Lecture Notes in Microeconomic Theory

This is a revised version of the book, first published in 2005.
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Please e-mail me with comments or corrections.
A solution manual is available upon request by instructors only (see my home-page).

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Lecture Notes in Microeconomic Theory

The Economic Agent

Second Edition

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Preface

This is the 2018 revision of the second edition of my lecture notes for the first quarter of a microeconomics course for PhD (or MA) economics students. The lecture notes were developed over a period of 20 years during which I taught the course at Tel Aviv University, Princeton University, and New York University.

I published this book for the first time in 2007 and have revised it annually since then. I was hesitant about writing this book since several superb books were already on the shelves. Foremost among them are those of David Kreps. Kreps (1990) pioneered the shift of the game-theoretic revolution from research papers to textbooks. His book covers the material in depth and includes many ideas for future research. His more recent book, Kreps (2013), is even better than his first and is now my clear favorite for a graduate microeconomics course.

Three other books are on my shortlist: Mas-Colell, Whinston and Green (1995) is a very comprehensive textbook; Bowles (2003) brings economics back to its authentic political economics roots; and Jehle and Reny (1997) with its very precise style. They constitute an impressive collection of textbooks for an advanced microeconomics course.

My book covers only the first quarter of the standard course. It does not aim to compete with these other books. I published it and continue updating it only because I think that it reflects a different point of view on economic theory and some of the didactic ideas presented might be beneficial to both students and teachers.

Downloading Updated Versions

As a matter of principle, the book is posted on the web and is available free of charge. I am grateful to Princeton University Press for allowing it to be downloaded for free immediately after publication.

Since 2007, I have updated the book annually, adding material and correcting mistakes. My plan is to continue revising the book annually as long as I teach the course. To access the latest electronic version go to: http://arielrubinstein.tau.ac.il.

Solution Manual

Teachers (and only teachers) of the course can also get an updated solution manual. I do my best to make the manual available only to teachers of a graduate course in microeconomics. Requests for the manual should be made at: http://gametheory.tau.ac.il/microtheory.

Gender

Throughout the book I use only male pronouns. This is my deliberate choice and does not reflect the policy of the editors or the publishers. I believe that continuous reminders of the he/she issue simply divert readers' attention. Language is of course important in shaping our thinking, but I feel it is more effective to raise the issue of discrimination against women in the discussion of "the issues" rather than raising flags on every page of a book on economic theory.

Acknowledgments

I would like to thank all of my teaching assistants who made helpful comments during the many years I taught the course prior to the first edition: Rani Spiegler, Kfir Eliaz, Yoram Hamo, Gabi Gayer, and Tamir Tshuva at Tel Aviv University; Bilge Yilmaz, Ronny Razin, Wojciech Olszewski, Attila Ambrus, Andrea Wilson, Haluk Ergin, and Daisuke Nakajima at Princeton; and Sophie Bade and Anna Ingster at NYU. Sharon Simmer and Rafi Aviav helped me with the English editing. Avner Shlain prepared the index.

Special thanks to Benjamin Bachi for his devoted work in producing the revised versions of the book for about a decade.

Introduction

As a graduate student just starting out, you are at the beginning of a new stage in your life. In a few months you will be overloaded with definitions, concepts, and models. Your teachers will be guiding you into the wonders of economics and will rarely stop to raise fundamental questions about what these models are supposed to mean. It is not unlikely that you will be brainwashed by the professionally sounding language and hidden assumptions. I am afraid I am about to initiate you into this inevitable process. Still, I want to pause for a moment and alert you to the fact that many economists have strong and conflicting views about what economic theory is. Some see it as a *set of theories* about the interaction between individuals in economic situations, theories that can be tested. Others see it as a *bag of tools* to be used by economic agents. Many see it as a *framework* through which academic economists view the world.

My own view may disappoint those of you who have come to this course with practical motives. In my view, economic theory is "just" an arena for the *investigation of concepts* we use in thinking about real-life economic situations. What makes a theoretical model "economics" is that the concepts we are analyzing are taken from real-life reasoning about economic issues. Through the investigation of these concepts, we try to better understand reality, and the models provide a language that enables us to think about economic interactions in a systematic way. But I do not view economic models as an attempt to describe the world, to provide tools for predicting the future or to prescribe how people should behave. I object to looking for an ultimate truth in economic theory, and I do not expect it to be the foundation for any policy recommendation. Nothing is "holy" in economic theory and everything in it is the creation of people like yourself.

Essentially, this course consists of a discussion of *concepts* and *models* related to the behavior of a single economic agent. Although we will be studying formal concepts and models, they will always be given an interpretation. An economic model differs substantially from a purely mathematical model in that it is a *combination* of a mathematical model and its *interpretation*. When mathematicians use terms such as "field" or "ring" that are in everyday use, it is only for the sake of convenience.

When they name a collection of sets a “filter”, they are doing so in an associative manner; in principle, they could call it “ice cream cone”. When they use the term “well-ordering”, they are not making an ethical judgment. In contrast to mathematics, interpretation is an essential ingredient of an economic model and the names of the mathematical objects are an integral part of an economic model.

The word “model” sounds more scientific than “fable” or “fairy tale”, but I don’t see much difference between them. The author of a fable draws a parallel to a situation in real life and often has some moral he wishes to impart to the reader. A fable is an imaginary situation that is somewhere between fantasy and reality. Any fable can be dismissed as being unrealistic or simplistic, but this is also its advantage. As something between fantasy and reality, a fable is free of extraneous details and annoying diversions. In this unencumbered state, we can clearly discern what cannot always be discerned in the real world. On our return to reality, we are in possession of some relevant and useful argument. We do exactly the same thing in economic theory. A good model in economic theory, like a good fable, identifies a number of themes and elucidates them. We perform thought exercises that are only loosely connected to reality and have been stripped of most of their real-life characteristics. However, in a good model, as in a good fable, something significant remains.

One can think about this book as an introduction of the characters that inhabit economic fables. Here, we feature the characters in isolation. In models of markets and games (not discussed in this textbook), we investigate the interactions between these characters.

It is my hope that you will not be just a user of what is currently called economic theory and that you will acquire alternative ways of thinking about economic and social interactions. At the very least, I would like to encourage you to ask hard (and probably painful) questions about economic models and whether they are relevant to real-life situations and not simply take for granted that they are the ”right models”.

Microeconomics

In this course we deal only with microeconomics, a collection of models in which the primitives are details about the behavior of units referred to as *economic agents*. An economic agent is the basic unit operating in the model. Most often, we have in mind that the economic agent is an individual, a person with one head, one heart, two eyes, and two ears.

Although, in some economic models, the agent is a group of people, a family, or a government. At other times, the “individual” is broken down into a collection of economic agents, each operating in different circumstances. However, the facade of generality in economic theory (and elsewhere) may be misleading. We have to be aware that when we take an economic agent to be a group of individuals, the reasonable assumptions about his behavior will differ from those in the case of a single individual. For example, although it is quite natural to talk about the will of a person, it is not clear what is meant by the will of a group when the members of the group differ in their preferences.

Bibliographic Notes

For an extensive discussion of my views about economic theory, see Rubinstein (2006a), and my semi-academic book Rubinstein (2012).

Lecture Notes in Microeconomic Theory

LECTURE 1

Preferences

Preferences

Our economic agent will soon be advancing to the stage of economic models. Which of his characteristics will we be specifying in order to get him ready? One might suggest his name, age and gender, personal history, brain structure, needs, cognitive abilities, and emotional state. However, in most of economic theory, we only specify his attitude toward the elements in some relevant set, and usually we assume that it can be expressed in the form of *preferences*.

We begin the course with a modeling "exercise" in which we seek to develop a "proper" formalization of the concept of preferences. Although we are on our way to constructing a model of rational choice, for now we will think about the concept of preferences independently of choice. This makes sense since we often use the concept of preferences in context other than choice. For example, we can talk about an individual's tastes over the paintings of the great masters even if he never makes a decision based on those preferences. We can talk about the preferences of an agent were he to arrive tomorrow on Mars or travel back in time and become King David, even if he does not believe in the supernatural. Actually, people often prefer elements that are forbidden for them to choose.

Imagine that you want to fully describe the preferences of an agent toward the elements in a given set X. For example, imagine that you want to describe your own attitude toward the universities you have applied to before finding out to which of them you have been admitted. What must the description include? What conditions must the description fulfill?

We take the approach that a description of preferences should fully specify the attitude of the agent toward each pair of elements in X. For each pair of alternatives, it should provide an answer to the question of how the agent compares between the two alternatives. We present two versions of this question. For each version, we formulate the consistency requirements necessary for the responses to be “preferences’. We then will examine the connection between the two formalizations.

The Questionnaire Q

Think about the preferences on a set X as *answers* to a long questionnaire Q that consists of all quiz questions of the type:

Q(x,y) (for all distinct x and y in X):

How do you compare x to y? Check one and only one of the following three options:

☐ I prefer x to y (denoted by $x \succ y$).
☐ I prefer y to x (denoted by $y \succ x$).
☐ I am indifferent (denoted by I).

A “legal” answer to the questionnaire is a response in which exactly one of the boxes is checked in each question. We do not allow refraining from answering a question or checking more than one answer. Furthermore, by allowing only the above three options we exclude plausible responses that demonstrate a lack of ability to make a comparison, such as:

☐ They are incomparable.
☐ I don’t know what x is.
☐ I have no opinion.
☐ I prefer both x over y and y over x.

or that involve dependence on other factors, such as:

☐ It depends on what my parents think.
☐ It depends on the circumstances (sometimes I prefer x and sometimes I prefer y).

or that involve the intensity of preferences, such as:

☐ I somewhat prefer x.
☐ I love x and I hate y.

The constraints that we place on the legal responses of the agents constitute our implicit assumptions. Particularly important are the assumption that the elements in the set X are all comparable and the fact that we ignore the intensity of preferences.

A legal answer to the questionnaire can be formulated as a function f, which assigns to any pair (x, y) of distinct elements in X exactly one of the three "values", $x \succ y$ or $y \succ x$ or I, with the interpretation that $f(x, y)$ is the answer to the question $Q(x, y)$. (Alternatively, we can use the notation of the soccer betting industry and say that $f(x, y)$ must be 1, 2, or $\times$ with the interpretation that $f(x, y) = 1$ means that x is preferred to y, $f(x, y) = 2$ means that y is preferred to x, and $f(x, y) = \times$ means indifference.)

Not all legal answers to the questionnaire Q qualify as *preferences over the set* X. We will adopt two "consistency" requirements:

First, the answer to $Q(x, y)$ must be identical to the answer to $Q(y, x)$. In other words, we want to exclude the "framing effect" by which people who are asked to compare two alternatives tend to somewhat prefer the first one.

Second, we require that the answers to $Q(x, y)$ and $Q(y, z)$ are consistent with the answer to $Q(x, z)$ in the following sense. If the answers to the two questions $Q(x, y)$ and $Q(y, z)$ are "x is preferred to y" and "y is preferred to z", then the answer to $Q(x, z)$ must be "x is preferred to z", and if the answers to the two questions $Q(x, y)$ and $Q(y, z)$ are "indifference", then so is the answer to $Q(x, z)$.

To summarize, following is my favorite formalization of the notion of preferences:

Definition 1

Preferences on a set X are a function f that assigns to any pair (x, y) of distinct elements in X one of the three "values" $x \succ y$, $y \succ x$, or I so that for any three different elements x, y, and z in X, the following two properties hold:

- *No order effect*: $f(x, y) = f(y, x)$.
- *Transitivity*:
  if $f(x, y) = x \succ y$ and $f(y, z) = y \succ z$, then $f(x, z) = x \succ z$ and if $f(x, y) = I$ and $f(y, z) = I$, then $f(x, z) = I$.

Note again that here I, $x \succ y$, and $y \succ x$ are merely symbols representing verbal answers. Needless to say, the choice of symbols is not an arbitrary one. (Is there any reason to use the notation I rather than $x \sim y$?)

A Discussion of Transitivity

Transitivity is an appealing property of preferences. How would you react if somebody told you he prefers x to y, y to z, and z to x? You would probably feel that his answers are "confused". Furthermore, it seems that, when confronted with intransitivity in their responses, people are embarrassed and want to change their answers.

On some occasions before giving this lecture, I have asked students to fill out a questionnaire similar to Q regarding a set X that contains nine alternatives, each specifying the following four characteristics of a travel package: location (Paris or Rome), price, quality of the food, and quality of the lodgings. The questionnaire included only thirty-six questions since for each pair of alternatives x and y, only one of the questions, $Q(x, y)$ or $Q(y, x)$, was randomly selected to appear in the questionnaire (thus the dependence on the order of an individual's response was not checked within the experimental framework). Out of 1300 students who responded to the questionnaire, only 15% had no intransitivities in their answers, and the median number of triples in which intransitivity existed was 6. Many of the violations of transitivity involved two alternatives that were actually the same but differed in the order in which the characteristics appeared in the description: "A weekend in Paris at a 4-star hotel with food quality of Zagat 17 for \$574", and "A weekend in Paris for \$574 with food quality of Zagat 17 at a 4-star hotel". All students expressed indifference between the two alternatives, but in a comparison of these two alternatives to a third alternative—"A weekend in Rome at a 5-star hotel with food quality of Zagat 18 for \$612"— a quarter of the students gave responses that violated transitivity.

Interestingly, I observed a negative correlation between the time it took a student to respond to the questionnaire and whether he satisfies transitivity of came close to it. The explanation is that most of the students who had no cycles of intransitivities formed their answers by applying a simple and easy-to-execute rule (such as I prefer Paris to Rome and I compare two vacation packages in the same city according to price). Students who compared the alternatives holistically without a simple guiding rule were more likely to have cycles of intransitivity.

In spite of the appeal of the transitivity requirement, note that when we assume that the attitude of an individual toward pairs of alternatives is transitive, we are excluding from the discussion individuals who base their judgments on procedures that cause systematic violations of transitivity. The following are two such examples:

1. *Aggregation of considerations as a source of intransitivity.* Sometimes, an individual's attitude is derived from the aggregation of more basic considerations. Consider, for example, a case where $X = \{a, b, c\}$ and the individual has three primitive considerations in mind. The individual finds an alternative x to be better than an alternative y if a majority of considerations supports x. This aggregation process can yield intransitivities. For example, if the three considerations rank the alternatives as $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$, and $c \succ_3 a \succ_3 b$, then the individual determines a to be preferred over b, b over c, and c over a, thus violating transitivity.

