder is rationalizable. We also provide an example (Claim 4) of a graph and a story builder who satisfies Story Consistency and is not rationalizable.
Claim 3. Assume that $G$ is a directed tree. If $F$ satisfies Story Consistency, then there exists a preference relation $\succsim$ on $S$ such that $F(E)$ is the $\succsim$-maximal story in $S_E$.

Proof: Given $F$, we say that the story $x$ is revealed to be better than the story $y$, denoted by $x \succsim y$, if there exists an evidence set $E$ such that $F(E) = x$ and $y \in S_E$. We show that the relation $\succsim$ is anti-symmetric and transitive and thus can be extended to a preference relation. Then, by definition, $F(E)$ is the $\succsim$-maximal element in $E$.

Anti-symmetry (Implied by Story Consistency even if $G$ is not a tree). Assume that $x \succsim y$ and $y \succsim x$, that is, there are evidence sets $E$ and $E'$ such that $x,y \in S_E \cap S_{E'}$, $F(E) = x$ and $F(E') = y$. Both $x$ and $y$ are in $S_{E \cup E'}$. Since $F(E) = x$ and $E' - E$ contains only events that appear in $x$, it follows from Story Consistency that $F(E \cup E') = x$. Similarly, $F(E \cup E') = y$, which implies that $x = y$.

Transitivity. Assume $x \succsim y$ and $y \succsim z$. Let $E$ be an evidence set such that $x,y \in S_E$ and $F(E) = x$, and let $E'$ be an evidence set such that $y,z \in S_{E'}$ and $F(E') = y$. Both $a_{xy}$ (the split event of the stories $x$ and $y$) and $a_{yz}$ are in $y$. The event $a_{xy}$ appears (weakly) before $a_{yz}$. Otherwise, since $G$ is a tree, $E'$ consists of events in $y$ that appears (weakly) before $a_{yz}$ and therefore $x \in S_{E'}$ implying $y \succsim x$ and contradicting the anti-symmetry of $\succsim$. Since $G$ is a tree, $E$ consists of events in $y$ that appears (weakly) before $a_{xy}$ and since $z$ splits from $y$ not before $a_{xy}$ the event $z$ is consistent with $E$, implying that $x \succsim z$.

Claim 4: There exists a graph and a story builder that satisfies Story Consistency but is not rationalizable.

Proof: Let $G = \langle A, \rightarrow \rangle$ be the graph where $A = \{1, 2, \ldots, T\} \times \{0, 1\}$ with $(t, 0) \rightarrow (t + 1, 1)$ for any $t, \delta$. An event $(t, \delta)$ can be thought of as "\delta happens at date $t$". Any event with index $t$ can be followed by any event with index $t + 1$. Obviously, $G$ is not a tree. A story $(O, (1, \delta_1), \ldots, (T, \delta_T))$ can be thought of as a sequence of zeroes and ones of length $T$ and thus can be simply denoted by $(\delta_1, \ldots, \delta_T)$.

Define $F(\{(t^1, \delta^1), \ldots, (t^K, \delta^K)\})$ where $1 \leq t^1 < t^2 < \ldots < t^K \leq T$ as the story $(a_1, \ldots, a_T)$ where $a_t = \delta^i$ for any $t^i \leq t < t^{i+1}$ and $a_t = \delta^K$ for $t < t^1$ or $T \geq t \geq t^K$. In other words, thinking about $1, \ldots, T$ as a cycle (1 comes after period $T$), any evidence that $\delta$ occurs at $t$ is taken as a proof that $\delta$ persists from $t$ on until contradictory evidence is received. In the case of the empty evidence set, $F(\emptyset) = (0, \ldots, 0)$. Thus, for example, when $T = 7$ we have $F(\{(2, 1), (4, 0), (6, 0)\}) = (0, 1, 1, 0, 0, 0, 0)$.
This story builder satisfies Property 1 but cannot be rationalized when \( T \geq 4 \). For example, if (for \( T = 4 \)) \( F \) was rationalized by \( \succeq \) then:

\[
\begin{align*}
F([(1,1),(3,0)]) &= (1,1,0,0) \succ (1,0,0,1) \\
F([(2,0),(4,1)]) &= (1,0,0,1) \succ (0,0,1,1) \\
F([(1,0),(3,1)]) &= (0,0,1,1) \succ (0,1,1,0) \\
F([(2,1),(4,0)]) &= (0,1,1,0) \succ (1,1,0,0), \text{ a contradiction.} 
\end{align*}
\]

