

# Strategic Tournaments

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## **Abstract**

A strategic (round-robin) tournament is a simultaneous  $n$ -player game built on top of a symmetric two-player game  $G$ . Each player chooses one action in  $G$  and is matched to play  $G$  against all other players. The winner of the tournament is the player who achieves the highest total  $G$ -payoff. The tournament has several interpretations as an evolutionary model, as a model of social interaction and as a model of competition between firms with procedurally rational consumers. We prove some general properties of the model and explore the intuition that a tournament encourages riskier behavior. (JEL C72, C73)

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In this paper, we study a game with a special structure to be referred to as a *tournament*. A tournament is built on top of a symmetric two-player game  $G$ . The tournament is based on the round-robin method, in which each player is matched against each of the other players once. In each match the players play the game  $G$ . A player in the tournament is assumed to employ the same action in all his matches and his total score in the tournament is the sum of his "payoffs" in each individual match. A prize is awarded to the player with the highest total score. In the case of a tie, the prize is randomly rewarded to one of the top scoring players. A player in the tournament wishes to maximize his probability of winning the prize. In particular, he does not care about his absolute total score and is indifferent between attaining the second-best score and the lowest score.

The following game  $G$  is used to illustrate the structure of a tournament:

Each player can attain at most 10 "points". He can sacrifice 1 or 2 points in order to "destroy" 4 or 8 points respectively of the other player.

|     |     |     |     |
|-----|-----|-----|-----|
|     | $a$ | $b$ | $c$ |
| $a$ | 10  | 6   | 2   |
| $b$ | 9   | 5   | 1   |
| $c$ | 8   | 4   | 0   |

The only symmetric Nash equilibrium of  $G$  is the dominant strategy  $a$ . However,  $a$  is not an equilibrium of the tournament based on  $G$  with, say, three players. In this case, if a player deviates to  $b$  he loses 1 point in each match but inflicts a loss of 4 on the other two players and thus becomes the sole winner. Thus, the deviation is profitable. The tournament with three players (and in fact with two as well) has a unique equilibrium in which all players play  $c$ . If all players choose  $c$ , each of them has the same chance ( $1/3$ ) of winning the prize. If a player deviates, he will increase his score by less than the increase in the other players' scores and thus will reduce his probability of winning to 0. When the number of players in the tournament is 5 or more,  $a$  becomes the equilibrium.

Our model differs substantially from the standard contest model, which is sometimes referred to as a tournament. (See, for example, Lazear and Rosen (1981), Green and Stokey (1983), Dixit (1987) and Konard (2009) which is a recent survey of the contest literature.) In a

contest, each agent chooses an effort level and attains a score that increases with the level chosen. His utility in that model depends on his relative position among the agents and the cost of his effort. In our strategic (round-robin) tournament model, the relative score depends on the interaction between the agent's choice and those of the other agents and there is no cost associated with an agent's actions. Therefore, it is not possible to embed one model within the other.

Another related contest model discussed in the literature is the elimination tournament. This model consists of several stages. In each stage, players are pair-wise matched and play an all-pay auction game. The winner of each match progresses to the next stage. This literature focuses on the strategic dynamic considerations. (For a recent example of this type of model see Grou, Moldovanu, Sela and Sunde (2012)).

Our tournament model has several interpretations:

*The straightforward interpretation:* Players participate in a round-robin tournament in which changing one's action is costly and therefore a player employs the same action in all matches. This is often the case in sports tournaments in which a team's management chooses which type of players to purchase and on which tactics to focus during practice (e.g. offensive or defensive tactics) already at the beginning of the season.

*The evolutionary interpretation:* Each agent is characterized by an action in  $G$  (determined by his genes). Occasionally, pairs of agents are randomly matched to interact. Evolutionary forces select the agents with the highest achievements and eliminate the others. This happens, for example, when the agents are males who occasionally fight each other and the female selects only the winner for reproduction. In the context of the evolution of human behavior, one can think of the tendency of human beings to imitate the most successful modes of behavior as an evolutionary force that eliminates the less successful ones. (For a model of choice with a related interpretation, see Dekel and Scotchmer (1999). For a model of knock-out tournament in which the chance of survival depends on how far a player progresses in the tournament see Broom, Cannings and Vickers (2000).) This is also related to the evolutionary approach (see Shaffer (1988)) which assumes that only the strategies attaining the highest score will survive and thus a "spiteful" strategy (in the sense that it reduces the payoffs of his opponents more than that of the player) can invade a population successfully. Note that

this evolutionary approach differs from the commonly-used game-theoretical evolutionary approach, which treats the  $G$  payoffs as units of fitness that determine the rates of reproduction.

*The social interpretation:* Individuals are occasionally involved in two-person interactions. An individual is indifferent to his absolute level of achievement and cares only about his ranking relative to other individuals. He simply aspires to be the best among them. This is an extreme assumption but so is the commonly made assumption that players care only about the absolute payoff while ignoring their relative ranking in the population. One can also think of a career politician who chooses a "philosophy" as he feels obliged to be consistent over the course of his career and who from time to time must debate other politicians. The successful politician is the one who is perceived to have the upper hand most frequently.

*The procedural decision interpretation:* A number of firms, each producing one variation of the same product, are competing to win over a particular consumer. The game  $G$  reflects the pairwise comparison that the consumer makes between two firms. The score of a firm in  $G$  could be, for example, the number of dimensions in which its product is superior to the product of the other firm. The consumer makes all pairwise comparisons and eventually chooses the firm with the highest average score.

The analysis of the tournament employs the solution concept of a symmetric mixed strategy equilibrium. That is, each player employs an action that is drawn once from a lottery over actions (a mixed strategy). The lottery is executed only once and the player employs the resulting action against all other players. This concept is appropriate to a situation in which the players in a tournament are drawn from a large population characterized by a distribution of actions, which is treated as a mixed strategy, and the interaction between players is anonymous. We apply the concept in order to examine the intuition that tournaments encourage participants to be more daring and to choose riskier or more aggressive actions, which may yield extreme payoffs and are often labeled as more competitive.

We have in mind two perspectives when comparing the Nash equilibrium of the tournament to that of  $G$ .

(a) Focus on the tournament. Calculating the equilibrium in a tournament based on  $G$  may be difficult and we attempt to determine whether one can use the concept of Nash equilibrium

in  $G$  as an approximation of the equilibria in the tournament (for a discussion of this approach, see Arad (2012)).

(b) Focus on  $G$ . A mixed-strategy Nash equilibrium is often justified as a stable distribution of actions over a large population of individuals who are occasionally matched to play  $G$  and strive to maximize their average payoff. We wish to examine the question of whether a Nash equilibrium of  $G$  also can be justified as a stable distribution of actions in environments where the individuals are interested in maximizing *the chances of being the best* rather than their average payoff. Such an alternative goal seems to be more appropriate to an evolutionary interpretation of equilibrium. If each equilibrium of  $G$  has a close equilibrium in the tournament with a large enough number of players, then such an interpretation is plausible. If only some equilibria have this property, then our approach is to be viewed as a selection criterion.

Some experimental game theorists have used the above tournament structure in experiments that aimed to study behavior in the game  $G$ . In such experiments, subjects are incentivized by a promise that the participant with the highest score will receive a prize (a noted early study of this kind is Axelrod (1984)'s olympiad of the repeated prisoner's dilemma). This method has been criticized on the grounds that the incentives of a player in a tournament are very different from those in the basic game  $G$  and thus a player's behavior may differ. Our paper attempts to examine, on a theoretical level, whether the Nash equilibrium in the basic game  $G$  is a good approximation of the equilibrium in a tournament. However, we do not answer the question of whether actual players behave differently in the basic game than in a tournament.

The paper has a classic game theoretical format. As in the model of repeated games, we study a class of games built on a basic normal-form game. We study the relationship between the equilibria of the tournament and those of the basic game. Whereas in the repeated games literature one of the main objectives is to identify conditions for the emergence of cooperation one of the main objectives here is to determine whether competition in the tournament encourages players to take risky actions more often than in the basic game  $G$ .

The structure of the paper is as follows: Section I formally presents the model. The analysis consists of three parts.