2. *The use of similarities as an obstacle to transitivity.* In some cases, an individual may express indifference in a comparison between two elements that are too "close" to be distinguishable. For example, let $X = \mathbb{R}$ (the set of real numbers). Consider an individual whose attitude toward the alternatives is "the larger the better"; however, he finds it impossible to determine whether a is greater than b unless the difference is at least 1. He will assign $f(x, y) = x \succ y$ if $x \geq y + 1$ and $f(x, y) = I$ if $|x - y| < 1$. This is not a preference relation because $1.5 \sim 0.8$ and $0.8 \sim 0.3$, but it is not true that $1.5 \sim 0.3$.

Did we require too little? A potential criticism of the definition is that our assumptions might be too weak and that we did not impose further reasonable restrictions on the concept of preferences. That is, there are other similar consistency requirements we might want to impose on a legal response in order for it to qualify it as a description of preferences. For example, if $f(x, y) = x \succ y$ and $f(y, z) = I$, we would naturally expect that $f(x, z) = x \succ z$. However, this additional consistency condition was not included in the above definition because it actually follows from the other conditions: if $f(x, z) = I$, then by the assumption that $f(y, z) = I$ and by the no-order effect, $f(z, y) = I$, and thus by transitivity $f(x, y) = I$ (a contradiction) and if $f(x, z) = z \succ x$, then by the no-order effect $f(z, x) = z \succ x$, and by $f(x, y) = x \succ y$ and transitivity $f(z, y) = z \succ y$ (a contradiction).

Similarly, note that for any preferences f, we have that if $f(x, y) = I$ and $f(y, z) = y \succ z$, then $f(x, z) = x \succ z$.

And by the way, this property of preferences is sometimes not attractive. The famous old pony-bicycle example demonstrates the point: Imagine a child who is indifferent between a pony and a bicycle. The child prefers a bicycle with a bell to a bicycle without one but is still indifferent between a bicycle with a bell and a pony.

The Questionnaire R

A second way to think about preferences is by means of an imaginary questionnaire R consisting of all questions of the type:

R(x,y) (for all $x, y \in X$, not necessarily distinct): Is x at least as preferred as y? Check one and only one of the following two boxes:

☐ Yes
☐ No

To be a "legal" response, we require that the respondent checks exactly one of the two boxes in each question. To qualify as preferences, a legal response must also satisfy two conditions:

1. The answer to at least one of the questions $R(x, y)$ and $R(y, x)$ must be Yes. (In particular, the "silly" question $R(x, x)$ that appears in the questionnaire must get a Yes response.)
2. For every $x, y, z \in X$, if the answers to the questions $R(x, y)$ and $R(y, z)$ are Yes, then so is the answer to the question $R(x, z)$.

We identify a response to this questionnaire with the binary relation $\succsim$ on the set X defined by $x \succsim y$ if the answer to the question $R(x, y)$ is Yes.

(*Reminder*: An *n-ary relation* on X is a subset of X^n. Examples: "Being a parent of" is a binary relation on the set of human beings; "being a hat" is an unary relation on the set of objects; "$x + y = z$" is a 3-ary relation on the set of numbers; "x is better than y more than x' is better than y'" is 4-ary relation on a set of alternatives, etc. An *n-ary relation* on X can be thought of as a response to a questionnaire regarding all n-tuples of elements of X where each question can get only a Yes/No answer.)

This brings us to the conventional definition of preferences:

Definition 2

Preferences on a set X is a binary relation $\succsim$ on X satisfying:

- *Reflexivity*: For any $x \in X$, $x \succsim x$.
- *Completeness*: For any distinct $x, y \in X$, $x \succsim y$, or $y \succsim x$.
- *Transitivity*: For any $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$.

The Equivalence of the Two Definitions

We will now discuss the sense in which the two definitions of preferences on the set X are equivalent. But first a reminder:

The function $f : X \to Y$ is a *one-to-one function* (or *injection*) if $f(x) = f(y)$ implies that $x = y$.

The function $f : X \to Y$ is an *onto function* (or *surjection*) if for every $y \in Y$ there is an $x \in X$ such that $f(x) = y$.

The function $f : X \to Y$ is a *one-to-one and onto function* (or *bijection*, or *one-to-one correspondence*) if for every $y \in Y$ there is a unique $x \in X$ such that $f(x) = y$.

When we think about the equivalence of two definitions in economics, we are thinking about much more than the existence of a one-to-one correspondence: The correspondence also has to *preserve the interpretation.* Note the similarity to the notion of an isomorphism in mathematics where a correspondence has to preserve "structure". For example, an isomorphism between two topological spaces X and Y is a one-to-one function from X onto Y that is required to preserve the open sets. In economics, the analogue to "structure" is the less formal notion of interpretation.

We will now construct a one-to-one and onto function, named *Translation*, between answers to Q that qualify as preferences by the first definition and answers to R that qualify as preferences by the second definition, such that the correspondence preserves the meaning of the responses to the two questionnaires.

To illustrate, imagine that you have two books. Each page in the first book is a response to the questionnaire Q that qualifies as preferences by the first definition. Each page in the second book is a response to the questionnaire R that qualifies as preferences by the second definition. The correspondence matches each page in the first book with a unique page in the second book, so that a reasonable person will recognize that the different responses to the two questionnaires reflect the same mental attitudes toward the alternatives.

Since we assume that the answers to all questions of the type $R(x, x)$ are Yes, the classification of a response to R as preferences requires only the specification of the answers to questions $R(x, y)$, where $x \neq y$. Table 1.1 presents the translation of responses.

This translation preserves the interpretation we have given to the responses. That is, if the response to the questionnaire Q exhibits that "I prefer x to y", then the translation to a response to the questionnaire R contains the statement "I find x to be at least as good as y, but I

Table 1.1

A response to: $Q(x,y)$ and $Q(y,x)$	A response to: $R(x,y)$ and $R(y,x)$	
$x \succ y$	Yes	No
I	Yes	Yes
$y \succ x$	No	Yes

don't find y to be at least as good as x" and thus exhibits the same meaning. Similarly, the translation of a response to Q that exhibits "I am indifferent between x and y" is translated into a response to R that contains the statement "I find x to be at least as good as y, and I find y to be at least as good as x" and thus exhibits the same meaning.

We will now prove that $Translation$ is indeed a one-to-one correspondence between the set of preferences as given by definition 1, and the set of preferences as given by definition 2.

Let f be a preference according to definition 1. First we check that $Translation(f)$ is a legal response to R. By the no-order effect assumption, for any two alternatives x and y, one and only one of the following three answers could have been given by f for both $Q(x,y)$ and $Q(y,x)$: $x \succ y$, I, and $y \succ x$. Thus, the responses to $R(x,y)$ and $R(y,x)$ are well-defined.

Next we verify that $Translation(f)$ is indeed a preference relation (by the second definition).

Completeness: In each of the three rows, the answers to at least one of the questions $R(x,y)$ and $R(y,x)$ is Yes.

Transitivity: Assume that the answers given by $Translation(f)$ to $R(x,y)$ and $R(y,z)$ are Yes. This implies that the answer to $Q(x,y)$ given by f is either $x \succ y$ or I, and the answer to $Q(y,z)$ is either $y \succ z$ or I. Transitivity of f implies that its answer to $Q(x,z)$ is $x \succ z$ or I, and therefore the answer to $R(x,z)$ must be Yes.

To see that $Translation$ is indeed a one-to-one function, note that for any two different responses to the questionnaire Q there must be a question $Q(x,y)$ for which the responses differ; therefore, the corresponding responses to either $R(x,y)$ or $R(y,x)$ must differ.

It remains to be shown that the range of $Translation$ includes all possible preferences as defined by the second definition. Let $\succsim$ (a response to R) be preferences by the second definition. We have to find a function f, which is preferences by the first definition, that is converted by $Translation$ into $\succsim$. Reading from right to left, the table provides us with a function f. The function is well-defined since by the completeness of $\succsim$, for any two elements x and y, one of the entries in the right-hand column is applicable (the fourth option, in which the two answers to $R(x,y)$ and $R(y,x)$ are No, is excluded). By definition, f satisfies the no order effect condition.

We still have to check that f satisfies the transitivity condition. If $f(x,y) = x \succ y$ and $f(y,z) = y \succ z$, then $x \succsim y$ and not $y \succsim x$ and $y \succsim z$ and not $z \succsim y$. By transitivity of $\succsim$, $x \succsim z$. In addition, not $z \succsim x$ since if $z \succsim x$, then the transitivity of $\succsim$ would imply $z \succsim y$. If $f(x,y) = I$ and $f(y,z) = I$, then $x \succsim y$, $y \succsim x$, $y \succsim z$, and $z \succsim y$. By transitivity of $\succsim$, both $x \succsim z$ and $z \succsim x$, and thus $f(x,z) = I$.

Summary

I could have replaced the entire lecture with the following two sentences: "Preferences on X are a binary relation $\succsim$ on a set X satisfying reflexivity, completeness and transitivity. Denote $x \succ y$ when both $x \succsim y$ and not $y \succsim x$, and $x \sim y$ when $x \succsim y$ and $y \succsim x$". However, the role of this chapter was not just to introduce a formal definition of preferences but also to conduct a modeling exercise and to make some methodological points:

1. When we introduce two formalizations of the same verbal concept, we have to make sure that they indeed carry the same meaning.
2. When we construct a formal concept, we make assumptions beyond those explicitly mentioned. Being aware of the implicit assumptions is important for understanding the concept and is useful in coming up with ideas for alternative formalizations.

Bibliographic Notes

Fishburn (1970) contains a comprehensive treatment of preference relations.

Problem Set 1

Problem 1. (*Easy*)
Let $\succsim$ be a preference relation on a set X. Define $I(x)$ to be the set of all $y \in X$ for which $y \sim x$.

Show that the set (of sets!) $\{I(x)|x \in X\}$ is a partition of X, that is,

- For all x and y, either $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$.
- For every $x \in X$, there is $y \in X$ such that $x \in I(y)$.

Problem 2. (*Standard*)
Kreps (1990) introduces another formal definition of preferences. His primitive is a binary relation P interpreted as "strictly preferred". He requires P to satisfy:

- *Asymmetry*: For no x and y do we have both xPy and yPx.
- *Negative Transitivity*: For all x, y, and $z \in X$, if xPy, then for any z either xPz or zPy (or both).

Explain the sense in which Kreps' formalization is equivalent to the traditional definition.

Problem 3. (*Difficult. Based on Kannai and Peleg (1984)*)
Let Z be a finite set and let X be the set of all nonempty subsets of Z. Let $\succsim$ be a preference relation on X (not Z). An element $A \in X$ is interpreted as a "menu", that is, "the option to choose an alternative from the set A". Consider the following two properties of preference relations on X:

1. If $A \succsim B$ and C is a set disjoint to both A and B, then $A \cup C \succsim B \cup C$, and
   if $A \succ B$ and C is a set disjoint to both A and B, then $A \cup C \succ B \cup C$.
2. If $x \in Z$ and $\{x\} \succ \{y\}$ for all $y \in A$, then $A \cup \{x\} \succ A$, and
   if $x \in Z$ and $\{y\} \succ \{x\}$ for all $y \in A$, then $A \succ A \cup \{x\}$.

a. Discuss the plausibility of the properties in the context of interpreting $\succsim$ as the attitude of the individual toward sets from which he will have to make a choice in a "second stage".
b. Provide an example of a preference relation that: (i) satisfies the two properties; (ii) satisfies the first but not the second property; (iii) satisfies the second but not the first property.

c. Show that if there are x, y, and $z \in Z$ such that $\{x\} \succ \{y\} \succ \{z\}$, then there is no preference relation satisfying both properties.

Problem 4. (*Moderately difficult*)
Let $\succ$ be an asymmetric binary relation on a finite set X that does not have a cycle, that is there is no finite sequence of elements $x_1, x_2, .., x_K$, where $K > 2$, such that $x_1 \succ x_2 \succ ... \succ x_K \succ x_1$. Show (by induction on the size of X) that $\succ$ can be extended to a complete ordering (i.e., a complete, asymmetric, and transitive binary relation).

Problem 5. (*Difficult*)
You have read an article in a "prestigious" journal about a decision maker (DM) whose mental attitude towards elements in a finite set X is represented by a binary relation $\succ$, which is asymmetric and transitive but not necessarily complete. The incompleteness is the result of the assumption that a DM is sometimes unable to compare between alternatives.

Then, the author states that he is going to make a stronger assumption: the DM uses the following procedure: he has n criteria in mind, each represented by an ordering $\succ_i$ $(i = 1, \dots, n)$ (which is asymmetric, transitive, and complete binary relation). He determines that $x \succ y$ if and only if $x \succ_i y$ for every i.

1. Verify that the relation $\succ$ generated by this procedure is asymmetric and transitive. Try to convince a reader of the paper that this is an attractive assumption by giving a "real life" example in which it is "reasonable" to assume that a DM uses such a procedure in order to compare between alternatives.

It is claimed that the additional assumption is vacuous: given any asymmetric and transitive relation, $\succ$, one can find a set of complete orderings $\succ_1, \dots, \succ_n$ such that $x \succ y$ iff $x \succ_i y$ for every i.

2. Demonstrate this claim for the binary relation on the set $X = \{a, b, c\}$ according to which only $a \succ b$ and the comparisons between [b and c] and [a and c] are not determined.
3. (Main part of the question) Prove this claim for the general case.

Problem 6. (*Fun*)
Listen to the illusion called the Shepard Scale. (Find it online at
https://www.youtube.com/watch?v=BzNzgsAE4F0 or
http://en.wikipedia.org/wiki/Shepard_tone.)
Any economic analogies?