A "slight" modification of the story builder described in the proof can be rationalized. Define \( F((t^1,\delta^1),...,(t^K,\delta^K)) \) where \( 1 \leq t^1 < t^2 < ... < t^K \leq T \) as the story \((a_1,\ldots,a_T)\) where \( a_t = \delta^l \) \((l = 1,\ldots,K-1)\) for every \( t^l \leq t < t^{l+1} \), \( a_t = \delta^1 \) for all \( t < t^1 \) and \( a_t = \delta^K \) for all \( t \geq t^K \). In other words, the story builder initially adopts \( a^1 = \delta^1 \). It proceeds to build the story so that \( a_t = a_{t-1} \) unless \( t = t^k \) for some \( k \) and \( a_{t-1} \neq \delta^k \) in which case he switches to \( a_t = \delta^k \). In addition, \( F(\emptyset) = (0,\ldots,0) \). Thus, for example, \( F([(2,1),(4,0),(6,0)]) = (1,1,0,0,0,0) \).

This modified story builder is rationalized by \( P(x_1,x_2,\ldots,x_T) = \sum_{t=2,\ldots,T} \lambda^t 1_{\{x_t = x_{t-1}\}} \) for \( \lambda < 1 \) close to 1. That is, \( P \) evaluates a story \((x_1,\ldots,x_T)\) by adding the weight \( \lambda^t \) (almost equal to 1) for every period \( t \) in which \( x_t = x_{t-1} \).

5. **Step-by-step story builders**

The story builders in the following examples construct a story in steps, starting from \( O \) and proceeding from one event to one of its immediate followers, by applying some "local" rule.

a. **Recursive Construction I: Progress along the most likely story line as long as you don't encounter a contradiction.**

The story builder has in mind an antisymmetric likelihood relation \( \succeq \) over all stories (that is, \( s \succeq s' \) means that \( s \) is more likely than \( s' \)). Given an evidence set \( E \), let \( C_k(E) \) be the set of all partial stories of length \( k \) that can be developed into a story consistent with \( E \). That is, \( C_k(E) = \{(O,a_1,\ldots,a_k) \mid \text{there is } s \in S_E \text{ that starts with } (O,a_1,\ldots,a_k)\} \).

The story is built recursively so that the story builder always has a full story in mind, one that is not necessarily consistent with the evidence. At \( k = 0 \), he keeps the most likely story in mind. He arrives at stage \( k + 1 \) with a story that starts with \((a_0,\ldots,a_k) \in \)
$C_k(E)$. He keeps the story in mind if it starts with $(a_0, \ldots, a_k, a_{k+1}) \in C_{k+1}(E)$. Otherwise, he replaces the story he has in mind with a new one: the most likely story from those that start with $(a_0, \ldots, a_k) \in C_k(E)$. He stops when he reaches a terminal event.

Consider Figure 3 with the relation $\triangleright$ represented by $p$. In the absence of any evidence, the story builder chooses $(O, b, c, d)$, the most likely story. Given the evidence set $\{e\}$, the story builder starts with the most likely story, i.e. $(O, b, c, d)$. He keeps it in mind at steps 1 and 2 since $(O, b) \in C_1(\{e\})$ and $(O, b, c) \in C_2(\{e\})$. At the third step, he realizes that $(O, b, c, d) \notin C_3(\{e\})$ and appends the event $e$ to $(O, b, c)$, thus ending up with the story $(o, b, c, e)$.