The first part (sections II and III) examines the basic relationship between the equilibria of

the tournament with  $n$  players  $T(G, n)$  and those of  $G$ . In Section II, we focus on the relationship between the equilibria of the tournament with a "large" number of players and the equilibria of  $G$ . In particular, we show that: (a) every limit of the equilibria of  $T(G, n)$  is a mixed-strategy equilibrium of  $G$ ; (b) for some games, there are equilibria of  $G$  that are not limits of any sequence of equilibria of  $T(G, n)$ ; and (c) a pure strict Nash equilibrium of  $G$  is an equilibrium of  $T(G, n)$  for large enough  $n$ . In Section III, we identify classes of games for which the set of equilibria of  $T(G, n)$  is identical to that of  $G$  for all  $n$ .

The second part (sections IV and V) investigates the common intuition that tournaments encourage aggressive actions and risk-taking. In Section IV, we analyze a novel game called "the challenge tournament" in which the tournament structure encourages players to take superfluous risks. In Section V, we analyze  $2 \times 2$  symmetric games and prove a general proposition regarding the direction from which equilibria of  $T(G, n)$  converge to an equilibrium of  $G$ . Based on this proposition, we demonstrate that tournaments do not always encourage daring behavior.

The third part of the paper (sections VI and VII) discusses two variations of the tournament structure. In Section VI, we explain the analytical differences between the structure of  $T(G, n)$  and that of a tournament in which a player chooses a mixed strategy that is executed independently in each match (rather than only once at the beginning of the tournament). In Section VII, we study a tournament procedure in which players are pair-wise matched once and the winner is the player with the highest score achieved in his individual match. We will see that the equilibria of the pair-wise matching tournament are quite different from those of the round-robin tournament and seem to exhibit more aggressive and risk taking behavior. The discussion in this part of the paper demonstrates the richness of possibilities for expanding the model to other natural procedures and the sensitivity of the results to the tournament procedure.

## I. The Model

Let  $G = \langle A, u \rangle$  be a two-player symmetric game: each player chooses an action from a finite set  $A$ ; when a player plays  $x$  and his opponent plays  $y$  the player's payoff (score) is given by  $u(x, y)$ . The tournament  $T(G, n)$  is a symmetric game with the set of players  $N = \{1, \dots, n\}$ . Each player chooses one element in  $A$ . In our framework, the game  $G$  is played between any two players. Given a profile of actions,  $(a_i)_{i \in N}$ , player  $i$ 's score in the tournament is the sum of the "payoffs" he collects in the  $n - 1$  individual interactions he participates in, that is  $\sum_{j \in N - \{i\}} u(a_i, a_j)$ . The winner in  $T(G, n)$  is the player with the highest total score. In the case of a tie, the winner is selected randomly from the set of top-scoring players. We assume that each player in the tournament wishes only to maximize his probability of winning the tournament.

The tournament is invariant to an affine transformation of the payoffs in  $G$ . Note that  $T(G, 2)$  is not identical to  $G$  since a player in  $T(G, 2)$  is only interested in his score exceeding that of the other player and does not care about its absolute value. Note also how this tournament differs from another intuitive tournament  $S(G, n)$  (see Laffond, Laslier and Le Breton, (2000)) in which a player's payoff is the sum of payoffs in the  $n - 1$  interactions (and as in  $T(G, n)$  a player in  $S(G, n)$  chooses one action which is employed in all his interactions).

Note the differences between this model and the contest model. In the latter, each player  $i$  chooses an action  $e_i$  (effort) that yields a random score which increases in player  $i$ 's effort. Player  $i$ 's payoff in the contest is his prize (which depends only on his relative score) less the cost of effort. In one sense, our model is simpler since it lacks the intrinsic cost of an action. However, it is more complex in the sense that a player's score also depends on the other players' actions.

We analyze the tournament  $T(G, n)$  as a one-shot symmetric game and use the standard concept of Symmetric Nash Equilibrium (with mixed strategies) which always exists. A mixed strategy in  $T(G, n)$  is a lottery over actions. The lottery is executed only once. This again contrasts with a different structure (which we will comment on later) in which a player chooses one mixed strategy and applies independent randomization in each match.

In order for a mixed strategy  $\alpha^*$  to be a symmetric equilibrium in  $T(G, n)$  it must satisfy the following two conditions:

(i) any action in its support wins the tournament with probability  $1/n$ , when all other  $(n - 1)$  players are faithful to the mixed strategy  $\alpha^*$ ; and

(ii) any action outside the support wins the tournament with probability not higher than  $1/n$  when the other  $(n - 1)$  players use the mixed strategy  $\alpha^*$ .

## II. The Limit of Tournaments Equilibria

We first examine the relationship between the equilibria of tournaments with a large number of participants and the equilibrium of the single game  $G$ . Our main result is that any limit of equilibria of  $T(G, n)$  (as  $n \rightarrow \infty$ ) is a symmetric mixed-strategy Nash equilibrium of  $G$ . However, the converse does not hold and we will bring examples of games that have symmetric equilibria that are not limits of any sequence of equilibria of the tournaments.

**Proposition 1:** Let  $\alpha^*$  be the limit of a subsequence of symmetric equilibria of  $T(G, n)$  when  $n \rightarrow \infty$ . Then  $\alpha^*$  is a symmetric equilibrium of  $G$ .

**Proof:** To keep the notation simple, assume that  $\alpha^*$  is a limit of a sequence  $(\alpha_n)$  of equilibria of  $T(G, n)$ . For an action  $x$  and a mixed strategy  $\alpha$ , denote by  $u(x, \alpha)$  the random variable that receives the value  $u(x, y)$  with probability  $\alpha(y)$ . Denote its expectation by  $Eu(x, \alpha) = \sum_y u(x, y)\alpha(y)$ .

Assume that  $\alpha^*$  is not a Nash equilibrium of  $G$ . Let  $b$  be an action in the support of  $\alpha^*$  and  $a$  another action such that  $Eu(a, \alpha^*) > Eu(b, \alpha^*)$ . There exists a number  $K > 0$  such that for any large enough  $n$ ,  $Eu(a, \alpha_n) > Eu(b, \alpha_n) + K$ . We distinguish between two cases:

Case (i):  $a$  is contained in the support of  $\alpha^*$ . It is sufficient to show that a player who plays  $b$  wins  $T(G, n)$  with probability in the order of  $1/n^2$ . Thus, it is less than  $1/n$ , which is the exact probability of winning for each action in the support of an equilibrium of  $T(G, n)$ . This follows from the following three observations:

(a) For large enough  $n$ , it is almost certain that the action  $a$  will be played by at least one player.

By Chernoff's inequality, for large enough  $n$  the probability that an action  $x$  in the support of  $\alpha^*$  appears in  $n$  realizations of the strategy  $\alpha_n$  with a proportion larger than  $\alpha^*(x)/2$  exceeds  $1 - q^n$  for some  $0 < q < 1$ . Thus:

(b) With a probability that rapidly approaches 1, there are at least  $\alpha^*(b)n/2$  players who end up playing  $b$ .



(c) The probability that an action  $b$  attains as high a score as the action  $a$  is in the order of  $c/n$  for some positive  $c$ . To see this, note that the action  $b$  attains a payoff as high as that attained by  $a$  if  $\sum_{k=1}^{n-2} u(b, \alpha_n^k) + u(b, a) \geq \sum_{k=1}^{n-2} u(a, \alpha_n^k) + u(a, b)$ , where  $\alpha_n^k$  is a copy of the  $T(G, n)$  equilibrium strategy  $\alpha_n$ . This inequality is equivalent to

$$\sum_{k=1}^{n-2} u(b, \alpha_n^k) - \sum_{k=1}^{n-2} u(a, \alpha_n^k) - (n-2)[Eu(b, \alpha_n) - Eu(a, \alpha_n)] \geq u(a, b) - u(b, a) - (n-2)[Eu(b, \alpha_n) - Eu(a, \alpha_n)].$$

For large enough  $n$ , this occurs when the value of the random variable  $\sum_{k=1}^{n-2} [u(b, \alpha_n^k) - u(a, \alpha_n^k)]$  exceeds its expectation  $(n-2)[Eu(b, \alpha_n) - Eu(a, \alpha_n)]$  by  $K(n-2) + L$ , where  $L = u(a, b) - u(b, a)$ . By Chebyshev's inequality, the probability of this event is less than  $(n-2)\sigma_n^2 / ((n-2)K + L)^2$ , where  $\sigma_n^2$  is the variance of  $[u(b, \alpha_n) - u(a, \alpha_n)]$ . This variance converges to the variance of  $[u(b, \alpha^*) - u(a, \alpha^*)]$ .