LECTURE 2

Utility

The Concept of Utility Representation

Think of examples of preferences. In the case of a small number of alternatives, we often describe a preference relation as a list arranged from best to worst. In some cases, the alternatives are grouped into a small number of categories, and we describe the preferences on X by specifying the preferences on the set of categories. But, in my experience, most of the examples that come to mind are similar to: "I prefer the taller basketball player", "I prefer the more expensive present", "I prefer a teacher who gives higher grades", "I prefer the person who weighs less". Common to all these examples is that they can naturally be specified by a statement of the form "$x \succsim y$ if $V(x) \geq V(y)$" (or $V(x) \leq V(y)$), where $V : X \to \mathbb{R}$ is a function that attaches a real number to each element in the set of alternatives X. For example, the preferences stated by "I prefer the taller basketball player" can be expressed formally by: X is the set of all conceivable basketball players, and $V(x)$ is the height of player x.

Note that the statement $x \succsim y$ if $V(x) \geq V(y)$ always defines a preference relation because the relation $\geq$ on $\mathbb{R}$ satisfies completeness and transitivity.

Even when the natural description of a preference relation does not involve a numerical evaluation, we are interested in an equivalent numerical representation. We say that *the function $U : X \to \mathbb{R}$ represents the preference $\succsim$* if for all x and $y \in X$, $x \succsim y$ if and only if $U(x) \geq U(y)$. If the function U represents the preference relation $\succsim$, we refer to it as a *utility function*, and we say that $\succsim$ has a *utility representation*.

It is possible to avoid the notion of a utility representation and to "do economics" with the notion of preferences. Nevertheless, we often use utility functions rather than preferences as a means of describing an economic agent's attitude toward alternatives, probably because we find it more convenient to talk about the maximization of a numerical function than of a preference relation.

Note that when defining a preference relation using a utility function, the function has an intuitive meaning that carries with it additional information. In contrast, when the utility function is formulated in order to represent an existing preference relation, the utility function has no meaning other than that of representing a preference relation. Absolute numbers are meaningless in the latter case; only relative order matters. If a preference relation has a utility representation, then it has an infinite number of such representations, as the following simple claim demonstrates:

Claim:

If U represents $\succsim$, then for any strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, the function $V(x) = f(U(x))$ represents $\succsim$ as well.

Proof:

$a \succsim b$
iff $U(a) \geq U(b)$ (since U represents $\succsim$)
iff $f(U(a)) \geq f(U(b))$ (since f is strictly increasing)
iff $V(a) \geq V(b)$.

Existence of a Utility Representation

If every preference relation could be represented by a utility function, then it would "grant a license" to use utility functions rather than preference relations with no loss of generality. Utility theory investigates the possibility of using a numerical function to represent a preference relation and the possibility of numerical representations carrying additional meanings (e.g., a is preferred to b more than c is preferred to d).

We will now examine the basic question of "utility theory": Under what assumptions do utility representations exist?

The first observation is quite trivial. When the set X is finite, there is always a utility representation. The (overly) detailed proof is presented here mainly to get you into the habit of analytical precision. We start with a lemma regarding the existence of minimal elements (an element $a \in X$ is *minimal* if $a \precsim x$ for any $x \in X$).

Lemma:

In any finite set $A \subseteq X$, there is a minimal element (similarly, there is also a maximal element).

Proof:

By induction on the size of A. If A is a singleton, then by reflexivity its only element is minimal. For the inductive step, let A be of cardinality $n+1$ and let $x \in A$. The set $A-\{x\}$ is of cardinality n and by the inductive assumption has a minimal element denoted by y. If $x \succsim y$, then y is minimal in A. If $y \succsim x$, then by transitivity $z \succsim x$ for all $z \in A-\{x\}$, and thus x is minimal.

Claim:

If $\succsim$ is a preference relation on a finite set X, then $\succsim$ has a utility representation with values that are natural numbers.

Proof:

We will construct a sequence of sets inductively. Let X_1 be the subset of elements that are minimal in X. By the above lemma, X_1 is not empty. Assume we have constructed the non-empty sets $X_1, \ldots, X_k$. If $X = X_1 \cup X_2 \cup \ldots \cup X_k$, we are done. If not, define X_{k+1} to be the set of minimal elements in $X - X_1 - X_2 - \cdots - X_k$. By the lemma $X_{k+1} \neq \emptyset$. Because X is finite, we must be done after at most $|X|$ steps. Define $U(x) = k$ if $x \in X_k$. Thus, $U(x)$ is the step number at which x is "eliminated". To verify that U represents $\succsim$, let $a \succ b$. Then $a \notin X_1 \cup X_2 \cup \cdots X_{U(b)}$ and thus $U(a) > U(b)$. If $a \sim b$, then $U(a) = U(b)$.

Without any further assumptions on the preferences, the existence of a utility representation is guaranteed when the set X is countable (recall that X is countable and infinite if there is a one-to-one function from the natural numbers onto X, namely, it is possible to specify an enumeration of all its members $\{x_n\}_{n=1,2,\ldots}$).

Claim:

If X is countable, then any preference relation on X has a utility representation with a range $(0, 1)$.

Proof:

Let $\{x_n\}$ be an enumeration of all elements in X. We will construct the utility function inductively. Set $U(x_1) = 1/2$. Assume that you have completed the definition of the values $U(x_1), \ldots, U(x_{n-1})$ so that $x_k \succsim x_l$ iff $U(x_k) \geq U(x_l)$. If x_n is indifferent to x_k for some $k < n$, then assign

$U(x_n) = U(x_k)$. If not, choose $U(x_n)$ to be between the two nonempty sets $\{U(x_k)|\ x_k \prec x_n\} \cup \{0\}$ and $\{U(x_k)|\ x_n \prec x_k\} \cup \{1\}$. This is possible since by transitivity all numbers in the first set are below all numbers in the second set. Thus, for any $k < n$ we have $x_n \succsim x_k$ iff $U(x_n) \geq U(x_k)$ and the function U extended to $\{x_1, \ldots, x_n\}$ represents the preferences on those elements.

To complete the proof that U represents $\succsim$, take any two elements, x and $y \in X$. For some k and l we have $x = x_k$ and $y = x_l$. The above applied to $n = \max\{k, l\}$ yields $x_k \succsim x_l$ iff $U(x_k) \geq U(x_l)$.

Lexicographic Preferences

Lexicographic preferences are the outcome of applying the following procedure for determining the ranking of any two elements in a set X. The individual has in mind a sequence of criteria that can be used to compare pairs of elements in X. The criteria are applied in a fixed order until a criterion is reached that succeeds in distinguishing between the two elements, in that it determines the preferred alternative. Formally, let $(\succsim_k)_{k=1,\ldots,K}$ be a K-tuple of preferences over the set X. The lexicographic preferences induced by those preferences are defined by $x \succsim_L y$ if (1) there is k^* such that for all $k < k^*$ we have $x \sim_k y$ and $x \succ_{k*} y$ or (2) $x \sim_k y$ for all k. Verify that $\succsim_L$ is a preference relation.

Example:

Let X be the unit square, that is, $X = [0,1] \times [0,1]$. Let $x \succsim_k y$ if $x_k \geq y_k$. The lexicographic preferences $\succsim_L$ induced from $\succsim_1$ and $\succsim_2$ are: $(a_1, a_2) \succsim_L (b_1, b_2)$ if $a_1 > b_1$ or both $a_1 = b_1$ and $a_2 \geq b_2$. (Thus, in this example, the left component is the primary criterion, whereas the right component is the secondary criterion.)

We will now show that the preferences $\succsim_L$ do *not* have a utility representation. The lack of a utility representation excludes lexicographic preferences from the scope of standard economic models, although they are derived from a simple and commonly used procedure.

Claim:

The lexicographic preference relation $\succsim_L$ on $[0,1] \times [0,1]$, induced from the sequence of relations $\{\succsim_k\}_{k=1,2}$, where $x \succsim_k y$ if $x_k \geq y_k$, does not have a utility representation.

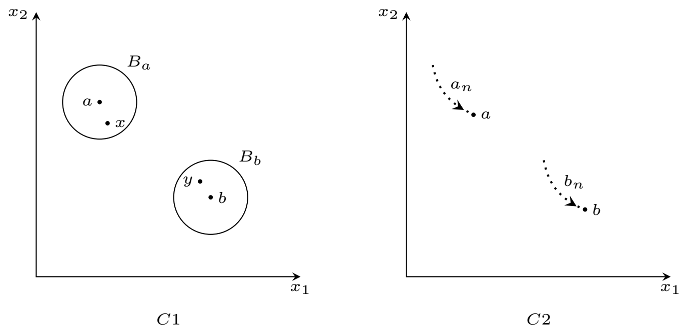

Figure 2.1
Two definitions of continuity of preferences.

Proof:

Assume by contradiction that the function $u : X \to \mathbb{R}$ represents $\succsim_L$. For any $a \in [0,1]$, $(a,1) \succ_L (a,0)$, we thus have $u(a,1) > u(a,0)$. Let $q(a)$ be a rational number in the nonempty interval $I_a = (u(a,0), u(a,1))$. The function q is a function from $[0,1]$ into the set of rational numbers. It is a one-to-one function since if $b > a$, then $(b,0) \succ_L (a,1)$ and therefore $u(b,0) > u(a,1)$. It follows that the intervals I_a and I_b are disjoint and thus $q(a) \neq q(b)$. But the cardinality of the rational numbers is lower than that of the continuum, a contradiction.

Continuity of Preferences

In economics we often take the set X to be an infinite subset of a Euclidean space. The following continuity condition guarantees the existence of a utility representation in such a case. The basic concept, captured by the notion of a continuous preference relation, is that if a is preferred to b, then "small" deviations from a or from b will not change the ordering.

In what follows we will refer to a *ball* around a in X with radius $r > 0$, denoted as $Ball(a, r)$, as the set of all points in X that are distanced less than r from a.

Definition C1:

A preference relation $\succsim$ on X is *continuous* if whenever $a \succ b$ (namely, it is not true that $b \succsim a$), there are balls (neighborhoods in the relevant topology) B_a and B_b around a and b, respectively, such that for all $x \in B_a$ and $y \in B_b$, $x \succ y$. (See fig. 2.1.)

Definition C2:

A preference relation $\succsim$ on X is *continuous* if the graph of $\succsim$ (i.e., the set $\{(x, y)|x \succsim y\} \subseteq X \times X$) is closed (with the product topology); that is, if $\{(a_n, b_n)\}$ is a sequence of pairs of elements in X satisfying $a_n \succsim b_n$ for all n and $a_n \to a$ and $b_n \to b$, then $a \succsim b$. (See fig. 2.1.)

Claim:

The preference relation $\succsim$ on X satisfies C1 if and only if it satisfies C2.

Proof:

Assume that $\succsim$ on X is continuous according to C1. Let $\{(a_n, b_n)\}$ be a sequence of pairs satisfying $a_n \succsim b_n$ for all n and $a_n \to a$ and $b_n \to b$. If it is not true that $a \succsim b$ (i.e., $b \succ a$), then there exist two balls B_a and B_b around a and b, respectively, such that for all $y \in B_b$ and $x \in B_a$, $y \succ x$. There is an N large enough such that for all $n > N$, both $b_n \in B_b$ and $a_n \in B_a$. Therefore, for all $n > N$, we have $b_n \succ a_n$, which is a contradiction.

Assume that $\succsim$ is continuous according to C2. Let $a \succ b$. Assume by contradiction that for all n there exist $a_n \in Ball(a, 1/n)$ and $b_n \in Ball(b, 1/n)$ such that $b_n \succsim a_n$. The sequence (b_n, a_n) converges to (b, a); by the second definition, (b, a) is within the graph of $\succsim$, that is, $b \succsim a$, which is a contradiction.

Remarks

1. If $\succsim$ on X is represented by a *continuous* function U, then $\succsim$ is continuous. To see this, note that if $a \succ b$, then $U(a) > U(b)$. Let $\varepsilon = (U(a) - U(b))/2$. By the continuity of U, there is a $\delta > 0$ such that for all x distanced less than δ from a, $U(x) > U(a) - \varepsilon$, and for all y distanced less than δ from b, $U(y) < U(b) + \varepsilon$. Thus, for x and y within the balls of radius δ around a and b, respectively, $x \succ y$.
2. The lexicographic preferences that were used as an example for non-existence of a utility representation are not continuous. This is because $(1, 1) \succ (1, 0)$, but in any ball around $(1, 1)$ there are points inferior to $(1, 0)$.
3. Note that the second definition of continuity can be applied to any binary relation over a topological space, not just to a preference relation. For example, the relation $=$ on the real numbers ($\mathbb{R}^1$) is continuous, whereas the relation $\neq$ is not.

Debreu's Theorem

Debreu's theorem, which states that continuous preferences have a *continuous* utility representation, is one of the classical results in economic theory. For a proof of the theorem, in a more general setting, see Debreu (1954, 1960).

In what follows, we will need the mathematical concept of a dense set. A set Y is said to be dense in X if every non-empty open set $B \subset X$ contains an element in Y. Any set $X \subseteq \mathbb{R}^n$ has a *countable dense subset.* (The standard topology in $\mathbb{R}^n$ has a countable base, that is, any open set is the union of subsets of the countable collection of open sets: $\{Ball(a, 1/m) |\, a \in \mathbb{R}^n$ and all its components are rational numbers; m is a natural number$\}$. For every set $Ball(q, 1/m)$ that intersects X, pick a point $y_{q,m} \in X \cap Ball(q, 1/m)$. The set that contains all of the points $\{y_{q,m}\}$ is a countable dense set in X.)

Proposition (Debreu):

Let $\succsim$ be a continuous preference relation on X, which is a convex subset of $\mathbb{R}^n$. Then $\succsim$ has a continuous utility representation.

Proof:

(Oren Danieli and Luke Levy-Moore suggested the following proof.)
For the case that $\succsim$ is the total indifference, any constant function represents the preferences. From here on, assume that $\succsim$ is not the total indifference.

Lemma 1:

If $x \succ y$, then there exists z in X such that $x \succ z \succ y$.