To demonstrate the psychological element behind this procedure, consider a person who is known to have a daily routine (home, cafe, office, gym, return home). The person was observed stopping at a bank and it is known that he could only have got there from either the cafe or the gym. According to the procedure, the story builder will believe (home, cafe, office, gym, bank, return home) rather than (home, cafe, bank, office, gym, return home) since he deviates from the most likely story only when he realizes that it does not have a consistent continuation.

This procedure satisfies Story Consistency. The story builder in the example depicted in Figure 3 is not rationalized by the likelihood relation $\triangleright$ represented by $p$ but is rationalized by the ordering $(O, b, c, d) \triangleright (O, b, c, e) \triangleright (O, a, c, d) \triangleright (O, a, f)$. More generally:

**Claim 5:** Let $G$ be a graph. Let $F$ be a story builder that follows "Recursive Construction I" with the likelihood relation $\triangleright$. Then, there is an ordering $\succsim$ on $S$ such that $F = F_{\succsim}$.

**Proof:** Recall that $l(x)$ is the length of the story $x$ and $j(x, y)$ is the length of the common head of the two stories. Define $x = (O, x_1, \ldots, x_{l(x)}) \succsim y = (O, y_1, \ldots, y_{l(y)})$ if there is a story $s$ starting with $(O, x_1, \ldots, x_{j(x,y)}, x_{j(x,y)+1})$ such that $s \succeq s'$ for all $s'$ starting with $(O, x_1, \ldots, x_{j(x,y)}, y_{j(x,y)+1})$. This relation is complete and antisymmetric. To verify transitivity assume $x \succsim y$ and $y \succsim z$. Let $s$ be a story starting with $(O, x_1, \ldots, x_{j(x,y)+1})$ that is $\triangleright$-preferred to any story starting with $(O, y_1, \ldots, y_{j(x,y)+1})$ and let $t$ be a story starting with $(O, y_1, \ldots, y_{j(y,z)+1})$ that is $\triangleright$-preferred to any story starting with $(O, z_1, \ldots, z_{j(x,y)+1})$. 

$p(O, a, f) = 0.2$

$p(O, a, c, e) = 0.35$

$p(O, b, c, d) = 0.45$

![Figure 3](image-url)
• Case (i): \(j(x, y) < j(y, z)\). Since \(j(x, y) = j(x, z) < j(y, z)\), we have \((O, y_1, \ldots, y_{j(x,y)+1}) = (O, z_1, \ldots, z_{j(x,y)+1})\) and \(s\) is also \(\succeq\)-preferred to any story starting with \((O, z_1, \ldots, z_{j(y,z)+1})\). Thus, \(x \succeq z\).

• Case (ii): \(j(x, y) > j(y, z)\). The story \(t\) is also consistent with \((O, x_1, \ldots, x_{j(x,y)+1})\) and since \(j(x, y) = j(x, z)\) we have \(x \succeq z\).

• Case (iii): \(j(x, y) = j(y, z) = j(x, z)\). The story \(s\) is \(\succeq\)-preferred to any story starting with \((O, y_1, \ldots, y_{j(x,y)+1})\) and, in particular, \(s \succeq t\). Thus, \(s\) is \(\succeq\)-preferred to any story starting with \((O, z_1, \ldots, z_{j(y,z)+1})\) and thus \(x \succeq z\).

• It is not possible that \(j(x, y) = j(y, z) < j(x, z)\) since if it was then there would be a story \(s\) consistent with \((O, x_1, \ldots, x_{j(x,y)+1})\) and \(\succeq\)-preferred to any story consistent with \((O, y_1, \ldots, y_{j(x,y)+1})\) including a story consistent with \((O, y_1, \ldots, y_{j(x,y)+1})\) and \(\succeq\)-preferred to any story consistent with \((O, z_1, \ldots, z_{j(y,z)+1}) = (O, x_1, \ldots, y_{j(y,z)+1})\) including the story \(s\).