Thus, the probability that a particular player who plays  $b$  will win  $T(G, n)$  is bounded from above by  $(c/n) / (\alpha^*(b)n/2)$ .

Case (ii):  $a$  is not contained in the support of  $\alpha^*$ .

It will be sufficient to show that for large enough  $n$  if a player deviates to  $a$  his probability of winning will be close to 1 (for  $n$  large enough) and thus above  $1/n$ .

From case (i), it follows that  $Eu(x, \alpha^*)$  is constant for all  $x$  in the support of  $\alpha^*$ . Observation (c) implies that if a player in  $T(G, n)$  plays  $a$  and the other  $n-1$  players play according to  $\alpha_n$ , then the probability that any action  $x$  in the support of  $\alpha^*$  will attain a score higher than  $a$  is bounded from above by  $c_x/n$  for some  $c_x$ . Let  $c_*$  be the maximum of  $c_x$ . Then, the probability that  $a$  wins the tournament is at least  $(1 - c_*/n)^{|A|}$  which is close to 1 and exceeds  $1/n$  for large enough  $n$ .

Later we will see that for relatively small  $n$  the tournament equilibrium is often significantly and systematically different from the equilibria of the single game. Furthermore, the set of equilibria of the single game and that of the tournament may differ, even in the limit.

**Proposition 2:** An equilibrium of  $G$  is not necessarily a limit of any sequence of equilibria of  $T(G, n)$  when  $n \rightarrow \infty$ .

**Proof:** We will present two games, each of which has an equilibrium that is not a limit of

any sequence of equilibria in the corresponding tournaments. Consider first the following "Rock, Paper, Scissors with an Outside Option" game:

|     |     |     |     |     |
|-----|-----|-----|-----|-----|
|     | $a$ | $b$ | $c$ | $d$ |
| $a$ | 0   | 1   | -1  | 0   |
| $b$ | -1  | 0   | 1   | 0   |
| $c$ | 1   | -1  | 0   | 0   |
| $d$ | 0   | 0   | 0   | 0   |

This is a zero-sum game. The strategy  $\alpha^*$ , which assigns probability  $1/4$  to all actions, is an equilibrium of  $G$  but is not a limit of equilibria of a sequence of tournaments. In fact there is no equilibrium of  $T(G, n)$  that assigns positive probability to all actions. To see it, assume that such an equilibrium exists. Then, the probability of a player who plays  $d$  to win the tournament must be  $1/n$ . However, whenever a player who plays  $d$  attains the highest score it must also be that all other players score 0. Therefore, whenever a player who plays  $d$  wins  $T(G, n)$  with probability  $1/n$ , it must be that all players tie with a score of 0. But, since the equilibrium has full support there is a positive probability that there is no tie with a score of 0, a contradiction.

Now consider the following "degenerate" game:

|     |     |     |
|-----|-----|-----|
|     | $a$ | $b$ |
| $a$ | 1   | 0   |
| $b$ | 1   | 0   |

The action  $a$  is a pure Nash equilibrium of the game. However, the action  $a$  wins the tournament only if all players play  $a$ . Thus, playing  $b$  is a dominating strategy in the tournament and the only equilibrium of  $T(G, n)$  for any  $n$  is  $b$ . It follows that there is no equilibrium of  $T(G, n)$  that assigns positive probability to  $a$ .

The last game demonstrates that a pure strategy Nash equilibrium of  $G$  might not be an equilibrium of any  $T(G, n)$ . However, the following proposition shows that if there are no  $a \neq x$  for which  $u(x, a) = u(a, a)$  then any pure strategy Nash equilibrium is an equilibrium of

$T(G, n)$  for large enough  $n$ .

**Proposition 3:** Let  $G$  be a game such that there is no  $a \neq x$  such that  $u(x, a) = u(a, a)$ .

If  $a$  is a Nash equilibrium of  $G$ , then there is  $N$  such that  $a$  is an equilibrium of  $T(G, n)$  iff  $n > N$ .

If  $a$  is not a Nash equilibrium of  $G$ , then the set of  $n$  for which  $a$  is an equilibrium of  $T(G, n)$  is a bounded interval (possibly empty).

**Proof:** If all players play  $a$ , the action  $x$  is a profitable deviation in  $T(G, n)$  iff  $(n - 1)u(x, a) > u(a, x) + (n - 2)u(a, a)$ , i.e. iff  $(n - 1)(u(x, a) - u(a, a)) > u(a, x) - u(a, a)$ .

Thus, if  $u(a, a) > u(x, a)$ ,  $x$  is a profitable deviation for all  $n < v_x$  for some number  $v_x$ . If  $u(a, a) < u(x, a)$ , then  $x$  is a profitable deviation if  $n > v_x$  for some number  $v_x$ . It follows that  $a$  is an equilibrium of  $T(G, n)$  for any  $n$  satisfying  $\max\{v_x | x \neq a \text{ and } u(a, a) > u(x, a)\} < n < \min\{v_x | x \neq a \text{ and } u(a, a) < u(x, a)\}$  (it could be that no such integer exists). Note that  $a$  is a Nash equilibrium of  $G$  iff  $\min\{v_x | x \neq a \text{ and } u(a, a) < u(x, a)\} = \infty$ .

**Comment:** Our concept of pure-strategy Nash equilibrium in  $T(G, n)$  coincides with that of generalized ESS a la Shaffer (1988) when restricted to pure strategies. The concept of a generalized ESS is based on a scenario with  $n$  players, each of whom chooses a mixed strategy. When two players with mixed strategies are matched, each attains his expected payoff. Each player collects the expected payoff from each of the individual  $n - 1$  matches. A mixed strategy  $\alpha^*$  is a generalized ESS if there is no strategy  $\alpha$  such that if a player deviates to it, the sum of his expected payoffs is higher than that of the players who stick to  $\alpha^*$ . Although this concept and ours coincide when restricted to pure strategies, they are very different when applied to mixed strategies. In particular, our concept explicitly treats a player's payoff in each match as a random variable and thus we define a mixed strategy as an equilibrium of the tournament if there is no deviating strategy that wins the tournament with probability higher than  $1/n$ .

### III. Sometimes the Equilibria of $G$ and $T(G, n)$ Coincide

For some classes of games, the equilibria of the basic game and of the tournament systematically coincide for any number of players in the tournament. An example is the set of coordination games.

**Proposition 4:** Let  $G = (A, u)$  be a coordination game where  $u(x, y) = 1$  if  $x = y$  and  $u(x, y) = 0$  if  $x \neq y$ . For any  $n$ ,  $\alpha$  is an equilibrium of  $T(G, n)$  iff  $\alpha$  is an equilibrium of  $G$ .

**Proof:** If  $\alpha$  is an equilibrium of  $G$ , then there is a non-empty set of actions  $B$  such that  $\alpha(x) = 1/|B|$  for any  $x \in B$ . Given that all players in the tournament play  $\alpha$ , any action outside of  $B$  will score 0 with probability 1 and will be among the winners only in realizations where all players score 0. Thus, its probability of winning is below  $1/n$ .

Now assume that  $\alpha$  is an equilibrium of  $T(G, n)$  and assume by contradiction that  $\alpha(a) > \alpha(b)$ . It is enough to show that  $a$  yields a probability of winning strictly above  $1/n$ .

Consider a player who plays against  $n - 1$  players who use  $\alpha$ . Define a configuration of the actions taken by the  $n - 1$  players as a function  $\mu$  that assigns a non-negative integer  $\mu(x)$  to each  $x \in A$  such that  $\sum_{x \in A} \mu(x) = n - 1$ .