Proof:

Assume not. Let I be the interval between x and y. By the convexity of X, $I \subseteq X$. Construct inductively two sequences of points in I, $\{x_t\}$ and $\{y_t\}$, in the following manner: First, define $x_0 = x$ and $y_0 = y$. Assume that the two points x_t and y_t are defined, belong to I, and satisfy $x_t \succsim x$ and $y \succsim y_t$. Consider m, the middle point between x_t and y_t. Either $m \succsim x$ or $y \succsim m$. In the former case, define $x_{t+1} = m$ and $y_{t+1} = y_t$, and in the latter case define $x_{t+1} = x_t$ and $y_{t+1} = m$. The sequences $\{x_t\}$ and $\{y_t\}$ are converging, and they must converge to the same point z because the distance between x_t and y_t converges to zero. By the continuity of $\succsim$, we have $z \succsim x$ and $y \succsim z$ and thus, by transitivity, $y \succsim x$, which contradicts the assumption that $x \succ y$.

Another simple proof would fit the more general case, in which the assumption that the set X is convex is replaced by the weaker assumption that X is a connected subset of $\mathbb{R}^n$ (i.e. a set which cannot be covered by two nonempty disjoint open sets): If there is no z such that $x \succ z \succ y$, then X is the union of two disjoint sets $\{a|a \succ y\}$ and $\{a|x \succ a\}$, which are open by the continuity of the preference relation. This contradicts the connectedness of X.

Lemma 2:

Let Y be dense in X. Then, for every $x, y \in X$, if $x \succ y$ there exists $z \in Y$ such that $x \succ z \succ y$.

Proof:

By Lemma 1, there exists $z \in X$ such that $x \succ z \succ y$. By continuity, there is a non-empty ball around z that is between x and y with respect to the preference relation and, by the denseness of Y, the ball contains an element of Y.

Lemma 3:

Let E be the set of $\succsim$ -maxima and $\succsim$ -minima in X. Let Y be a countable dense set in $X - E$. Then, $\succsim$ has a utility representation on Y, u with a range being the set of all dyadic rational numbers in $(0, 1)$ (namely all numbers that can be expressed as $k/2^l$ where k and l are natural numbers and $k < 2^l$). Note that this set is dense in $(0, 1)$.

Proof:

The set $X - E$ contain two points a and b such that $a \succ b$ and therefore it follows from Lemma 2 that the set Y is infinite. Let $\{y_n\}$ be an enumeration of Y.

Construct u by induction as follows: Start with $u(y_1) = 1/2$. Let $P(y_n) = \{y_1, .., y_{n-1}\}$, i.e., the set of elements that precedes y_n in the enumeration of Y. If $y_n \sim y_m$ for some $y_m \in P(y_n)$, let $u(y_n) = u(y_m)$. If $y_n \succ y_k$ where y_k is maximal in $P(y_n)$, set $u(y_n) = (1 + u(y_k))/2$. If $y_l \succ y_n$ where y_l is minimal in $P(y_n)$, set $u(y_n) = u(y_l)/2$. Otherwise, there are $y_i, y_j \in P(y_n)$ such that y_i is minimal among the elements in $P(y_n)$ that are preferred to y_n and y_j is maximal among the elements in $P(y_n)$ that are inferior to y_n. Let $u(y_n) = (u(y_i) + u(y_j))/2$. Note that by Lemma 2, for every element in the sequence there will always eventually be one element in the sequence that is above it and one that is below it and for every two elements in the sequence there will eventually be an element in the sequence that is sandwiched between the two. Therefore, the range of u is exactly all dyadic numbers in $(0, 1)$.

Completing the Proof:

Let Y be a countable dense set in $X - E$. Define u on Y according to Lemma 3. The function u can be extended to X by: (i) assigning the value 1 to all maxima points in X and the value 0 to all minima points and (ii) defining $u(x) = sup\{u(y) \mid x \succsim y \text{ } and \text{ } y \in Y\}$ for all $x \notin E$. This function represents the preference relation since by definition if $x \sim z$ we have $u(x) = u(z)$ and if $x \succ z$ then by Lemma 2 there are y_1 and y_2 in Y such that $x \succ y_1 \succ y_2 \succ z$ and thus $u(x) \geq u(y_1) > u(y_2) \geq u(z)$.

In order to prove the continuity of u, consider a point $x \notin E$ (a similar proof applies to extreme points). Let $\varepsilon > 0$. By Lemma 3, there are y_1 and y_2 in Y such that $u(x) - \varepsilon < u(y_1) < u(x) < u(y_2) < u(x) + \varepsilon$. Thus, $y_2 \succ x \succ y_1$. By twice applying the definition of the continuity of $\succsim$, we obtain a ball B around x that is between y_1 and y_2 with respect to the preference relation. By definition, elements in this ball receive u values between $u(y_1)$ and $u(y_2)$ and thus are not further than ε from $u(x)$.

Bibliographic Notes

Fishburn (1970) covers the material in this lecture very well. The example of lexicographic preferences appears in Debreu (1959) (see also Debreu (1960), in particular chapter 2, which is available online at http://cowles.econ.yale.edu/P/cp/p00b/p0097.pdf.)

Problem Set 2

Problem 1. (*Easy*)
The purpose of this problem is to make sure that you fully understand the basic concepts of utility representation and continuous preferences.

a. Is the following statement correct: "if both U and V represent $\succsim$, then there is a *strictly* monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that $V(x) = f(U(x))$"?
b. Can a continuous preference relation be represented by a discontinuous utility function?
c. Show that in the case of $X = \mathbb{R}$, the preference relation that is represented by the discontinuous utility function $u(x) = \lfloor x \rfloor$ (the largest integer n such that $x \geq n$) is not a continuous relation.
d. Show that the two definitions of a continuous preference relation (C1 and C2) are equivalent to:

   Definition C3: For any $x \in X$, the upper and lower contours $\{y| \ \ y \succsim x\}$ and $\{y| \ \ x \succsim y\}$ are closed sets in X,

   and

   Definition C4: For any $x \in X$, the sets $\{y| \ \ y \succ x\}$ and $\{y| \ \ x \succ y\}$ are open sets in X.

Problem 2. (*Moderately difficult*)
Give an example of preferences over a countable set in which the preferences cannot be represented by a utility function that returns only integers as values.

Problem 3. (*Easy*)
Let $\succsim$ be continuous preferences on a set $X \subseteq \mathbb{R}^n$ that contains the interval connecting the points x and z. Show that if $x \succsim y \succsim z$, then there is a point m on the interval connecting x and z such that $y \sim m$.

Problem 4. (*Moderately difficult*)
Consider the sequence of preference relations $(\succsim^n)_{n=1,2,\ldots}$, defined on $\mathbb{R}^2_+$ where $\succsim^n$ is represented by the utility function $u_n(x_1, x_2) = x_1^n + x_2^n$. We will say that the sequence $\succsim^n$ converges to the preferences $\succsim^*$ if for every x and y, such that $x \succ^* y$, there is an N such that for every $n > N$ we have $x \succ^n y$. Show that the sequence of preference relations $\succsim^n$ converges to the preferences $\succ^*$, which are represented by the function $max\{x_1, x_2\}$.

Problem 5. (*Moderately difficult*)

Let X be a set and let $(\succsim, \succ\succ)$ be a pair where $\succsim$ is a preference relation; let $\succ$ be induced from $\succsim$, and let $\succ\succ$ be a transitive subrelation of $\succ$ (by subrelation, we mean that $x \succ\succ y$ implies $x \succ y$.) The meaning of x$\succ\succ$ y is that x is much preferred over y. Assume that the pair satisfies the following extended transitivity condition: If $x \succ\succ y$ and $y \succsim z$, or if $x \succsim y$ and $y \succ\succ z$, then $x \succ\succ z$.

We say that a pair $(\succsim, \succ\succ)$ is represented by a function u if:

$u(x) = u(y)$ iff $x \sim y$,
$u(x) - u(y) > 0$ iff $x \succ y$, and
$u(x) - u(y) > 1$ iff $x \succ\succ y$.

Show that if X is finite every extended preference $(\succsim, \succ\succ)$ can be represented by a function u.

Problem 6. (*Moderately difficult*)

Utility is a numerical representation of preferences. One can also think about the numerical representation of other abstract concepts. Here, we discuss a numerical representation of the concept "approximately the same" (see Luce (1956) and Rubinstein (1988)). For simplicity, let X be the interval $[0, 1]$.

Consider the following six properties of the binary relation S:

(S-1) For any $a \in X$, aSa.
(S-2) For all $a, b \in X$, if aSb, then bSa.
(S-3) Continuity (the graph of the relation S in $X \times X$ is a closed set).
(S-4) Betweenness: If $d \geq c \geq b \geq a$ and dSa, then also cSb.
(S-5) For any $a \in X$, there is an open interval around a such that xSa for every x in the interval.
(S-6) Denote $M(a) = max\{x|xSa\}$ and $m(a) = min\{x|aSx\}$. Then, M and m are (weakly) increasing functions and are strictly increasing whenever they do not have the values 0 or 1.

a. Do these assumptions capture your intuition about the concept "approximately the same"?
b. Show that the relation S_ε, defined by $aS_\varepsilon b$ if $|b - a| \leq \varepsilon$ (for positive ε), satisfies all the assumptions.
c. (*Difficult*) Let S be a binary relation that satisfies the above six properties and let ε be a strictly positive number. Show that there is a strictly increasing and continuous function $H : X \to \mathbb{R}$ such that aSb if and only if $|H(a) - H(b)| \leq \varepsilon$.

LECTURE 3

Choice

Choice Functions

Until now we have avoided any reference to behavior. We have talked about preferences as a summary of the individual's mental attitude toward a set of alternatives. But economics is about action, and therefore we now move on to modeling "agent behavior".

An economic agent is described in our models as an entity that responds to a scenario called a *choice problem*, where the agent must make a choice from a set of available alternatives.

Consider a *grand set* X of possible alternatives. A *choice problem* is a nonempty subset of X, and we refer to a choice from $A \subseteq X$ as specifying one of A's members.

Modeling a choice scenario as a set of alternatives carries with it implicit assumptions of rationality according to which the agent's choice does not depend on the way the alternatives are presented. For example, if the alternatives appear in a list, he ignores the order in which they are presented and the number of times an alternative appears in the list. If there is an alternative with a default status, he ignores that as well. As a rational agent he is attentive only to the set of alternatives available to him.

In some contexts, not all choice problems are relevant. Therefore we allow that the agent's behavior be defined only on a restricted set D of subsets of X. We will refer to a pair (X, D) as a *context*.

Example:

1. Imagine that we are interested in a student's choice from the set of universities to which he has been admitted. Let $X = \{x_1, \dots, x_N\}$ be the set of all universities with which the student is familiar. A choice problem A is interpreted as the set of universities to which he has been admitted. If the fact that the student was admitted to some universities does not mean he will be admitted to others, then D contains the $2^N - 1$ nonempty subsets of X. But if, for example, the universities are listed

according to difficulty of getting into (x_1 being the most difficult) and if the fact that the student is admitted to x_k means that he is admitted to all less "prestigious" universities, that is, to all x_l with $l > k$, then D will consist of the N sets $A_1, \ldots, A_N$ where $A_k = \{x_k, \ldots, x_N\}$.

2. Imagine a scenario in which a decision maker is choosing whether to remain with the status quo s or choose an element in a set Y. We formalize such a context by defining $X = Y \cup \{s\}$ and identifying the domain D with the set of all subsets of X that contain s.

3. The set of potential candidates for a political position is X. The set is partitioned into R and D. A choice function domain is the set of all sets of size two where one alternative is in R and another is in D.

The economic agent in this framework is characterized by a full description of his behavior in all scenarios that we imagine he might confront in a certain context. We might think about an agent's behavior as a hypothetical response to a questionnaire that contains questions of the following type, one for each $A \in D$:

Q(*A*): Assume you must choose from a set of alternatives A. Which alternative do you choose?

A permissible response to this questionnaire requires that the agent single out a unique element in A for every question Q(A). We implicitly assume that the agent cannot give any other answer such as "I choose either a or b"; "the probability of my choosing $a \in A$ is $p(a)$"; or "I don't know".

Formally, given a context (X, D), a *choice function* C assigns to each set $A \in D$ a unique element of A with the interpretation that $C(A)$ is the element chosen from the set A. A decision maker behaving in accordance with the function C will choose $C(A)$ if he has to make a choice from a set A. Typically, we cannot observe choice functions. At most, we might observe some particular choices made by the decision maker in some instances. Thus, a choice function is a description of hypothetical behavior.

Rational Choice Functions

In most contexts of economic theory, the deliberation process is referred to as *rational* choice. The agent goes through a three-step process:

1. Determining the desirability of all the alternatives that might be relevant in the context.
2. Determining which alternatives are feasible.

3. Choosing the most desirable alternative from among the feasible alternatives.

Note the order of the stages. In particular, the stage during which preferences are shaped precedes the stage in which feasible alternatives are recognized; the rational economic agent's preferences are independent of the set of alternatives.

Rationality in economics does not involve making any judgments about preferences. A rational agent can have preferences that are universally viewed, probably also by the agent himself, as being against his best interest.

Until recently the practice of most economists was to make assumptions that capture the materialistic desires of the economic agent while dismissing the role of psychological motives. This practice has changed somewhat during the past decade with the development of Behavioral Economics' approach.

Note that even if most people's behavior in most contexts is inconsistent with rationality, the investigation of rational economic agents is still of interest. It is meaningful to talk about the concept of "being good" even in a society where all people are evil; similarly, it is meaningful to talk about the concept of a "rational man" and about the interactions between rational economic agents even if all people systematically behave in a non-rational manner. Furthermore, we often refer to rational decision making as a normative process

Formally, the rational economic agent has in mind a preference relation $\succsim$ on the set X and, given any choice problem A in D, he chooses an element in A that is $\succsim$ optimal. Assuming that it is well-defined, we define the *induced choice function* $C_{\succsim}$ as the function that assigns to every nonempty set $A \in D$ the $\succsim$-best element of A. As mentioned earlier, the preferences are fixed, that is, they are independent of the choice set being considered.

Dutch Book Arguments

Some of the justifications for the assumption that choice is determined by "rational deliberation" are normative, that is, they reflect a perception that people should be rational in this sense and, if they are not, then they should be. One interesting class of arguments supporting this approach is referred to in the literature as "Dutch book arguments". The claim is that an economic agent who behaves according to a choice function that is not derived from the maximization of a preference relation will not survive.