• If \(j(x, y) > j(y, z)\), there is a story \(t\) consistent with \((O, y_1, \ldots, y_{j(y,z)+1})\) which is \(\succeq\)-preferred to any story consistent with \((O, z_1, \ldots, z_{j(x,y)+1})\). The story \(t\) is also consistent with \((O, x_1, \ldots, x_{j(x,y)+1})\) and since \(j(x, y) = j(x, z)\) we have \(x \succeq z\).

Finally, we verify that \(F = F_\succ\). Let \(s = F(E)\) and \(t = F_\succ(E)\). By definition of \(j(s, t)\), \((O, t_1, \ldots, t_{j(s,t)}) = (O, s_1, \ldots, s_{j(s,t)})\) and \(s_{j(s,t)+1} \neq t_{j(s,t)+1}\). Then \(t \succeq x\) for any story \(x\) consistent with \(E\) and, in particular, \(t \succeq s\). By definition of \(\succeq\), there is a story starting with \((O, t_1, \ldots, t_{j(s,t)+1})\) that is \(\succeq\)-preferred to a story starting with \((O, t_1, \ldots, t_k, s_{j(s,t)+1})\), a contradiction of the definition of \(s_{j(s,t)+1}\). ■
b. **Recursive Construction II: Advance to the most likely event given the evidence and the partial story.** The story builder has in mind a probability measure $p$ on $S$. For an evidence set $E$, let $p_E$ be the conditional of $p$ on $E$. The story builder constructs a story recursively. At stage 0, he sets $a_0 = O$. He approaches stage $m + 1$ after constructing a path $(a_0, a_1, ..., a_m)$. He appends to the path the event that, given the evidence and given $(a_0, a_1, ..., a_m)$, is the most likely follower of $a_m$. In other words, he chooses $a_{m+1}$ to be a maximizer of $\phi(a) = p_E(\{ s \mid s \text{ starts with } (a_0, a_1, ..., a_m, a) \})$.

Such a procedure may not satisfy Story Consistency. For example, consider the graph and probability measure $p$ depicted in Figure 4. In that case $F(\emptyset) = (O, a, b)$: in the absence of any evidence, the story builder advances from $O$ to $a$ which is a more likely continuation than $b$ and then proceeds to $b$ since the path $(O, a, b)$ is more likely than $(O, a, c)$. However, if he receives the evidence $\{b\}$ the story builder advances from $O$ to $b$ since $(O, b)$ is more likely than $(O, a, b)$ and thus $F(\{b\}) = (O, b)$.

c. **Recursive Construction III: Advance to the most likely event given the evidence but independently of the partial story.** The story builder has in mind a probability measure $p$ on $S$. He builds the story recursively. At stage 0, he sets $a_0 = O$. He approaches stage $m + 1$ after constructing a partial path $(a_0, a_1, ..., a_m)$. He continues by adding the most likely event following $a_m$ conditional on the evidence and ignoring the partial path he has constructed so far. In other words, he chooses $a_{m+1}$ to be a maximizer of $\psi(a) = p_E(\{ s \mid a \text{ is in } s \})$. This procedure is similar to the previous one with the difference that when constructing the story recursively, the appended event $a_{m+1}$ is selected independently of what the story builder believes to be the path that led to $a_m$. Note that the test for choosing $a_{m+1}$ is based on assessing $p_E(\{ s \mid a \text{ is in } s \})$ rather than $p_E(\{ s \mid s \text{ contains } a_m \to a \})$.

Such a story builder $F$ may not satisfy Story Consistency. Consider Figure 3. In this case, $F(\emptyset) = (O, a, c, d)$. The story builder continues from the partial path $(O, a, c)$ to $d$ since it is a more frequent event than $e$, although the path $(O, a, c, d)$ is a zero probability story. Given the evidence $\{a\}$ he reaches the conclusion that $F(\{a\}) = (O, a, c, e)$.

d. "**Backward induction**". The story builder applies a rule to select a terminal event $a
consistent with the evidence (for example, he holds a probabilistic belief on the terminal events and chooses \( a \) to be the most likely one consistent with the evidence). He then applies one of the procedures mentioned earlier to the modified graph obtained by reversing all arrows and placing \( a \) as the origin.