Let  $W(y, \mu)$  be the probability that a player who chooses  $y$  wins the tournament given that the realization of actions taken by the other players is summarized by  $\mu$ . If all other players play  $\alpha$ , denote by  $\text{Prob}_\alpha(\mu)$  the probability that the realized configuration of actions is  $\mu$ . Then, the probability that an action  $y$  wins the tournament is  $\sum_\mu W(y, \mu) \text{Prob}_\alpha(\mu)$ .

Now, for any  $\mu$  define a configuration  $\mu_{a,b}$  as  $\mu_{a,b}(b) = \mu(a)$ ,  $\mu_{a,b}(a) = \mu(b)$  and  $\mu_{a,b}(x) = \mu(x)$  for all  $x \neq a, b$ . Clearly,  $(\mu_{a,b})_{a,b} = \mu$ .

The set of configurations can be partitioned into  $\{\{\mu, \mu_{a,b}\} | \mu\}$ , where each cell contains one or two configurations.

To show that  $\sum_\mu W(a, \mu) \text{Prob}_\alpha(\mu) > \sum_\mu W(b, \mu) \text{Prob}_\alpha(\mu)$

it is sufficient to show that for any  $\mu$

$$W(a, \mu) \text{Prob}_\alpha(\mu) + W(a, \mu_{a,b}) \text{Prob}_\alpha(\mu_{a,b}) \geq W(b, \mu) \text{Prob}_\alpha(\mu) + W(b, \mu_{a,b}) \text{Prob}_\alpha(\mu_{a,b})$$

with at least one strict inequality.

This is equivalent to:

$$(W(a, \mu) - W(b, \mu)) \text{Prob}_\alpha(\mu) + (W(a, \mu_{a,b}) - W(b, \mu_{a,b})) \text{Prob}_\alpha(\mu_{a,b}) \geq 0.$$

For any  $\mu$ , we have  $W(a, \mu) = W(b, \mu_{a,b})$  (note that the scores for all other strategies in  $\mu$

are the same as in  $\mu_{ab}$ ).

Thus, the last inequality is equivalent to:

$$(W(a, \mu) - W(b, \mu))(\text{Prob}_\alpha(\mu) - \text{Prob}_\alpha(\mu_{a,b})) \geq 0.$$

This inequality holds since given that  $\alpha(a) > \alpha(b)$  for any  $\mu$ ,

$\text{Prob}_\alpha(\mu) - \text{Prob}_\alpha(\mu_{a,b}) > 0$  iff  $\mu(a) > \mu(b)$ .

Furthermore,  $\mu(a) > \mu(b)$  iff  $W(a, \mu) - W(b, \mu) > 0$ .

Clearly, for any  $\mu$  such that  $\mu \neq \mu_{a,b}$ , this is a strict inequality.

**Comment:** Proposition 4 can be generalized in a straightforward manner to include any game  $G$  in which (i) for every action  $a$  there is a unique action  $\sigma(a)$  such that  $u(a, b) = 1$  for  $b = \sigma(a)$  and  $u(a, b) = 0$  otherwise; and (ii)  $\sigma(\sigma(a)) = a$  for every  $a$ . In particular, the uniform mixed strategy is the only equilibrium of any tournament  $T(G, n)$  based on:

|     |     |     |
|-----|-----|-----|
|     | $T$ | $B$ |
| $T$ | 0   | 1   |
| $B$ | 1   | 0   |

The following is another example of a game  $G$  for which the set of equilibria of  $T(G, n)$  coincides with those of  $G$ .

**Example 1: A Variation of Rock, Paper, Scissors**

|     |     |     |     |
|-----|-----|-----|-----|
|     | $R$ | $P$ | $S$ |
| $R$ | 0   | 0   | 1   |
| $P$ | 1   | 0   | 0   |
| $S$ | 0   | 1   | 0   |

**Claim 1:**  $\alpha$  is an equilibrium of  $T(G, n)$  iff  $\alpha$  is an equilibrium of  $G$ .

**Proof:** One side is straightforward. Now assume that  $\alpha$  is an equilibrium of  $T(G, n)$  that is not uniform. Clearly, it has full support.

There are only two cases to explore:

- (a) the most frequent action beats the least frequent action, i.e. without loss of generality

$$\alpha(S) \geq \alpha(R) \geq \alpha(P);$$

(b) the least frequent action beats the most frequent action, i.e. without loss of generality  $\alpha(S) \leq \alpha(R) \leq \alpha(P)$ .

We will show for (a) that  $R$  is strictly better than  $S$  against  $n - 1$  players playing  $\alpha$ . (Similarly, one can show for (b) that  $S$  is strictly better than  $R$  against  $n - 1$  players playing  $\alpha$ .)

As before,  $\mu$  denotes a configuration. Consider the function that assigns to each configuration  $\mu$  the configuration  $\mu'$  defined by  $\mu'(S) = \mu(P)$ ,  $\mu'(R) = \mu(S)$  and  $\mu'(P) = \mu(R)$ . This function is a one-to-one and onto.  $R$  wins in  $\mu'$  with a particular number of co-winners iff  $S$  wins in  $\mu$  with the same number of co-winners. The probability of  $S$  winning the tournament is  $\sum_{\mu} W(S, \mu) \text{Prob}_{\alpha}(\mu)$ . The probability of  $R$  winning is  $\sum_{\mu} W(R, \mu') \text{Prob}_{\alpha}(\mu')$ . Given that  $W(R, \mu') = W(S, \mu)$ , we need to show that

$$\sum_{\mu} W(S, \mu) \text{Prob}_{\alpha}(\mu') > \sum_{\mu} W(S, \mu) \text{Prob}_{\alpha}(\mu).$$

Recall that if our player plays  $S$ , he will be among the winners in any realization of the  $n - 1$  strategies that yields  $\mu$  for which  $W(S, \mu) > 0$ . Partition the set of configurations for which  $W(S, \mu) > 0$  into two subsets:

$$\Lambda_1 = \{\mu \mid \mu(R) = 0\}$$

$$\Lambda_2 = \{\mu \mid \mu(P) \geq \mu(S) + 1 \text{ and } \mu(P) \geq \mu(R) \geq 1\}.$$

To show that the probability of  $R$  winning is above that of  $S$ , it is sufficient to show two statements:

1.  $\sum_{\mu \in \Lambda_1} W(S, \mu) \text{Prob}_{\alpha}(\mu') \geq \sum_{\mu \in \Lambda_1} W(S, \mu) \text{Prob}_{\alpha}(\mu)$  with strict inequality if  $\alpha(R) > \alpha(P)$ .

For any  $\mu \in \Lambda_1$ , we have  $W(S, \mu) = 1$ . Thus, the left hand side of the inequality is the probability of all configurations  $\mu$  with  $\mu(R) = 0$ . The right hand side is the probability of all configurations  $\mu$  with  $\mu(P) = 0$ . Given  $\alpha(R) \geq \alpha(P)$ , the former is at least as likely as the latter and is strictly more likely if  $\alpha(R) > \alpha(P)$ .

2. For any  $\mu \in \Lambda_2$ ,  $\text{Prob}_{\alpha}(\mu') \geq \text{Prob}_{\alpha}(\mu)$ , with strict inequality for some configurations.

Since  $\alpha$  is not the uniform mixed strategy one of the two must hold:

(i)  $\alpha(S) > \alpha(R) \geq \alpha(P)$ ; then

$$\text{Prob}_{\alpha}(\mu') / \text{Prob}_{\alpha}(\mu) = \alpha(S)^{\mu(P) - \mu(S)} \alpha(R)^{\mu(S) - \mu(R)} \alpha(P)^{\mu(R) - \mu(P)} \geq \alpha(S)^{\mu(P) - \mu(S)} \alpha(R)^{\mu(S) - \mu(P)} > 1.$$

(ii)  $\alpha(S) \geq \alpha(R) > \alpha(P)$ ; then

$$\text{Prob}_{\alpha}(\mu') / \text{Prob}_{\alpha}(\mu) \geq \alpha(R)^{\mu(P) - \mu(R)} \alpha(P)^{\mu(R) - \mu(P)} \geq 1 \text{ (with strict inequalities for some of the$$

configurations).