The following is a "sad" story about a monkey in a forest with three trees: a , b, and c. The monkey is about to pick a tree to sleep in. Assume that the monkey can assess only two alternatives at a time and that his choice function is $C(\{a, b\}) = b$, $C(\{b, c\}) = c$, $C(\{a, c\}) = a$. Obviously, his choice function cannot be derived from a preference relation over the set of trees. Assume that whenever he is on tree x it occurs to him to jump to one of the other trees; namely, when he is on x he considers the set $\{x, y\}$ where y is one of the two other trees. This induces the monkey to perpetually jump from one tree to another – a catastrophic mode of behavior in the jungle.

Another argument – which is more appropriate to the case of human beings – is called the "money pump" argument. Assume that a decision maker behaves like the monkey with respect to three alternatives: a, b, and c. Assume that, for all x and y, the choice $C(x, y) = y$ is firm enough so that whenever the decision maker is about to choose alternative x and somebody gives him the option to also choose y, he is ready to pay one cent for the opportunity to do so.

Now imagine a manipulator who presents the agent with the choice problem $\{a, b, c\}$. Whenever the decision maker is about to make the choice a, the manipulator allows him to revise his choice to b for one cent. Similarly, every time he is about to choose b (or c), the manipulator sells him the opportunity to choose c (or a) for one cent. The decision maker will cycle through the intentions to choose a, b, and c until his pockets are emptied or until he learns his lesson and changes his behavior.

The above arguments are open to criticism. The elimination of patterns of behavior that are inconsistent with rationality requires an environment in which the economic agent is indeed confronted with the above sequence of choice problems naturally or they are presented by an outsider who coerces the individual to make the choices. The arguments are presented here simply as interesting ideas and not as convincing arguments for assuming rationality.

Rationalizing

Economists have often been criticized for making the assumption that decision makers maximize a preference relation. It would seem that most decisions are made without explicitly maximizing a preference relation. The most common response to this criticism is that we don't really need

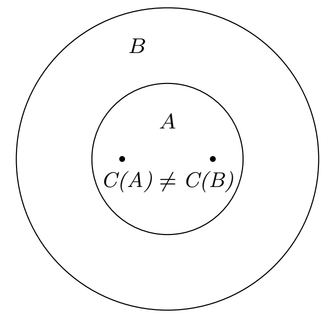

Figure 3.1
Violation of condition α.

to assume that the individual follows the rational man procedure. All we need to assume is that he can be described *as if* he were maximizing some preference relation.

Accordingly, we will say that a choice function C *can be rationalized* if there is a preference relation $\succsim$ on X so that $C = C_{\succsim}$ (i.e., $C(A) = C_{\succsim}(A)$ for any A in the domain of C).

We will now identify a condition under which a choice function can indeed be presented as if derived from some preference relation (i.e., can be rationalized).

Condition α:

We say that C satisfies condition α if for any two problems A,$B \in D$, if $A \subset B$ and $C(B) \in A$, then $C(A) = C(B)$. (See fig. 3.1)

Note that if $\succsim$ is a preference relation on X, then $C_{\succsim}$ (defined on a set of subsets of X that have a single most preferred element) satisfies condition α: if $C(B) \succsim x$ for all $x \in B$ the it is also true that $C(B) \succsim x$ for all $x \in A$.

As an example of a choice procedure that does not satisfy condition α, consider the *second-best choice function* C: the decision maker has in mind an ordering $\succsim$ of X (i.e., a complete, asymmetric and transitive binary relation) and $C(A)$ is the element from A that is the $\succsim$-maximal from among the non-maximal alternatives. The choice function C does not satisfy condition α: Let b be the $\succsim$-maximal element in B, and let $A = B - \{b\}$. Then, $C(B) \in A \subset B$ but $C(A) \neq C(B)$ since $C(A)$ is the third $\succsim$–best in B.

We will show now that condition α is a sufficient condition for a choice function to be formulated *as if* the decision maker is maximizing some preference relation.

Proposition:

Assume that C is a choice function defined on a domain containing at least all subsets of X with size of at most 3. If C satisfies condition α, then there is a preference $\succsim$ on X so that $C = C_{\succsim}$.

Proof:

Define a binary relation $\succsim$ by $x \succsim y$ if $x = C(\{x, y\})$.

Let us first verify that the relation $\succsim$ is a preference relation.

Completeness and Reflexivity: Follows from the fact that $C(\{x, y\})$ is always well defined.

Transitivity: If $x \succsim y$ and $y \succsim z$, then $C(\{x, y\}) = x$ and $C(\{y, z\}) = y$. By condition α and $C(\{x, y\}) = x$, $C(\{x, y, z\}) \neq y$. By condition α and $C(\{y, z\}) = y$, $C(\{x, y, z\}) \neq z$. Therefore, $C(\{x,y, z\}) = x$ and by condition α we must have $C(\{x, z\}) = x$, that is $x \succsim z$.

We still need to show that $C(B) = C_{\succsim}(B)$ for all B. Let $C(B) = x$ and let $y \in B - \{x\}$. Condition α implies that $C(\{x, y\}) = x$. Thus, $x \succsim y$ for all $y \in B$.

Following is a different version of the above proposition:

Proposition:

Let C be a choice function with a domain D satisfying that if $A, B \in D$, then $A \cup B \in D$. If C satisfies condition α, then there is a preference relation $\succsim$ on X such that $C = C_{\succsim}$.

Proof:

Define a binary relation as xRy if x and y are distinct and there is a set $A \in D$ such that $y \in A$ and $c(A) = x$. Note that R is not necessarily complete. We will see that the relation R does not have cycles.

If xRy and yRz, then there is $A \in D$ containing y such that $C(A) = x$ and there is $B \in D$ containing z such that $C(B) = y$. The set $A \cup B$ is a member of D. The element $C(A \cup B)$ is in either A or B and thus by condition α it is either x or y. It is not y since if $C(A \cup B) = y \in A$ then by condition α, $C(A \cup B) = C(A) = y$. Thus, $C(A \cup B) = x$. Since $C(B) = y$ then by condition α we have $z \neq x$ and therefore xRz. Thus, the relation R is transitive and hence does not have a cycle.

A well-known proposition in Set Theory (which you proved for finite sets in Problem Set 1- Problem 4) guarantees that the acyclic and antisymmetric relation R extends to a preference relation $\succsim$. By definition, $c(A) \succsim x$ for all $x \in A$ and thus $C(A) = C_{\succsim}(A)$.

The Satisficing Procedure

The fact that if D is wide enough we can present any choice function satisfying condition α as an outcome of the optimization of some preference relation supports the view that the scope of microeconomic models goes beyond situations in which agents carry out explicit optimization. But have we indeed expanded the scope of these models significantly? Are there reasonable choice functions satisfying condition α which can naturally be described without reference to maximization of some preference relation?

Consider the following "decision scheme", which was named *satisficing* by Herbert Simon. Let $v : X \to \mathbb{R}$ be a valuation of the elements in X, and let $v^* \in \mathbb{R}$ be a threshold of satisfaction. Let O be an ordering of the alternatives in X. Given a set A, the decision maker arranges the elements of this set in a list $L(A, O)$ according to the ordering O. He then chooses the first element in $L(A, O)$ that has a v-value at least as large as v^*. If there is no such element in A, the decision maker chooses the last element in $L(A, O)$.

We now show that the choice function induced by this procedure satisfies condition α. Assume that a is chosen from B and is also a member of $A \subset B$. The list $L(A, O)$ is obtained from $L(B, O)$ by eliminating all elements in $B - A$. If $v(a) \geq v^*$, then a is the first satisfactory element in $L(B, O)$ and is also the first satisfactory element in $L(A, O)$. Thus, a is chosen from A. If all elements in B are unsatisfactory, then a must be the last element in $L(B, O)$. Since A is a subset of B, all elements in A are unsatisfactory and a is the last element in $L(A, O)$. Thus, a is chosen from A.

A direct proof that the procedure is rationalizeable is obtained by explicitly constructing an ordering that rationalizes the satisficing procedure. Let $\succsim$ be the ordering that places the elements that satsifice, (namely, the members of $\{x | v(x) \geq v^*\}$) on top, ordered according to O. The other alternatives are placed at the bottom, ordered according to the reversed ordering O. For any set A, maximizing $\succsim$ will yield the first element (according to O) which is satisficing and if there isn't one then maximization will choose the last element in A (according to O).

Note, however, that even a "small" modification of the procedure no longer satisfies condition α. For example:

Satisficing using two orderings: Let X be a population of university graduates who are potential candidates for a job. If there are less than 5, order them alphabetically. If there are at least 5, order them by their social security number. Whatever ordering is used, choose the first candidate whose undergraduate average is above 85. If there are none, choose the last student on the list.

Condition α might not be satisfied. It may be that a is the first candidate with a satisfactory grade in a long list of students ordered by their social security numbers. Still, a might not be the first candidate with a satisfactory grade on a list of only three of the candidates appearing on the original list when they are ordered alphabetically.

To summarize, the satisficing procedure, though stated in a way that seems unrelated to the maximization of a preference relation or utility function, can be described as if the decision maker maximizes a preference relation. I know of no other interesting very general choice procedure that satisfy condition α other than the "rational man" and the satisficing procedures. Later on, in the discussion of consumer theory, we will come across several appealing examples of demand functions that can be rationalized, though they appear to be unrelated to the maximization of a preference relation.

What is an Alternative

Some of the cases in which rationality is violated can be attributed to the incorrect specification of the space of alternatives. Consider the following example taken from Luce and Raiffa (1957): A diner in a restaurant will choose *chicken* from the menu of *steak tartare*, *chicken* but will choose *steak tartare* from the menu of *steak tartare*, *chicken*, *frog legs*. At first glance, it appears not to be rational (since his choice violates condition α). His choice makes sense if he believes that the existence of *frog legs* is a positive indication of the quality of the chef. Thus, if the dish *frog legs* is on the menu, the cook must be a real expert, and the decision maker is happy ordering *steak tartare*, which requires expertise to make. If the menu lacks *frog legs*, the decision maker does not want to take the risk of choosing *steak tartare*.

Rationality is "restored" if we make the distinction between "*steak tartare* served in a restaurant where *frog legs* are also on the menu (and the cook must then be a real chef)" and "*steak tartare* in a restaurant where *frog legs* are not served (and the cook is likely a novice)". Such a distinction makes sense because the *steak tartare* is not the same in the two choice sets.

Note that if we define an alternative to be (a, A), where a is a physical description and A is the choice problem from which it is chosen, any choice function C can be trivially rationalized by a preference relation satisfying $(C(A), A) \succsim (a, A)$ for every $a \in A$.

The lesson from the above discussion is that care must be taken in specifying the term "alternative". An alternative a must have the same meaning in every choice problem A which contains a.

Choice Correspondences

The choice function definition we have been using requires that a single element be assigned to each choice problem. If the decision maker follows the rational man procedure using a preference relation with indifferences, the previously defined induced choice function $C_{\succsim}(A)$ might be undefined because for some choice problems there would be more than one maximal element. This is one of the reasons that sometimes we use the alternative framework of choice correspondence.

A *choice correspondence* C is required to assign to every nonempty $A \in D$ a nonempty *subset* of A, that is, $\emptyset \neq C(A) \subseteq A$. According to our interpretation of a choice problem, a decision maker selects a unique element from every choice set. Thus, $C(A)$ cannot be interpreted as the choice made by the decision maker when he chooses from A. The revised interpretation of $C(A)$ is the *set* of all elements in A that are satisfactory in the sense that if the decision maker is about to make a decision and chooses $a \in C(A)$, then he has no desire to move away from it. In other words, the induced choice correspondence reflects an "internal equilibrium": if the decision maker facing A considers an alternative outside $C(A)$, he will continue searching for another alternative. If he happens to consider an alternative inside $C(A)$, then that is the one he will choose.

A related interpretation of $C(A)$ involves viewing it as the set of all elements in A that might be chosen under any of the many possible circumstances not included in the description of the set A. Formally, let (A, f) be an extended choice set where f is the frame that accompanies the set A (such as the default alternative or the order of the alternatives). Let $c(A, f)$ be the choice of the decision maker from the choice set A given the frame f. The (extended) choice function c induces a choice correspondence $C(A) = \{x | x = c(A, f) \text{ for some } f\}$.

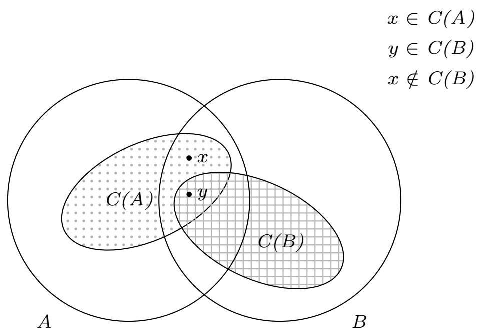

Figure 3.2
Violation of the weak axiom.

Given a preference relation $\succsim$, we define the induced choice correspondence (assuming it is never empty) as $C_{\succsim}(A) = \{x \in A \mid x \succsim y \ for\ all\ y \in A\}$.

When $x, y \in A$ and $x \in C(A)$, we say that x is revealed to be at least as good as y. If, in addition, $y \notin C(A)$, we say that x is revealed to be strictly better than y. Condition α is now replaced by condition WA, which requires that if x is revealed to be at least as good as y, then y is not revealed to be strictly better than x.

The Weak Axiom of Revealed Preference (WA):
We say that C satisfies WA if whenever $x, y \in A \cap B$, $x \in C(A)$, and $y \in C(B)$, it is also true that $x \in C(B)$ (fig. 3.2).

Notice that if $C(A)$ contains all the elements that are maximal according to some preference relation, then C satisfies WA.

The Weak Axiom trivially implies two properties of choice correspondences that are often discussed in the literature:

Condition α: If $a \in A \subset B$ and $a \in C(B)$, then $a \in C(A)$.

Condition β: If $a, b \in A \subset B$, $a, b \in C(A)$, and $b \in C(B)$, then $a \in C(B)$.

Verify that conditions α and β together are equivalent to WA for any choice correspondence with a domain satisfying that if A and B are included in the domain, then so is their intersection.