6. The story builder deliberates on the source of the evidence

Story builders differ in how they assess the fact that a particular evidence set has reached them. We distinguish between non-strategic and strategic approaches.

**Non-strategic approach:** The story builder treats the source of the evidence as nature. He has in mind a function \( \mu \) that assigns to each story \( s \) a belief over the evidence sets that are consistent with \( s \). That is, \( \mu(s)(E) \) is the probability he assigns to receiving the evidence set \( E \) when the truth is \( s \).

Given an evidence set \( E \), the story builder applies Bayesian reasoning to update his beliefs about \( s \) and then adopts the most likely story. Such a story builder may not satisfy the invariance property (and thus not Story Consistency either). Consider the graph presented in Figure 5 and assume that the story builder believes that the evidence set includes the event \( b \) if and only if the true story ends with \( z_2 \). Then, for any \( E \subseteq \{a, b\} \), \( F(E) = (O, a, b, z_2) \) if \( b \in E \) and \( F(E) = (O, a, b, z_1) \) if \( b \notin E \).

This story builder does not satisfy the invariance property since \( S_{\{a\}} = S_{\{b\}} \) while \( F(\{a\}) \neq F(\{b\}) \).

**Strategic approach:** The story builder views the evidence as originating from a source with preferences over the story builder’s conclusions. That is, the story builder views the situation within some speaker-listener framework.

We say that a story builder is *Gricean* if he believes that the speaker: (i) knows the truth; (ii) is interested in the story builder arriving at the true story, and (iii) is not better off presenting only a strict subset of \( E \) if he holds the evidence set \( E \). Formally, we say that a story builder \( F \) is Gricean if there is no \( s \in S \) and an evidence set \( E \) that is consistent with \( s \), such that \( F(E) \neq s \) and there is an \( E' \subset E \) with \( F(E') = s \).
This definition of the Gricean story builder is in line with Grice’s Cooperative Principle which underlies the natural interpretation of utterances in the course of a conversation (see Grice (1989)). The speaker provides evidence in order to serve the common interest that the listener will arrive at the truth (the maxim of quality). Notice that (iii) does not contradict Grice’s maxim of quantity since it allows for the speaker to present only part of the evidence he possesses when it is sufficient to convey the truth.

Claim 6: $F$ satisfies Story Consistency if and only if $F$ is Gricean.

Proof: Assume that $F$ does not satisfy Story Consistency. That is, there is an evidence set $E'$ and an event $a \notin E'$ such that $s \in S_{E' \cup \{a\}}$ and $F(E' \cup \{a\}) = s' \neq s = F(E')$. Then, $s$ is consistent with $E = E' \cup \{a\}$, $F(E) \neq s$ and $F(E') = s$, (that is, if the speaker holds the evidence set $E$ at $s$ he prefers to present the evidence set $E'$) and therefore $F$ is not Gricean.

Assume that $F$ is not Gricean. That is, there are two sets $E \supset E'$, and an event $s$ consistent with $E$, such that $F(E) \neq s$ and $F(E') = s$. Then, there exist $E \supset E'' \supset E'$ and $a \in E - E''$ such that $F(E'') = s$ but $F(E'' \cup \{a\}) \neq s$. Then, $F(E'') = s$ and $a$ is consistent with $s$ but $F(E'' \cup \{a\}) \neq s$, contradicting Story Consistency. ■

We say that the story builder is suspicious if he believes that the speaker: (i) has some preferences about the story that the story builder constructs; (ii) knows the truth and is able to present the entire true story or any part of it as evidence; and (iii) chooses the evidence that leads the story builder to believe in the best story from the speaker’s perspective. Claim 7 states that if the story builder $F$ is “rational” then $F$ is consistent with him believing that the speaker is thinking strategically. Formally, we say that the story builder $F$ is suspicious if there is a preference relation $\succsim_s$ on $S$ such that $F(E) \succsim_s F(E')$ for any $E'$ consistent with $F(E)$.