#### **IV. Tournaments Encourage Superfluous Risk Taking: The Challenge Tournament**

We now examine the common intuition that tournaments encourage risk-taking. In this section, we present an extended example which shows that tournaments may lead to superfluous risk-taking. The example illustrates a phenomenon observed in real-life, where some activities are primarily motivated by the desire to compete for its own sake. An extreme example is the popular children's card game "war" which has no strategic component and whose outcome is determined purely by luck. Many adult betting activities (such as "let's bet on whether there will be a war next year") have no purpose but to satisfy people's desire to win.

In the tournament analyzed below, a player decides whether or not to be a challenger, where challenging means forcing a fair bet on all other players. A player's only aim is to maximize the probability that his total net payoff from betting will be the highest among the players. The tournament is built on a zero-sum game  $G$ , in which each player chooses either 1 (challenge) or 0 (don't challenge). One of the two players is chosen randomly (with equal probabilities) to be the winner. A number of points equal to the sum of the number of challenges (0 if no one challenges, 1 if only one player challenges and 2 if both players challenge) is transferred from the loser to the winner. In other words, players in  $G$  take fair bets and the size of the bet depends on the "heat" of the match. Note that in this example  $u(x,y)$  is a random variable. Of course, any mixed strategy, including "challenge" and "don't challenge", is a symmetric equilibrium of  $G$ . We will see that only one of these equilibria will prevail in the tournament.

The tournament with  $n$  players involves three types of randomizations:

(a) The  $n$  independent randomizations of the mixed strategies. Each randomization yields 0 or 1. Note again that this randomization is done only once and determines a player's action in each of his  $n - 1$  matches.

(b) The  $C(n,2)$  randomizations which determine the winner in each of the two-player matches.

(c) A lottery which selects the winner from among the players with the highest score.

In what follows, we will use the letter  $\alpha$  to denote the mixed strategy according to which a player chooses to challenge with probability  $\alpha$ .

The tournament  $T(G,3)$  has three equilibria:  $\alpha = 0$ ,  $1/2$ , and  $1$ . "Don't challenge" is an equilibrium since if one player deviates and challenges he reduces his probability of winning from  $1/3$  to  $1/4$ . Similarly, "challenge" is an equilibrium since a player who deviates and does not challenge will reduce his probability of winning to  $1/4$ . For any other mixed strategy where a player challenges with probability  $1 > \alpha > 0$ , if a player challenges while the others play  $\alpha$  then his probability of winning is  $\alpha^2(1/3) + (1 - \alpha)^2(1/4) + 2\alpha(1 - \alpha)(3/8)$ , which is  $1/3$  only for  $\alpha = 1/2$ .

A direct calculation shows that  $T(G,4)$  has only one equilibrium:  $\alpha = 1$ .

**Proposition 5:** Let  $T(G,n)$  be the challenge tournament.

(a) For all  $n$ ,  $\alpha = 1$  is an equilibrium of  $T(G,n)$ .

(b) For  $n > 3$ ,  $\alpha = 0$  is not an equilibrium of  $T(G,n)$ .

**Proof:** (a) Consider a player who confronts  $n - 1$  players who play  $1$ . We will show that the probability that this player wins the tournament if he plays  $1$  is higher than if he plays  $0$ .

Consider a realization of the randomizations in the matches such that if our player plays  $0$ , he will score  $W$  and be among  $w$  co-winners. It must be that each of the other  $n - 1$  players receives an odd number of points (since they either win or lose 2 points in each match among themselves and either win or lose one point in each match with our player). Thus, it must be that  $W > 0$  ( $W$  cannot be  $0$  since then all other  $n - 1$  players must have a negative score, which makes the sum of scores negative). For such a realization, our player receives a total score of  $2W$  if he switches to playing  $1$  while any other player increases his score by at most  $1$ . Thus, if  $W > 1$ , our player becomes the sole winner. If  $W = 1$ , he remains among the winners (and scores  $2$ ) and shares the prize with a subset of the  $w$  winners with whom he shares the prize if he plays  $0$ . Thus, our player's probability of winning if he plays  $1$  is higher than if he plays  $0$ .

(b) If a player deviates to  $1$  and all the others play  $0$ , then his score is distributed  $\text{Bin}(n - 1, 1/2)$ . For  $n = 4$ , he will win with probability  $1/8 + (1/2)(3/8) = 5/16 > 1/4$ . For  $n \geq 5$  the probability that his score is at least  $2$  (and thus he is the sole winner) is higher than



$1/n$  (and converges to  $1/2$ ).

Simulations suggest that for any  $n > 3$ , "challenge" is the only equilibrium of  $T(G, n)$  and thus the pure desire to win the tournament leads in this model to the maximal degree of superfluous activity. Note that this would follow from the conjecture below which states that in any case that some players end up playing 0 and some end up playing 1, a player who plays 1 wins the tournament with probability greater than  $1/n$ .

**Conjecture:** Assume  $n > 3$  and let  $N_0$  and  $N_1$  be a non-degenerate partition of the set of players. Assume that all players in  $N_0$  play 0 and all players in  $N_1$  play 1. Then, the probability that a player in  $N_1$  wins the tournament is strictly greater than  $1/n$ .

We have confirmed this conjecture analytically for  $n = 4$  and by simulations for  $n \leq 40$  but were unable to formulate a complete proof. Xiaosheng Mu (Yale University) has proved that the conjecture is true for large enough  $n$ .

## V. Do Tournaments Always Encourage Daring Behavior?

In the previous section, we studied a game which confirmed the intuition that competition in a tournament encourages players to take more risks. In this section, we continue the analysis using two classes of  $2 \times 2$  symmetric games.

We start by analyzing a general  $2 \times 2$  symmetric game:

|     |          |          |
|-----|----------|----------|
|     | $a$      | $b$      |
| $a$ | $u_{aa}$ | $u_{ab}$ |
| $b$ | $u_{ba}$ | $u_{bb}$ |

Denote by  $\alpha$  a mixed strategy that assigns probability  $\alpha$  to the action  $a$ . For simplicity, we restrict ourselves to "generic" payoffs in the sense that two actions never tie in the tournament (i.e. there are no integers  $w, x, y$  and  $z$  such that  $wu_{aa} + xu_{ab} = yu_{ba} + zu_{bb}$ ).

It follows from Proposition 3 that for any generic  $2 \times 2$  game  $G$ : (i) if  $a$  is a Nash equilibrium of  $G$  then there is  $N_*$  such that  $a$  is an equilibrium of  $T(G, n)$  iff  $n > N_*$  and (ii) if

$a$  is not a Nash equilibrium of  $G$  then there is an  $N^*$  (possibly 1) such that  $a$  is an equilibrium of  $T(G, n)$  iff  $n < N^*$ .

**Proposition 6:** Let  $G$  be a symmetric  $2 \times 2$  game with a unique non-degenerate mixed strategy equilibrium  $\alpha^*$ . Then, for large enough  $n$  the tournament has a unique symmetric equilibrium  $\alpha(n)$  which lies between  $\alpha^*$  and  $1/2$ .

**Proof:** Since  $G$  does not have a symmetric pure strategy equilibrium,  $u_{aa} < u_{ba}$  and  $u_{bb} < u_{ab}$ .

The action  $a$  attains a top score in  $T(G, n)$  whenever the number of other players who play  $a$ , denoted by  $x$ , satisfies  $x = n - 1$  or

$$(x)u_{aa} + (n - 1 - x)u_{ab} > (x + 1)u_{ba} + (n - 2 - x)u_{bb}, \text{ which is equivalent to}$$

$$(x)(u_{aa} - u_{ba} + u_{bb} - u_{ab}) > u_{ba} + (n - 2)u_{bb} - (n - 1)u_{ab} =$$

$$(u_{ba} - 2u_{bb} + u_{ab}) + n(u_{bb} - u_{ab}).$$

Since  $(u_{aa} - u_{ba} + u_{bb} - u_{ab}) < 0$  and  $\alpha^* = (u_{ab} - u_{bb})/(u_{ba} - u_{aa} + u_{ab} - u_{bb})$ , the action  $a$  will be a top score if  $x = n - 1$  or if

$$x/n < ((u_{ba} - 2u_{bb} + u_{ab})/(u_{aa} - u_{ba} + u_{bb} - u_{ab}))/n + \alpha^* = q^*(n).$$

In equilibrium the probability that a player who plays  $a$  will win must be  $1/n$ . Thus, denoting  $m = \lceil nq^*(n) \rceil$  we get:

$$\alpha^{n-1}/n + \sum_{k=0, \dots, m-1} \alpha^k (1 - \alpha)^{n-1-k} C(k, n-1)/(k+1) = 1/n.$$

Since  $nC(k, n-1)/(k+1) = C(k+1, n)$  we have

$$\sum_{k=0, \dots, m-1} \alpha^{k+1} (1 - \alpha)^{n-1-k} C(k+1, n) + \alpha^n = \alpha, \text{ or in other words:}$$

$$\sum_{k=0, \dots, m} \alpha^k (1 - \alpha)^{n-k} C(k, n) - (1 - \alpha)^n + \alpha^n = \alpha$$

which is equivalent to  $P(\alpha) = Q(\alpha)$ , where  $P(\alpha) = \text{prob}[\text{Bin}(n, \alpha) \leq m]$  and  $Q(\alpha) = \alpha + (1 - \alpha)^n - \alpha^n$ .