Note the following two examples of choice correspondences (with domains including all subsets of X):

1. The decision maker has in mind a strict preference relation and $C(A)$ contains the top two elements in A. The correspondence satisfies condition α but not β.
2. The decision maker has in mind a complete directed graph $\rightarrow$. If the set A contains an element a such that $a \rightarrow x$ for all $x \in A - \{a\}$, then $C(A) = \{a\}$. Otherwise, $C(A) = A$. Unless $\rightarrow$ is transitive, the correspondence satisfies condition β but not α.

For the next proposition, we could have made do with a weaker version of WA, which includes the same condition applied only to any two sets $A \subset B$ where A is a set of two elements.

Proposition:

Assume that C is a choice correspondence with a domain that includes at least all subsets of size at most 3. Assume that C satisfies properties α and β. Then, there is a preference $\succsim$ such that $C = C_{\succsim}$.

Proof:

Define $x \succsim y$ if $x \in C(\{x, y\})$. We will now show that the relation is a preference:

Reflexivity: Follows from $C(\{x\}) = x$.

Completeness: Follows from $C(\{x, y\}) \neq \emptyset$.

Transitivity: If $x \succsim y$ and $y \succsim z$, then $x \in C(\{x, y\})$ and $y \in C(\{y, z\})$. If $y \in C(\{x, y, z\})$ then by condition α we get $y \in C(\{x, y\})$ and by condition β implies that $x \in C(\{x, y, z\})$. Similarly, if $z \in C(\{x, y, z\})$, then $y \in C(\{x, y, z\})$. Thus, in any case, $x \in C(\{x, y, z\})$. By condition α, $x \in C(\{x, z\})$ and thus $x \succsim z$.

It remains to be shown that $C(B) = C_{\succsim}(B)$.

Assume that $x \in C(B)$. By condition α for every $y \in B$ we have $x \in C(\{x, y\})$ and thus $x \succsim y$. It follows that $x \in C_{\succsim}(B)$.

Assume that $x \in C_{\succsim}(B)$. Let $y \in C(B)$. If $y \neq x$, then $x \in C(\{x, y\})$ and by condition α we have $y \in C(\{x, y\})$. From condition β we conclude that $x \in C(B)$.

Psychological Motives Not Included within the Framework

A more modern attack on the standard approach to modeling economic agents was launched by psychologists, most notably Amos Tversky and Daniel Kahneman. They have provided us with ample examples demonstrating not only that rationality is often violated but that there are systematic reasons for the violation resulting from certain elements of our decision procedures. Following are a few examples:

Framing

The following experiment (see Tversky and Kahneman (1986)) illustrates that the way in which alternatives are framed may affect decision makers' choices. Subjects were asked to imagine being confronted by the following dramatic choice problem:

An outbreak of disease is expected to cause 600 deaths in the United States. Two mutually exclusive programs are expected to yield the following results:

a. 400 people will die.
b. With probability 1/3, 0 people will die, and with probability 2/3, 600 people will die.

In the original experiment, a different group of subjects was given the same background information and asked to choose from the following alternatives:

c. 200 people will be saved.
d. With probability 1/3, all 600 will be saved, and with probability 2/3, none will be saved.

Whereas 78% of the first group chose b, only 28% of the second group chose d. These are "problematic" results since all agree that a and c are identical alternatives, as are b and d. Thus, the choice from $\{a, b\}$ should be consistent with the choice from $\{c, d\}$.

Both questions were presented in the above order to 8,250 students taking game theory courses with the result that 72% chose b and 50% chose d. It seems plausible that many students kept in mind their answer to the first question while responding to the second one, and therefore the level of inconsistency was reduced. Nonetheless, a large proportion of students gave different answers to the two problems, which makes the findings even more striking.

The results demonstrate the sensitivity of choice to the framing of the alternatives. What is more basic to rational decision making than taking the same choice when only the manner in which the problems are stated is different?

Simplifying the Choice Problem and the Use of Similarities

The following experiment was also conducted by Tversky and Kahneman. One group of subjects was presented with the following choice problem:

Choose one of the two roulette games a or b. Your prize is the one corresponding to the outcome of the chosen roulette game as specified in the following tables:

	Color	White	Red	Green	Yellow
(a)	Chance %	90	6	1	3
	Prize $	0	45	30	-15

	Color	White	Red	Green	Yellow
(b)	Chance %	90	7	1	2
	Prize $	0	45	-10	-15

A different group of subjects was presented the same background information and asked to choose between:

	Color	White	Red	Green	Blue	Yellow
(c)	Chance %	90	6	1	1	2
	Prize $	0	45	30	-15	-15

and

	Color	White	Red	Green	Blue	Yellow
(d)	Chance %	90	6	1	1	2
	Prize $	0	45	45	-10	-15

In the original experiment, 58% of the subjects in the first group chose a, whereas no one in the second group chose c. When the two problems were presented, one after the other, to 3,250 students, 51% chose a and 7% chose c. Interestingly, the median response time among the students who answered a was 53 seconds, whereas among the students who answered b to was 89 seconds.

The results demonstrate a common procedure used by people when confronted with a complicated choice problem. We often transform a complicated problem into a simpler one by "canceling" similar elements.

Although d clearly dominates c, the comparison between a and b is not as easy. Many subjects "cancel" the probabilities of White, Yellow, and Red and are left with comparing the Green prizes, a process that leads them to choose a.

Incidentally, several times in the past when I have presented these choice problems in class, I have had students (some of the best ones, in fact) who chose c. According to their explanations they had identified the second problem with the first and used the following procedural rule: "I chose a from $\{a, b\}$. The alternatives c and d are identical to the alternatives a and b, respectively. It is only natural then that I choose c from $\{c, d\}$." This observation brings to our attention the fact that the model of rational man does not allow for dependence of choice on previous choices made by the decision maker.

Reason-Based Choice

Making choices sometimes involves finding reasons to pick one alternative over another. When the deliberation involves reasons strongly associated with the problem at hand ("internal reasons"), we often find it difficult to reconcile the choice with the rational man paradigm.

Imagine, for example, a European student who would choose *Princeton* from {*Princeton*, *LSE*} and *LSE* from {*Princeton*, *Chicago*, *LSE*}. According to his explanation he prefers an American university as long as he does not have to choose between American schools – a choice he deems to be harder. When choosing from {*Princeton*, *Chicago*, *LSE*}, he finds it difficult to decide between *Princeton* and *Chicago* and therefore chooses not to cross the Atlantic. His choice does not satisfy condition α, not because of a careless specification of the alternatives (as in the restaurant menu example discussed previously), but because his reasoning involves an attempt to avoid a more difficult decision.

A better example was suggested to me by former NYU student Federico Filippini: "Imagine there's a handsome guy called Albert, who is looking for a date to take to a party. Albert knows two girls that are crazy about him, both of whom would love to go to the party. The two girls are called Mary and Laura. Of the two, Albert prefers Mary. Now imagine that Mary has a sister, and this sister is also crazy about Albert. Albert must now choose between the three girls, Mary, Mary's sister, and Laura. With this third option, I bet that if Albert is rational, he will be taking Laura to the party."

Another example follows Huber, Payne, and Puto (1982):

Let $x = (x_1, x_2)$ be "a holiday package of x_1 days in Paris and x_2 days in London". Choose one of the four vectors $a = (7, 4)$, $b = (4, 7)$, $c = (6, 3)$, and $d = (3, 6)$.

All subjects in the experiment agreed that a day in Paris and a day in London are desirable goods. Some of the subjects were asked to choose between the three alternatives a, b, and c; others to choose between a, b, and d. The subjects exhibited a tendency toward choosing a out of the set $\{a, b, c\}$ and b out of the set $\{a, b, d\}$.

A related experiment is reported in Shafir, Simonson, and Tversky (1993). A group of subjects was asked to imagine they have to choose between a camera priced \$170 and a better camera, produced by the same company, which costs \$240. Another group of subjects was asked to imagine they have to choose between three cameras – the two described above and a third, much more sophisticated camera, priced at \$470. The addition of the third alternative significantly increased the proportion of subjects who chose the \$240 camera. The commonsense explanation for this choice is that subjects faced a conflict between two desires: to buy a better camera and to pay less. They resolved the conflict by choosing the "compromise alternative".

To conclude, decision makers look for reasons to prefer one alternative over the other. Typically, making decisions by using "external reasons" (which do not relate to the properties of the choice set) will not lead to violations of rationality. However, applying "internal reasons" such as "I prefer the alternative a over the alternative b since a clearly dominates the other alternative c while b does not" might result violations of condition α.

Mental Accounting

The following example is taken from Kahneman and Tversky (1984). Members of one group of subjects were presented with the following question:

1. Imagine that you have decided to see a play and paid \$20 for a ticket. As you enter the theater, you discover that you have lost the ticket. The seat was not marked and the ticket cannot be reissued. Would you pay \$20 for another?

Members of another group were asked to answer the following question:

2. Imagine that you have decided to see a play where the tickets cost \$20. As you arrive at the theater, you discover that you have lost a \$20 bill. Would you still pay \$20 for a ticket to the play?

If the rational man cares only about seeing the play and his wealth, he should realize that there is no difference between the consequence of replying Yes to both question 1 and question 2 (in both cases he will own a ticket and will be poorer by \$20). Similarly, there is no difference between the consequence of replying No to question 1 and replying No to question 2. Thus, the rational agent should give the same answer to both questions. Nonetheless, only 46% said they would buy another ticket after they had lost the first one, whereas 88% said they would buy a ticket after losing the \$20 bill. In the data I collected (from about 2,400 participants) the gap is significant but is much smaller: 66% and 79% respectively. It is likely that in this case, many of the participants made a calculation in which they compared the "mental price" of a ticket to its subjective value. Many of those who decided not to buy another ticket after losing the first attributed a price of \$40 to the ticket and attributed the price of \$20 to the ticket after they lost the \$20 bill. This example demonstrates that those "mental calculations" may be inconsistent with rationality.

Modeling Choice Procedures

There is a large body of evidence that decision makers systematically use procedures of choice that violate the classical assumptions and that the rational man paradigm is lacking. As a result economists have introduced models in which economic agents are assumed to use alternative procedures of choice. This section demonstrates one line of research that attempts to incorporate such decision makers into economic models.

We wish to enrich the concept of a choice problem to include not only the set of alternatives but also a *default option*, which is irrelevant to the interests for the decision maker though it may nevertheless affect his choice. The statement $c(A, a) = b$ would mean that when facing the choice problem A with a default alternative a the decision maker chooses the alternative b. Experimental evidence and introspection tell us that a default option is often viewed positively by a decision maker, a phenomenon known as the *status quo bias*.

Let X be a finite set of alternatives. Define an *extended choice function* to be a function that assigns a unique element in A to every pair (A, a) where $A \subseteq X$ and $a \in A$.

Following are some examples of extended choice functions that demonstrate the richness of the concept:

1. The decision maker has in mind a partial ordering D where aDb is interpreted as "a clearly dominates b" and an additional ordering $\succsim$ interpreted to be the real preference relation of the decision maker. The alternative $C(A,a)$ is the $\succsim$-best element in the set of alternatives that dominate a, as well as the default alternative (i.e.,$\{x|\ xDa\} \cup \{a\}$).
2. Let d be a distance function on X. The decision maker has in mind a preference relation $\succsim$. The element $C(A,a)$ is the $\succsim$-best alternative that is not too far from a (i.e., it lies within $\{x \mid d(x,a) \le d^*\}$ for some d^*).
3. The decision maker has in mind a preference relation $\succsim$ on the a set of alternatives X, each with an exclusive English name. The element $C(A,a)$ is the first alternative in A after a, which is $\succsim$-better than the default alternative a (and in the absence of such an alternative he sticks with the default).
4. Buridan's donkey: The decision maker has a preference relation in mind. If there is a unique alternative which is better than the default, then it is chosen. If not, then the decision maker stays with the default option (since he cannot make up his mind) (see http://en.wikipedia.org/wiki/Buridan's_ass).
5. Just not the status quo: The decision maker has in mind preferences $\succsim$ on X. Given (A,a) he chooses the $\succsim$ -best alternative in $A-\{a\}$.
6. A default bias: The decision maker is characterized by a utility function u and a "bias function" β, which assigns a non-negative number to each alternative. The function u is interpreted as representing the "true" preferences. The number $\beta(x)$ is interpreted as the bonus attached to x when it is a default alternative. Given an extended choice problem (A,a), the procedure denoted by $DBP_{u,\beta}$, selects:

$$DBP_{u,\beta}(A,a) = \begin{cases} x \in A-\{a\} & if \quad u(x) > u(a)+\beta(a)\ and\ u(x) > u(y) \\ & \qquad for\ any\ y \in A-\{a,x\} \\ a & if \quad u(a)+\beta(a) \ge u(x), \forall x \in A-\{a\} \end{cases}.$$

In the rest of the section we characterize the set of extended choice functions that can be described as $DBP_{u,\beta}$ for some u and β. We will adopt two assumptions:

The Weak Axiom (WA)

We say that an extended choice function c satisfies the Weak Axiom if there are no sets A and B, $a, b \in A \cap B$, $a \neq b$ and $x, y \notin \{a, b\}$ (x and y are not necessarily distinct) such that:

1. $c(A, a) = a$ and $c(B, a) = b$ or
2. $c(A, x) = a$ and $c(B, y) = b$.

The Weak Axiom states that:

1. If a is revealed to be better than b in a choice problem where a is the default, then there cannot be any choice problem in which b is revealed to be better than a when a is the default.
2. If a is revealed to be better than b in a choice problem where neither a nor b is a default, then there cannot be any choice problem in which b is revealed to be better than a when again neither a nor b is the default.

Comment:

The set of all subsets of X that contain a satisfies the union property and thus WA (which implies property α) implies that for every $a \in X$ there is a preference relation $\succsim_a$ such that $c(A, a)$ is the $\succsim_a$-maximal in A. However, $\succsim_a$ might be different from $\succsim_b$.

Default Tendency (DT)

We say that an extended choice function c satisfies Default Tendency if for every set A, if $c(A, x) = a$, then $c(A, a) = a$.

This assumption states that if the decision maker chooses a from a set A when $x \neq a$ is the default, he does not change his mind if x is replaced by a as the default alternative.

Proposition:

An extended choice function c satisfies WA and DT if and only if it is a default-bias procedure.