Claim 7: If $F$ is rational, then $F$ is suspicious.

Proof: Since $F$ is rational there is an ordering $\succsim_l$ such that if $F(E) = s$ then $s \succsim_l s'$ for all $s' \in S_{E} - \{s\}$. Define $\succsim_s$ by $x \succsim_s y$ iff $y \succsim_l x$. If $F(E) = s$ and $E'$ is an evidence set consistent with $s$ so that $F(E') = s' \neq s$, then $s' \succsim_l s$ and therefore $s \succsim_l s'$. ■
7. A Robust Failure of coordination between different types of story builders

Developing the concept of a story builder leads to the study of interactions among different types of story builders. To demonstrate the potential of the analysis, consider the following simple example: a "large" number of individuals are randomly pair-wise matched and play a coordination game in which each tells a story represented by a path in the graph shown in Figure 6.

Before telling a story, each individual builds a point-wise belief (a story) about his partner’s choice. Assume that no individual has any evidence of his partner’s actual choice. The population is partitioned into three types of story builders as described below. All individuals use the same distribution of stories \( p \) as a parameter in their belief formation procedure. In equilibrium, this distribution is the actual distribution of stories in the population.

Following are the three types of story builders:
Type \( a \) chooses the most likely story given \( p \).
Type \( b \) uses the Recursive II procedure (building the story by advancing along the graph in steps, each time moving to the most likely next event, given \( p \) and given the path he has chosen so far).
Type \( c \) uses the Recursive III procedure (choosing the most likely next event independently of the path he has chosen so far).

Denote the frequencies of these three types in the population by \( \alpha, \beta \) and \( \gamma \), respectively. Assume that \( \beta + \gamma > \alpha > \delta > \gamma \).

A candidate for equilibrium is a profile \( (s_a, s_b, s_c) \) where the story \( s_i \) is chosen by the individuals of type \( i \). Since each individual wishes to coordinate with his matched partner, we require that in equilibrium \( s_i \) be identical to the story built by a type \( i \) story builder about his partner, given \( p \). In addition we require that \( p \) be the distribution of stories in the population, induced from \( (s_a, s_b, s_c) \) and the distribution of types \( (\alpha, \beta, \gamma) \).

Under these assumptions, \( s_a = (O, a, c) \), \( s_b = (O, b, f) \), \( s_c = (O, b, c) \) is an equilibrium. The \( a \) type finds \( (O, a, c) \) to be the most likely story. The \( b \) type chooses \( s_b \) since the event \( b \) is the most likely event that follows \( O (\beta + \gamma > \alpha) \) and \( f \) is the most likely event.
given the path \((O, b) (\beta > \gamma)\). The \(c\) type chooses \(s_c\) since the event \(b\) is more likely than \(a\) \((\beta + \gamma > \alpha)\) and the event \(c\) is more likely than \(f\) \((\alpha + \gamma > \beta)\).

In this equilibrium, when two individuals of different types are matched they will fail to coordinate. This failure differs from that in a non-degenerate mixed strategy Nash equilibrium, in a standard coordination game. In such an equilibrium, each player is indifferent between several alternatives. The mixed strategy equilibrium is not robust in the sense that a small change in the action distribution will push the players away from the equilibrium. In our case, the behavior of each type is robust to "slight changes" in the distribution of stories in the population.

Note that such a robust equilibrium with coordination failure will not exist if all individuals were of the same type or if only two of the three types (with nonidentical proportions) existed in the population.

One can view a story builder in this example as a problem solver in the sense of Glazer and Rubinstein (2018) whose task is to come up with a story that will match his partner’s choice. In this respect, the model is one of interaction between different problem solvers.
References