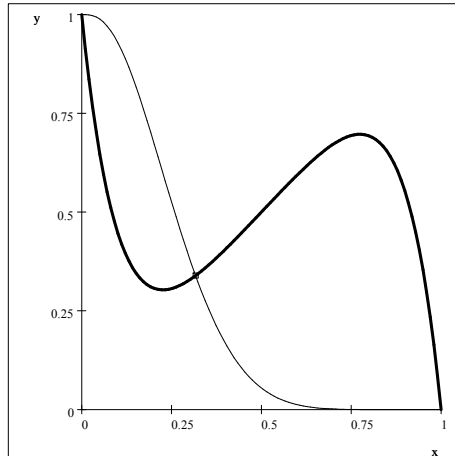


Figure 1: Illustration of  $P$  (non-bold line) and  $Q$  (bold line) for  $n = 10$ ,  $m = 2$  (intersection at  $\alpha = 0.317$ )

Assume that  $\alpha^* < 1/2$  (the case of  $\alpha^* > 1/2$  is analogous). For large enough  $n$ ,  $\alpha(n) < 1/2$  (by Proposition 1 and the uniqueness of equilibrium of  $G$ ). The function  $P$  starts at 1 and stays close to it until dropping rapidly to the vicinity of 0. The function  $Q$  is convex in  $[0, 1/2]$ ,  $Q(x) = 1 - Q(1 - x)$ ,  $Q(1/2) = 1/2$  and  $Q(0) = 1$  (see Figure 1). For large  $n$ ,  $Q$  is almost identical to  $\alpha$  at almost throughout the interval.

An equilibrium of  $T(G, n)$  is located at the intersection of  $P$  and  $Q$ . For large enough  $n$  the intersection must be above  $\alpha^*$  since by Hoeffding's inequality  $P(\alpha^*)$  – the probability that  $\text{Bin}(n, \alpha^*)/n < \alpha^*$  – is at least  $1/2$ , i.e.  $P(\alpha^*) \geq 1/2$ . Thus, for large enough  $n$  the intersection of  $P$  and  $Q$  must be between  $\alpha^*$  and  $1/2$ .

We now apply the previous analysis to two classes of games. For simplicity we focus on generic games.

### Example 2: Taking risks

Let  $G(z)$  ( $z > 0$ ) be the following game

|     |     |     |
|-----|-----|-----|
|     | $a$ | $b$ |
| $a$ | 0   | 0   |
| $b$ | $z$ | -1  |

We interpret the action  $a$  as the safe action and the action  $b$  as the risky action. The strategy  $a$  is never a Nash equilibrium of the tournament. The action  $b$  is a Nash equilibrium as long as  $n < 2 + z$ .

**Claim 2:** If  $z < 1$  then for  $n$  large enough the risky action is chosen in equilibria of  $T(G(z), n)$  more often than in the Nash equilibrium of  $G(z)$ .

If  $z > 1$  then for  $n$  large enough the risky action is chosen in equilibria of  $T(G(z), n)$  less often than in the Nash equilibrium of  $G(z)$ .

**Proof:** It follows from the previous section that when  $z < 1$  ( $\alpha^* = 1/(1+z) > 1/2$ ), the equilibria of  $T(G(z), n)$  approach  $\alpha^*$  from below. Thus, for large enough  $n$  the risky action is chosen more often than in the Nash equilibrium of the single game.

On the other hand, when  $z > 1$  ( $\alpha^* < 1/2$ ) the equilibria of  $T(G(z), n)$  approach  $\alpha^*$  from above. In other words, for a large number of players in the tournament the risky action is chosen with a lower probability than in the  $G(z)$  equilibrium.

Note that, in a tournament with a small number of players, the risky action may be played more frequently than in the game  $G(z)$ . For example, when  $z$  is slightly less than 1.5, the equilibrium probability of playing  $a$  in  $G$  is slightly higher than 0.4, whereas in the tournaments with  $n = 3, \dots, 10$  the probabilities are: (0, 0.38, 0.33, 0.35, 0.4, 0.37, 0.42, 0.39).

The case of  $z = 1$  is not generic but is of particular interest since  $\alpha = 1/2$ . A similar proof to that of Proposition 6 implies that for  $n > 3$ , the equilibria of  $T(G(1), n)$  are all below 1/2. In other words, all equilibria involve more risk-taking than  $G(1)$  does.

### Example 3: Battle of the Sexes

Let  $G(z)$  ( $z > 1$ ) be the following "Battle of the Sexes" game

|     |     |     |
|-----|-----|-----|
|     | $a$ | $b$ |
| $a$ | 0   | $z$ |
| $b$ | 1   | 0   |

Playing  $a$  can be interpreted as demanding the preferred outcome while playing  $b$  expresses a willingness to give up. The only symmetric equilibrium of the game is  $\alpha^* = z/(z + 1) > 1/2$ . The intuition that a tournament encourages more "aggressive behavior" is not confirmed here.

**Claim 3:** For  $n$  large enough, the "aggressive action"  $a$  is chosen in equilibria of  $T(G(z), n)$  less often than in the Nash equilibrium of  $G(z)$ .

**Proof:** By Proposition 6, for  $n$  large enough, the tournament  $T(G(z), n)$  has a unique mixed strategy equilibrium in which the probability of playing  $a$  is within the interval  $(1/2, \alpha^*)$ .

Calculations for various values of  $z$  suggest that also for small numbers of players the equilibrium probability of playing the aggressive action in the tournament tends to be lower than in the game  $G$  (though not always). To illustrate, for  $z = 2$  (which is not a generic case) we obtain the following equilibrium probabilities of playing  $a$  for  $n = 3, \dots, 9$ : (1, 0.62, 0.67, 0.65, 0.6, 0.64, 0.63). For  $z = 1.5$ , we obtain the following sequence of probabilities for  $n = 3, \dots, 9$ : (0.5, 0.62, 0.596, 0.56, 0.602, 0.54, 0.58). Note, however, that for a large value of  $z$ , playing  $a$  is the unique Nash equilibrium of the tournament for a wide range of numbers of players (as long as  $n < z + 1$ ). That is, the tournament triggers more aggressiveness in this range than  $G$  does.

## VI. A Variation of the Tournament Structure: Local vs. Global Randomizations

A player in our model uses what we refer to as global randomization, i.e. he uses the same action in all interactions after executing the mixed strategy once. An alternative model would

be one in which a player is allowed to use local randomizations. Namely, he chooses a mixed strategy and executes it independently against each of his opponents.

Recall that in the Challenge Game (analyzed in section IV), the three-player tournament with global randomization has three equilibria. In one of them, the players don't challenge. If a player deviates and "challenges", he reduces his probability of winning from  $1/3$  to  $1/4$ . Now suppose that he can apply local randomizations. If he chooses to challenge each player independently with probability  $p$ , he will end up challenging both players with probability  $p^2$ , challenging one player with probability  $2p(1-p)$  and not challenging any of them with probability  $(1-p)^2$ . Thus, the probability that he will win the tournament (given that the two other players employ the strategy of not challenging) is  $p^2/4 + 2p(1-p)/2 + (1-p)^2/3$ , which reaches a maximum of  $0.4$  at  $p = 0.4$ . Thus, local randomization can be beneficial in tournaments. Therefore, it is worthwhile analyzing the symmetric Nash equilibrium in round-robin tournaments with local randomization strategies. The calculation of such equilibria is typically not trivial and we will make do with two simple examples:

**Example 4:**

Consider the three-player tournament based on the following game:

|     |     |     |
|-----|-----|-----|
|     | $a$ | $b$ |
| $a$ | 0   | 1   |
| $b$ | 1   | 0   |

**Claim 4:** The local randomization, in which a player plays  $a$  with probability  $1/2$  in each match, is the unique equilibrium of the tournament with local randomizations.