Proof:

Consider a default-bias procedure c characterized by the functions u and β. It satisfies:

- DT: if $c(A, x) = a$ and $x \neq a$, then $u(a) > u(y)$ for any $y \neq a$ in A. Thus, also $u(a) + \beta(a) > u(y)$ for any $y \neq a$ in A and $c(A, a) = a$.

- WA: for any two sets A, B, $a, b \in A \cap B$, $a \neq b$:
  1. if $c(A, a) = a$ and $c(B, a) = b$, then both $u(a) + \beta(a) > u(b)$ and $u(b) > u(a) + \beta(a)$.
  2. if $c(A, x) = a$ and $c(B, y) = b$ ($x, y \notin \{a, b\}$), then both $u(a) > u(b)$ and $u(b) > u(a)$.

In the other direction, let c be an extended choice function satisfying WA and DT. Define a relation $\succ$ on $X \times \{0, 1\}$ as follows:

- For any pair (A, x) for which $c(A, x) = x$ and for any $y \in A - \{x\}$, define $(x, 1) \succ (y, 0)$.
- For any pair (A, x) for which $c(A, x) = y \neq x$ and for any $z \in A - \{x, y\}$, define $(y, 0) \succ (x, 1)$ and $(y, 0) \succ (z, 0)$.
- For all $x \in X$, $(x, 1) \succ (x, 0)$.

The relation is not complete and might not be transitive, but by WA it is asymmetric. We will see that $\succ$ can be extended to a full ordering over $X \times \{0, 1\}$ denoted by $\succ^*$. Then. let v be a utility function representing $\succ^*$. Define $u(x) = v(x, 0)$ and $\beta(x) = v(x, 1) - v(x, 0)$ to obtain the result.

1. If $c(A, a) = a$, then $(a, 1) \succ (x, 0)$ for all $x \in A - \{a)$ and thus $u(a) + \beta(a) > u(x)$ for all x, that is, $c(A, a) = DBP_{u,\beta}(A, a)$.
2. If $c(A, a) = x$, then $(x, 0) \succ (a, 1)$ and $(x, 0) \succ (y, 0)$ for all $y \in A - \{a, x\}$ and therefore $u(x) > u(a) + \beta(a)$ and $u(x) > u(y)$ for all $y \in A - \{a, x\}$. Thus, $c(A, a) = DBP_{u,\beta}(A, a)$.

Using problem 4 in Problem Set 1, we only need to show that the relation does not have cycles.

First note that:

a. For no x and y is $(x, 0) \succ (y, 0) \succ (x, 1)$ since otherwise there is a set A containing x and y and another alternative $z \in A$ such that $c(A, z) = x$. By DT, also $c(A, x) = x$ and thus $(x, 1) \succ (y, 0)$, contradicting WA.

Assume that $\succ$ has a cycle and consider a shortest cycle. By WA, there is no cycle of length two, and therefore shortest cycle has to be at least of length three. Steps (b) and (c) establish that it is impossible for the shortest cycle to contain a consecutive pair $(x, 0) \succ (y, 0)$.

b. Assume that the cycle contains a consecutive segment $(x, 0) \succ (y, 0) \succ (z, 1)$. There is a set A such that $c(A, z) = y$. By (a), $z \neq x$. By WA $c(A \cup (x\}, z) \in \{x, y\}$ and since $(x, 0) \succ (y, 0)$, $c(A \cup$

$(x\}, z) = x$ and $(x, 0) \succ (z, 1)$. Therefore, we can shorten the cycle.

c. Assume that the cycle contains a consecutive segment of the type $(x, 0) \succ (y, 0) \succ (z, 0)$. By WA, $z \neq x$. Since $(y, 0) \succ (z, 0)$, there exists a set A containing y and z and a different $a \in A$ such that $c(A, a) = y$. By (a), $a \neq x$ and then $c(A \cup \{x\}, a) = x$ and $(x, 0) \succ (z, 0)$, allowing us to shorten the cycle.

The next two steps establish that it is impossible for the shortest cycle to contain a consecutive pair $(x, 0) \succ (y, 1)$.

d. $(x, 0) \succ (y, 1) \succ (z, 0)$ and $y \neq z$. By WA, $z \neq x$. Therefore we get, $c(\{x, y, z\}, y) = x$ and $(x, 0) \succ (z, 0)$, thus allowing us to shorten the cycle.
e. $(x, 0) \succ (y, 1) \succ (y, 0) \succ (z, 1)$. By DT, $z \neq x$ and by definition $z \neq y$. Consider $c\{\{x, y, z\}, z\}$. By WA and $(y, 0) \succ (z, 1)$, it cannot be z. If it is x, then $(x, 0) \succ (y, 0)$ and we can shorten the cycle. If it is y, then $(y, 0) \succ (x, 0)$ and we can again shorten the cycle.

A Comment on the Significance of Axiomatization

There is something aesthetically attractive about axiomatizing choice procedures. However, I disagree with the approach that such an axiomatization is necessary in order to develop a model in which the procedure appears. Most often the axioms make sense only when we have in mind the choice scheme we want to axiomatize.

I take the view that a necessary condition for an axiomatization of a choice scheme to be of interest is the possibility of coming up with examples of sensible procedures of choice that satisfy the axioms and are not specified explicitly in the language of the procedure we are axiomatizing. Can one find such a procedure for the above axiomatization? I myself am unable to. Indeed, many of the axiomatizations in this field lack such examples, and therefore, in spite of their aesthetic value (and although I have done some such axiomatizations myself), I find them to be futile exercises.

Bibliographic Notes

An excellent book on Choice Theory is Kreps (1988). On consistency in choice and revealed preference assumptions, see Samuelson (1948), Houthakker (1950), and Richter (1966). Simon (1955) is the source for the discussion of satisficing. For a discussion of the bounded rationality approach to choice, see Rubinstein (1998). Sen (1993) provides a more philosophical discussion of choice. An excellent introduction to the Dutch Books arguments is Yaari (1985). Kahneman and Tversky (2000) is a definitive textbook on the psychological criticism of the economic approach to rationality. Rabin (1998) surveys the modern Economics and Psychology approach. The DBP procedure was axiomatized by Massatliglu and Ok (2005); the simpler axiomatization presented here is due to Rubinstein and Salant (2006b).

Problem Set 3

Problem 1. (*Easy*)
The following are descriptions of decision-making procedures. Discuss whether the procedures can be described in the framework of the choice model presented in this lecture and whether they are compatible with the rational man paradigm.

a. The decision maker goal is to maximize another person's suffering.
b. The decision maker asks his two children to rank the alternatives (1,2,3...) and then chooses the alternative that gets the lowest average score.
c. The decision maker has an ideal point in mind and chooses the alternative that is closest to it according to some distance function.
d. The decision maker looks for the alternative that appears most often in the choice set.
e. The decision maker has in mind an ordering of the alternatives in X and always chooses the median element.

Problem 2. (*Moderately difficult*)
A choice correspondence C satisfies the *path independence* property if for every set A and a partition of A into A_1 and A_2 ($A_1, A_2 \neq \emptyset$, $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$) we have $C(A) = C(C(A_1) \cup C(A_2))$. (The definition applies also to choice functions.)

a. Show that the rational decision maker satisfies path independence.
b. Find examples of choice procedures that do not satisfy this property.
c. Show that if a choice function satisfies path independence, then it satisfies condition α.
d. Find an example of a choice correspondence satisfying path independence that cannot be rationalized.

Problem 3. (*Easy*)
Let X be a finite set. Check whether the following three choice correspondences, defined over all non-empty subsets of X, satisfy WA:

$C(A) = \{x \in A|$ *the number of* $y \in X$ *for which* $V(x) \geq V(y)$ *is at least* $|X|/2\}$, and if the set is empty, then $C(A) = A$.

$D(A) = \{x \in A|$ *the number of* $y \in A$ *for which* $V(x) \geq V(y)$ *is at least* $|A|/2\}$ $E(A) = \{x \in A | x \succsim_1 y$ *for every* $y \in A$ *or* $x \succsim_2 y$ *for every* $y \in A\}$ where $\succsim_1$ and $\succsim_2$ are two strict preferences over X.

Problem 4. (*Moderately difficult*)
Consider the following choice procedure: A decision maker has a strict ordering $\succsim$ over the set X and assigns to each $x \in X$ a natural number $class(x)$ to be interpreted as the "class" of x. Given a choice problem A, he chooses the best element in A from those belonging to the most common class in A (i.e., the class that appears in A most often). If there is more than one most common class, he picks the best element from the members of A that belong to a most common class with the highest class number.

a. Define the relation: xPy if x is chosen from $\{x, y\}$. Show that the relation P is a strict ordering (complete, asymmetric, and transitive).
b. Is the procedure consistent with the rational man paradigm?

Problem 5. (*Moderately difficult. Based on Kalai, Rubinstein, and Spiegler (2002).*)
Consider the following two choice procedures: Interpret them so that they will "make sense". Determine whether any are consistent with the rational man paradigm.

a. The primitives of the procedure are two numerical (one-to-one) functions u and v defined on X and a number v^*. For any given choice problem A, let $a^* \in A$ be the maximizer of u over A and let b^* be the maximizer of v over A. The decision maker chooses a^* if $v(a^*) \geq v^*$ and b^* if $v(a^*) < v^*$.
b. The primitives of the procedure are two numerical (one-to-one) functions u and v defined on X and a number u^*. For any given choice problem A, the decision maker chooses the element $a^* \in A$ that maximizes u if $u(a^*) \geq u^*$, and the element $b^* \in A$ that maximizes v if $u(a^*) < u^*$.

Problem 6. (*Moderately difficult. Based on Rubinstein and Salant (2006a)*)
The standard economic choice model assumes that a choice is made from a *set*. Here we assume that the choice is made from a *list*. (Note that the list $< a, b >$ is distinct from $\langle a, a, b\rangle$ and $\langle b, a\rangle$.)

Let X be a finite grand set. A *list* is a nonempty finite vector of elements in X. A *choice function* C is a function that assigns a single element from $\{a_1, \ldots, a_K\}$ to each vector $L = \langle a_1, \ldots, a_K\rangle$. Denote by $\langle L_1, \ldots, L_m\rangle$ the concatenation of the m lists $L_1, \ldots, L_m$ (if the length of L_i is k_i, the length of the concatenation is $\Sigma_{i=1,\ldots,m} k_i$). We say that L' *extends* the list L if there is a list M such that $L' = \langle L, M\rangle$.

A choice function C satisfies Property I if for all $L_1, \ldots, L_m$, we have $C(\langle L_1, \ldots, L_m\rangle) = C(\langle C(L_1), \ldots, C(L_m)\rangle)$.

a. Interpret Property I. Give two examples of choice functions that satisfy I and two examples that do not.
b. Define formally the following two properties of a choice function:
Order Invariance: A change in the order of the elements in the list does not alter the choice.
Duplication Invariance: Deleting an element that appears elsewhere in the list does not change the choice.
Show that Duplication Invariance implies Order Invariance.
c. Characterize the choice functions that satisfy Duplication Invariance and property I.

Assume now that at the back of the decision maker's mind there is a value function u defined on the set X (such that $u(x) \neq u(y)$ for all $x \neq y$). For any choice function C, define $v_C(L) = u(C(L))$.

We say that C *accommodates a longer list* if, whenever L' extends L, $v_C(L') \geq v_C(L)$ and there is a pair of lists L' and L such that L' extends L and $v_C(L') > v_C(L)$.

d. Give two interesting examples of choice functions that accommodate a longer list.
e. Give two interesting examples of choice functions that satisfy property I but do not accommodate a longer list.

Problem 7. (*Difficult. Based on Rubinstein and Salant (2006)*)
Let X be a finite set. We say that a choice function c is *lexicographically rational* if there exists a profile of preference relations $\{\succ_a\}_{a\in X}$ (not necessarily distinct) and an ordering O over X such that for every set $A \subset X$, $c(A)$ is the $\succ_a$-maximal element in A, where a is the O-maximal element in A.

A decision maker who follows this procedure is attracted by the most notable element in the set (as described by O). If a is that element, he applies the ordering $\succ_a$ and chooses the $\succ_a$-best element in the set.

We say that c satisfies the *reference point property* if, for every set A, there exists $a \in A$ such that if $a \in A'' \subset A' \subset A$ and $c(A') \in A''$, then $c(A'') = c(A')$.

a. Show that a choice function c is lexicographically rational if and only if it satisfies the *reference point property*.
b. Find an example of a choice function satisfying the reference point axiom that is not stated explicitly in the language of the lexicographically rational choice function.

Problem 8. (*Difficult. Based on Cherepanov, Fedderson, and Sandroni (2008).*)
Consider a decision maker who has in mind a set of rationales and an asymmetric complete relation over a finite set X. Given $A \subset X$, he chooses the best alternative in that he can rationalize.

Formally, we say that a choice function c is *rationalized* if there is an asymmetric complete relation $\succ$ (*not* necessarily transitive!) and a set of partial orderings (asymmetric and transitive) $\{\succ_k\}_{k=1\ldots K}$ (called rationales) such that $c(A)$ is the $\succ$ -maximal alternative from among those alternatives found to be maximal in A by at least one rationale (given a binary relation $\succ$ we say that x is $\succ$ -maximal in A if $x \succ y$ for all $y \in A$). Assume that the relations are such that the procedure always leads to a solution.

We say that a choice function c satisfies The *Weak Weak Axiom of Revealed Preference (WWARP)* if for all $\{x, y\} \subset B_1 \subset B_2$ $(x \neq y)$ and $c\{x, y\} = c(B_2) = x$, then $c(B_1) \neq y$.

a. Show that a choice function satisfies WWARP if and only if it is rationalized. For the proof, construct one rationale for each choice problem.

Consider the "warm glow" procedure: The decision maker has two orderings in mind: one moral $\succsim_M$ and one selfish $\succsim_S$. He chooses the most moral alternative m as long as he doesn't "lose" too much by not choosing the most selfish alternative. Formally, for every alternative s there is some alternative $l(s)$ such that if the most selfish alternative is s, then he is willing to choose m as long as $m \succsim_S l(s)$. If $l(s) \succ_S m$, he chooses s.