**Proof:** Assume that players 1 and 2 use this local randomization. The following table presents the probabilities that player 3 wins in various events: each row represents one profile of the realizations of his local strategy (for example, the second row represents the case in which he ends up playing  $a$  against player 1 and  $b$  against player 2) and each column represents the pair of actions selected by the two other players to use against him (for example, the second column represents the case in which player 1 ends up playing  $a$  and player 2 plays  $b$  against player 3). Each entry in the table is the average of the probabilities that player 3 wins in the two equally likely events in which 1 and 2 mismatch their actions in their own match and

in the event that they fail to mismatch their actions.

|                 | <i>players 1 and 2</i> |             |             |             |
|-----------------|------------------------|-------------|-------------|-------------|
| <i>player 3</i> | <i>a, a</i>            | <i>a, b</i> | <i>b, a</i> | <i>b, b</i> |
| <i>a, a</i>     | 1/6                    | 1/4         | 1/4         | 2/3         |
| <i>a, b</i>     | 1/4                    | 1/6         | 2/3         | 1/4         |
| <i>b, a</i>     | 1/4                    | 2/3         | 1/6         | 1/4         |
| <i>b, b</i>     | 2/3                    | 1/4         | 1/4         | 1/6         |

If player 3 chooses the local randomization strategy of choosing  $a$  with probability  $p$ , then the first row occurs with probability  $p^2$ , the second and third with probability  $p(1-p)$  and the fourth with probability  $(1-p)^2$ . It turns out that for any  $p$  in  $[0, 1]$  the probability that player 3 wins is  $1/3$ . Therefore,  $p = 1/2$  is a best response. This is also the only equilibrium: if players 1 and 2 choose  $p < 1/2$ , then player 3 can choose  $a$  and win with probability:

$$(1/3)(p^2 + (1-p)^2)p^2 + (1/2)(p^2 + (1-p)^2)2p(1-p) + (1-2p(1-p)(2/3))(1-p)^2 > p^2/6 + p(1-p)/2 + 2(1-p)^2/3 = \frac{1}{3}p^2 - \frac{5}{6}p + \frac{2}{3} > 1/3. \text{ Similarly, } p > 1/2 \text{ is not an equilibrium.}$$

**Example 5:**

It is quite common that there is no equilibrium in the case of local randomization. Consider the three-player tournament based on the following game (where  $\epsilon$  is a small positive number):

|     |                |     |
|-----|----------------|-----|
|     | $a$            | $b$ |
| $a$ | 0              | 2   |
| $b$ | $1 + \epsilon$ | 0   |

**Claim 5:** The game has no equilibrium with local randomization,

**Proof:** Suppose that players 1 and 2 choose the local randomization strategy in which each plays  $a$  with probability  $p$  in every match. The following table, which is structured like that in Example 4, presents the probabilities that player 3 wins in different events: the rows represent all the profiles of realizations of player 3's actions in his matches and the columns represent the profiles of actions taken by players 1 and 2 in their matches with player 3. The term

$\mu = p^2 + (1 - p)^2$  is the probability that in the match between players 1 and 2 they both receive 0 (otherwise one receives 2 and the other receives  $1 + \varepsilon$ ):

|          | players 1 and 2 |                     |                     |         |
|----------|-----------------|---------------------|---------------------|---------|
| player 3 | $a, a$          | $a, b$              | $b, a$              | $b, b$  |
| $a, a$   | $\mu/3$         | $\mu$               | $\mu$               | 1       |
| $a, b$   | 0               | $\mu/3$             | $\mu + (1 - \mu)/6$ | $\mu$   |
| $b, a$   | 0               | $\mu + (1 - \mu)/6$ | $\mu/3$             | $\mu$   |
| $b, b$   | $\mu$           | 0                   | 0                   | $\mu/3$ |

The payoff of the player who chooses  $a$  with probability  $z$  in each match when the other two players plays  $a$  with probability  $p$  is:

$$z^2[p^2\mu/3 + 2p(1 - p)\mu + (1 - p)^2] + (1 - z)^2[p^2\mu + (1 - p)^2\mu/3] + 2z(1 - z)[p(1 - p)\mu/3 + p(1 - p)(5\mu + 1)/6 + (1 - p)^2\mu].$$

To see that the game has no equilibrium with local randomization, take the first derivative of the expression with respect to  $z$  and then substitute  $z = p$ . This yields an equation with no solution.

The reason that there is no equilibrium with local randomization, in this game and in many other games as well, is that a player's probability of winning in the tournament may exhibit non-convexity in the local randomization strategy space. Consider, for example the global randomization strategy  $\alpha = 1/2$  (which is the only equilibrium of the tournament with global randomization). Given that players 1 and 2 behave according to this strategy, player 3 is indifferent between playing "always  $a$ " and "always  $b$ ". However, it is strictly harmful for player 3 to use a non-degenerate local randomization. This can be seen from the following table which presents the probability that player 3 wins in various events, where each row represents a realization of his local randomizations and each column represents the realization of the actions chosen by players 1 and 2, which applies to all matches:



|          | players 1 and 2 |        |        |        |
|----------|-----------------|--------|--------|--------|
| player 3 | $a, a$          | $a, b$ | $b, a$ | $b, b$ |
| $a, a$   | 1/3             | 0      | 0      | 1      |
| $a, b$   | 0               | 0      | 0      | 1      |
| $b, a$   | 0               | 0      | 0      | 1      |
| $b, b$   | 1               | 0      | 0      | 1/3    |

For player 3, using "always  $a$ " or "always  $b$ " yields a probability of winning of  $1/3$ . Any non-degenerate local randomization  $p$  of player 3 will give some weight to the middle two rows and will decrease his probability of winning.

### VII. Another Tournament Procedure: The Pair-Wise Matching Tournament

In the basic tournament model, each player is matched with all other participants in the tournament. According to the evolutionary interpretation, evolutionary forces favor only these actions that do the best on average. One can think of real-life situations in which matches are rare and a player is evaluated on the basis of his performance in a single random match. The following *pair-wise matching tournament* (with even  $n$ ) captures such situations. Players are divided into pairs and each pair plays a game  $G$  once. The scores (payoffs from  $G$ ) of all players are compared and those with the highest score share the chance to win the tournament. We denote this type of tournament as  $PWT(G, n)$  where  $G$  is a symmetric game and  $n = 2m$  is the number of players.

The equilibria of  $PWT(G, n)$  are very different from those of  $T(G, n)$ . First, note that since each player is involved in a single play of the game and a winner is drawn from those players who obtain the highest score, the  $G$ -payoffs have only ordinal meaning and  $PWT(G, n)$  is invariant to any order-preserving transformation of the  $G$ -payoffs. In contrast, in  $T(G, n)$  (with  $n > 1$ ) the  $G$ -payoffs have cardinal meaning and the set of equilibria of  $T(G, n)$  is sensitive to the  $G$ -payoffs beyond just in the ordinal sense. In this subsection, we examine some examples that demonstrate further differences between the two.

In the following two examples, the equilibria of  $PWT(G, n)$  are independent of  $n$  and are

not Nash equilibria of  $G$ :

**Example 6: The Pair-Wise Matching Tournament with the Battle of the Sexes**

Let us look again at the following "Battle of the Sexes" game,  $G(z)$  ( $z > 1$ ):

|     |     |     |
|-----|-----|-----|
|     | $a$ | $b$ |
| $a$ | 0   | $z$ |
| $b$ | 1   | 0   |

We have seen in Example 3 that in equilibrium of  $T(G(z), n)$  the action  $a$  is played with positive probability which converges from below to  $z/(z + 1)$  as  $n \rightarrow \infty$ . We concluded that the tournament structure leads to less aggressive behavior than in  $G(z)$ . In contrast, the aggressive action  $a$  is the the unique equilibrium of any pair-wise matching tournament.

**Claim 6:** Playing  $a$  is the unique equilibrium of any  $PWT(G(z), n)$  for any  $n$ .

**Proof:** The fact that playing  $a$  is an equilibrium follows from the following more general statement: For any two-action game  $G = \langle \{a, b\}, u \rangle$ , if  $u_{xy} > u_{yx}$  then the action  $x$  is an equilibrium of  $PWT(G, n)$  (a deviation will only reduce the probability of the deviator winning to 0). Clearly, playing  $b$  is not an equilibrium.

If all other players play some non-degenerate mixed strategy then a player who plays  $b$  wins the tournament only in the event that he plays against a player whose realized action is  $b$  and the outcomes in all other  $m - 1$  matches are either  $(a, a)$  or  $(b, b)$ . In other words, whenever the player is among the top-scoring players all  $n$  players are top-scorers. Thus, the probability that a player wins a prize if he plays  $b$  is strictly below  $1/n$ , whereas in equilibrium this probability must be  $1/n$ .

**Example 7:**

|     |     |     |
|-----|-----|-----|
|     | $a$ | $b$ |
| $a$ | 0   | 0   |
| $b$ | 1   | -1  |

The game  $G$  was discussed at the end of Example 2. The action  $a$  is played in equilibrium of  $T(G,n)$  with positive probability but below  $1/2$  as  $n \rightarrow \infty$ . We concluded that the tournament structure encourages more risky behavior than in  $G$ . The result in the corresponding pair-wise matching tournament is more extreme as playing the "risky" action  $b$  is the unique equilibrium for any  $n$ .

**Claim 7:** Playing  $b$  is the only equilibrium of  $PWT(G,n)$  for any  $n$ .

**Proof:** For any  $n$ , playing  $b$  is an equilibrium of the tournament. If all other players play  $b$ , a player who deviates to  $a$  will lose with certainty since the player against whom he is matched will score 1 and will be the sole winner.

Consider any mixed strategy where  $a$  is played with positive probability. It is sufficient to show that it is strictly better for a player to play  $b$  than  $a$  when the other  $n - 1$  players play this mixed strategy.

In the event that the player with whom he is matched plays  $a$ , playing  $b$  yields 1, the highest possible payoff, without increasing any other player's payoff. Thus, in this positive probability event, playing  $b$  rather than  $a$  increases his chances of winning.

In the event that the player is matched against a player who plays  $b$ , playing  $a$  means he will lose with certainty. Thus, playing  $b$  cannot decrease his probability of winning.

The final example is a game with three actions, which demonstrates that a sequence of (non-degenerate) mixed strategy equilibria may converge to a (pure) strategy that is not an equilibrium of  $G$ .

### **Example 8: The 18-20 Money Request Game**

The baseline game in this example is a version of the *Money Request Game* (introduced in Arad and Rubinstein (2012)). In this game, each player requests a sum of money: 18, 19 or 20 dollars. He receives the amount and in addition receives a "bonus" of \$20 if he requests exactly one dollar less than the other player:

|      |      |      |      |
|------|------|------|------|
|      | "20" | "19" | "18" |
| "20" | 20   | 20   | 20   |
| "19" | 39   | 19   | 19   |
| "18" | 18   | 38   | 18   |

Denote an equilibrium strategy as  $(\alpha, \beta, \gamma)$  where  $\alpha, \beta$  and  $\gamma$  are the probabilities of playing 20, 19 and 18, respectively. Clearly,  $PWT(G, n)$  has no pure strategy equilibrium. Neither does it have an equilibrium with only two strategies in the support: if  $\gamma = 0$ , then in  $PWT(G, n)$  19 dominates 20 and thus,  $\beta = 1$ . Similarly, if  $\alpha = 0$  then  $\gamma = 1$  and if  $\beta = 0$ , then  $\alpha = 1$ . All these cases lead to contradiction of the conclusion that  $PWT(G, n)$  has no pure strategy equilibria.

**Claim 8:** Any sequence  $(\alpha_n, \beta_n, \gamma_n)$  of equilibria of  $PWT(G, n)$  converges to  $(0, 0, 1)$ .

**Proof:** Any equilibrium of  $PWT(G, n)$  has full support. Consider a subsequence of  $(\alpha_n, \beta_n, \gamma_n)$  that converges to  $(\alpha^*, \beta^*, \gamma^*)$ .

If  $\alpha^* \beta^* > 0$ , then the probability that the winning score is less than 39 is of the order  $(1 - 2\alpha^* \beta^*)^m$ . If  $\beta^* \gamma^* > 0$ , then the probability that the winning score is less than 38 is of the order  $(1 - 2\alpha^* \beta^* - 2\beta^* \gamma^*)^m$ . In both cases, the probability that 20 wins  $PWT(G, n)$  is lower than  $1/n$  for high enough  $n$  and cannot be in the support of an equilibrium of  $PWT(G, n)$ . Thus,  $\alpha^* \beta^* = \beta^* \gamma^* = 0$ , i.e. either  $\beta^* = 0$  or  $\beta^* = 1$ . In order to conclude that  $\gamma^* = 1$ , we must exclude the following two cases:

Case (i):  $\alpha^* > 0$  and  $\beta^* = 0$ .

A player who plays 19 in  $PWT(G, n)$  will receive 39 with probability  $\alpha_n$ . He will share the prize with about  $n\alpha_n\beta_n$  players. Thus, his chances of winning the tournament are approximately  $1/(n\beta_n)$ . Since  $\beta_n \rightarrow 0$ , this probability is above  $1/n$  for large enough  $n$ .

Case (ii):  $\beta^* = 1$ .

Consider a player who plays against  $n - 1$  players who adopt  $(\alpha_n, \beta_n, \gamma_n)$ . We will show that for large enough  $n$  the action 18 yields a higher probability of winning than the action 20. We distinguish between two events regarding the realizations in the other  $m - 1$  matches:

(a) In one of the other  $m - 1$  matches, one of the players attains a score of 38 or 39. In this case, the action 20 will not win and the action 18 is at least as good.

(b) In all other matches the players score at most 20. By playing 18, the player winning the tournament will be the sole winner with probability  $\beta_n$ , while by playing 20 he will win the tournament with probability not higher than  $1 - \beta_n$ . Thus, for large enough  $n$  the action 18 is strictly better than the action 20, contradicting the assumption that 20 is in the support of  $(\alpha_n, \beta_n, \gamma_n)$ .

### VIII. Conclusion

In this paper, we have introduced and studied a particular  $n$ -player game, referred to as a tournament, which is constructed on top of a basic normal form game  $G$ . The tournament structure can be thought of as an explicit model of the interaction between identical agents who are involved in a specific pair-wise symmetric game. We assume each agent strives to be the best or that evolution selects those who are the best. Under these assumptions, we have shown that the equilibrium of the tournament in a large population is indeed approximated by a Nash equilibrium of  $G$ . However, not every Nash equilibrium of  $G$  is a limit of the tournaments' equilibria. Thus, when the population is large, we obtain a criterion for equilibrium selection.

It is also interesting to study the equilibrium of the tournament when the number of players is small. For several classes of games, we showed that the equilibrium of the tournament is systematically related to the equilibrium of  $G$ . For example, in the Battle of the Sexes, the "preferred" action appears in the tournament equilibrium (for almost all values of  $n$ ) with strictly lower frequency than in the equilibrium of  $G$ . In the Challenge Game, we conjecture that in any equilibrium with more than 3 players, players play only the superfluous action. In coordination games (with the outcomes 0 and 1), the set of equilibria of the tournament and that of  $G$  are identical.

We also studied a variation of the tournament model in which each player is randomly matched with a single participant in the tournament and plays  $G$  only once. As in the basic model, the prize is given to one of the highest-scoring players. This type of tournament exhibits different equilibria. For example, when  $G$  is the Battle of the Sexes, the only equilibrium of such a "single interaction tournament" is playing the "preferred" action. The analysis can be extended to include other tournament procedures and alternative assumptions

regarding players' goals in the tournament.

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