The function l satisfies $(i) s \succsim_S l(s)$ and $(ii) s \succsim_S s'$ implies $l(s) \succsim_S l(s')$.

b. Show that WWARP is satisfied by this procedure.
c. Show directly that the "warm glow" procedure is rationalized (in the sense of the definition presented in this problem).

Problem 9. (*Moderately difficult. Based on Luce (1959)*)
A decision maker's behavior is probabilistic. Define a choice function on a set X to be a function that assigns a positive number $C(a|A)$ to every $a \in A$ and every $A \subseteq X$ such that $\sum_{a \in A} C(a|A) = 1$. For any $B \subset A$ define $C(B|A) = \sum_{a \in B} C(a|A)$. We say that the function satisfies *path independence* (PI) if for any $a \in B \subset A$, $C(a|A) = C(a|B)C(B|A)$.

a. Explain the PI property. Give two examples of reasonable probabilistic choice functions that do not satisfy PI.
b. Prove that if the probabilistic choice function satisfies PI then there is a function $v : X \to R_+$ such that $C(a|A) = \frac{v(a)}{\sum_{x \in A} v(x)}$.

LECTURE 4

Consumer Preferences

The Consumer's World

Up to this point, we have dealt with the basic concepts of preferences and choice in the most abstract setting. In this lecture, we will discuss a special case of an economic agent: the *consumer*. We have in mind an economic agent who makes choices between available combinations of commodities. As usual, I have a certain image in mind: my late mother going to the marketplace with money in hand and coming back with a bundle of fruit and vegetables .

As before, we will begin with a discussion of consumer preferences and utility and only then discuss consumer choice. The first step is to move from an abstract treatment of the set X to a more detailed structure. We take X to be $\mathbb{R}_+^K = \{x = (x_1, \ldots, x_K) |$ for all k, $x_k \geq 0\}$. An element of X is called a *bundle*. A bundle x is interpreted as a combination of K commodities where x_k is the quantity of commodity k.

Given this special interpretation of X, we impose some conditions on the preferences in addition to those assumed for preferences in general. The additional three conditions are based on the mathematical structure of the space X: monotonicity is based on the orderings on the axis (the ability to compare bundles according to the the amount of any particular commodity); continuity is based on the topological structure (the ability to talk about closeness); convexity is based on the algebraic structure (the ability to speak of the sum of two bundles and the multiplication of a bundle by a scalar). Often we demonstrate properties of the consumer's preferences by referring to the map of indifference curves, where an indifference curve is a set of the type $\{y | y \sim x\}$ for some bundle x (see problem 1 in Problem Set 1).

Monotonicity

Monotonicity is a property that gives commodities the meaning of "goods". It is the condition that more is better. Increasing the amount of some commodities cannot hurt, and increasing the amount of all commodities is strictly desired. Formally:

The relation $\succsim$ satisfies *monotonicity* if for all $x, y \in X$,

if $x_k \geq y_k$ for all k, then $x \succsim y$, and
if $x_k > y_k$ for all k, then $x \succ y$.

In some cases, we will further assume that the consumer is strictly happier with any additional quantity of any commodity.

The relation $\succsim$ satisfies *strong monotonicity* if for all $x, y \in X$,

if $x_k \geq y_k$ for all k and $x \neq y$, then $x \succ y$.

In the case that preferences are represented by a utility function, preferences satisfying monotonicity (or strong monotonicity) can be represented by an increasing (or strongly increasing) utility function.

Examples:

- The preferences represented by $min\{x_1, x_2\}$ satisfy *monotonicity* but not *strong monotonicity*.
- The preferences represented by $x_1 + x_2$ satisfy *strong monotonicity*.
- A preference relation is said to be *nonsatiated at the bundle y* if for any $\varepsilon > 0$ there is some $x \in X$ that is less than the distance ε away from y, so that $x \succ y$. Every preference relation that is monotonic at y is also nonsatiated at y, but the reverse is, of course, not true. The preferences represented by $u(x) = -\sqrt{\sum(x_k - x_k^*)^2}$ do not satisfy monotonicity but are nonsatiated at every point besides x^*.

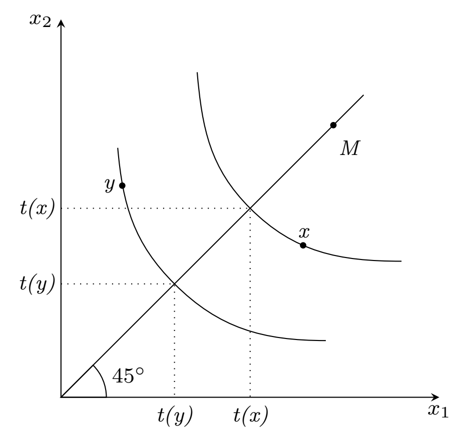

Figure 4.1
Construction of the utility function.

Continuity

We will use the topological structure of $\mathbb{R}_+^K$ (with the standard distance function d) in order to apply the definition of continuity discussed in Lecture 2. We say that the preferences $\succsim$ satisfy *continuity* if for all $a, b \in X$, $a \succ b$ implies that there is an $\varepsilon > 0$ such that $x \succ y$ for any x and y such that $d(x, a) < \varepsilon$ and $d(y, b) < \varepsilon$.

Existence of a Utility Representation

Debreu's theorem guarantees that any continuous preference relation is represented by some (continuous) utility function. If we assume monotonicity as well, we then have a simple and elegant proof:

Claim:

Any consumer preference relation satisfying monotonicity and continuity can be represented by a continuous utility function.

Proof:

Let us first show that for every bundle x, there is a bundle on the main diagonal (having equal quantities of all commodities), such that the consumer is indifferent between that bundle and the bundle x (see fig. 4.1). The bundle x is at least as good as the bundle $0 = (0, \ldots, 0)$ and the bundle $M = (max_k\{x_k\}, \ldots, max_k\{x_k\})$ is at least as good as x. Both 0 and M are on the main diagonal. By continuity, there is a bundle on the main diagonal that is indifferent to x (see Problem Set 2, problem 3). By monotonicity, this bundle is unique; denote it by $(t(x),\ldots,t(x))$. Let $u(x)=t(x)$. To verify that the function u represents the preferences, note that, by transitivity of the preferences, $x \succsim y$ iff $(t(x),\ldots,t(x)) \succsim (t(y),\ldots,t(y))$, and by *monotonicity* this is true iff $t(x) \geq t(y)$.

To see that u is continuous, let (x^n) be a sequence converging to x and assume by contradiction that $t(x^n) \nrightarrow t(x)$. Since $x^n \to x$ there is a number M^* such that $(M^*,..,M^*) \succ x^n$ for all n. All members of the sequence $(t(x^n))$ are in the compact interval $[0,M^*]$. As $t(x^n) \nrightarrow t(x)$ there is an $\varepsilon > 0$ such that for every N there is $n > N$ such that $t(x^n)$ is outside the interval $[t(x)-\varepsilon, t(x)+\varepsilon]$. Thus, there is an infinite subsequence of (x^n) with all $t-$values in $[t(x)+\varepsilon, M^*]$ (or in $[0, t(x)-\varepsilon]$)). This subsequence has an infinite subsequence (y^n) such that $(t(y^n))$ converges to some $t' > t(x)$. Now, $y^n \to x$ (since (y^n) is a subsequence of (x^n) that converges to x) and $(t(y^n),..,t(y^n)) \to (t',....,t')$. By the continuity of the preferences, $x \sim (t',..,t')$ but $x \sim (t(x),..,t(x))$ and $t' > t(x)$, a contradiction.

Convexity

Consider a scenario in which the alternatives are candidates running for office. The candidates are positioned in a left to right array as follows:

——a—b——c——d——e—.

Under normal circumstances, if we know that a voter prefers b to d, then we tend to conclude that:

- he prefers c to d, but not necessarily a to d (the candidate a may be too extreme).
- he prefers d to e (namely, we do not find it plausible that he views moving both to the right and left from d as improvement).

The notion of convex preferences captures two similar intuitions that are suited to situations where there exists a "geography" of the set of alternatives, in the sense that we can talk about one alternative being between two others:

- If x is preferred to y, then going part of the way from y to x is also an improvement over y.
- If z is between x and y, then it is impossible that both x and y are better than z.

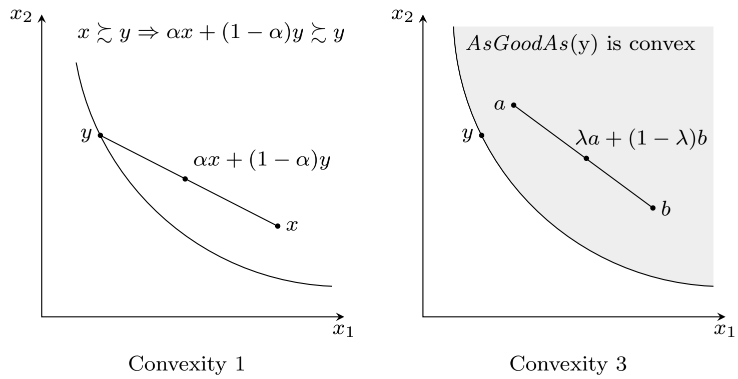

Figure 4.2
Definitions of convexity: Convexity 1 on the left, Convexity 3 on the right.

Convexity is appropriate for a situation in which the argument "if a move is an improvement, then so is any part of the move" is legitimate, whereas the argument "if a move is harmful, then so is any part of that move" is not.

The two intuitions can be stated formally:

Convexity 1:
The preference relation $\succsim$ satisfies *convexity 1* if $x \succsim y$ and $\alpha \in (0,1)$ implies that $\alpha x + (1-\alpha)y \succsim y$ (fig. 4.2).

Convexity 2:
The preference relation $\succsim$ satisfies *convexity 2* if for all x, y, and z such that $z = \alpha x + (1-\alpha)y$ for some $\alpha \in (0,1)$, $z \succsim x$ or $z \succsim y$.

The third definition of convexity, utilizes the notion of a convex set. (Recall that a set A is convex if $v\lambda a + (1-\lambda)b \in A$ for all $a, b \in A$ and all $\lambda \in [0,1]$).

Convexity 3:
The preference relation $\succsim$ satisfies *convexity 3* if for all y the set $AsGoodAs(y) = \{z \in X | z \succsim y\}$ is convex (fig. 4.2).

This definition captures the intuition that if both z_1 and z_2 are better than y, then the average of z_1 and z_2 is also preferred to y.

We proceed to show that the three definitions are equivalent.

Claim:

If the preference relation $\succsim$ satisfies one of the conditions convexity 1, convexity 2, and convexity 3, then it satisfies the other two as well.

Proof:

Assume that $\succsim$ satisfies convexity 1 and let $x, y, z \in X$ such that $z = \alpha x + (1-\alpha)y$ for some $\alpha \in (0,1)$. Without loss of generality, assume that $x \succsim y$. By convexity 1, we have $z \succsim y$. Thus, $\succsim$ satisfies convexity 2.

Assume that $\succsim$ satisfies convexity 2 and let $z, z' \in \mathit{AsGoodAs}(y)$. Then, by convexity 2, $\alpha z + (1-\alpha)z'$ is at least as good as either z or z' (or both). In any case, by transitivity, $\alpha z + (1-\alpha)z' \succsim y$, that is, $\alpha z + (1-\alpha)z' \in \mathit{AsGoodAs}(y)$, and thus $\succsim$ satisfies convexity 3.

Assume that $\succsim$ satisfies convexity 3. If $x \succsim y$, then both x and y are in $\mathit{AsGoodAs}(y)$ and thus $\alpha x + (1-\alpha)y \in \mathit{AsGoodAs}(y)$, which means that $\alpha x + (1-\alpha)y \succsim y$. Thus, $\succsim$ satisfies convexity 1.

Convexity also has a stronger version:

Strict Convexity

The preference relation $\succsim$ satisfies *strict convexity* if $a \succsim y$, $b \succsim y$, $a \neq b$, and $\lambda \in (0,1)$ imply that $\lambda a + (1-\lambda)b \succ y$.

Examples

The preferences represented by $\sqrt{x_1} + \sqrt{x_2}$ satisfy strict convexity. The preference relations represented by $min\{x_1, x_2\}$ and $x_1 + x_2$ satisfy convexity but not strict convexity. The lexicographic preferences satisfy strict convexity. The preferences represented by $x_1^2 + x_2^2$ do not satisfy convexity.

We now look at the properties of the utility representations of convex preferences. First, a reminder:

Concavity and Quasi-Concavity

Recall that a function u is *concave* if for all x, y, and $\lambda \in [0,1]$ we have $u(\lambda x + (1-\lambda)y) \geq \lambda u(x) + (1-\lambda)u(y)$ and is *quasi-concave* if for all y

the set $\{x|\, u(x) \geq u(y)\}$ is convex. Any function that is concave, is also quasi-concave.

If a preference relation is represented by a utility function, then it is convex iff the utility function is quasi-concave. However, the convexity of $\succsim$ does not imply that a utility function representing $\succsim$ is concave. Furthermore, continuous and convex preferences might not have a utility representation by any concave function. For example, consider the relation on the set $\mathbb{R}$ defined by $x \succsim y$ if $x \geq y$ or $y < 0$.

Special Classes of Preferences

We usually discuss a consumer while assuming some versions of monotonicity, continuity, and convexity. We will refer to such a consumer as a "classical consumer". However, often we also assume that the consumer's preferences satisfy additional properties.

The reader might have thought that the reason for abandoning the "generality" of the classical consumer is because empirically we observe only certain kinds of consumers who are described by special classes of preferences. Rather, stronger assumptions are needed in economic models in order to develop interesting models, just as an engaging story of fiction cannot be based on a hero about which the reader knows very little.

Following are some examples of "popular" classes of preference relations discussed in the literature:

Homothetic Preferences

A preference $\succsim$ is *homothetic* if $x \succsim y$ implies that $\alpha x \succsim \alpha y$ for all $\alpha \geq 0$ (see fig. 4.3).

Preferences represented by $\Pi_{k=1,\ldots,K} x_k^{\beta_k}$, where β_k is positive, are homothetic. More generally, any preference relation represented by a utility function u that is homogeneous of any degree λ (that is $u(\alpha x) = \alpha^\lambda u(x)$) is homothetic. This is because $x \succsim y$ iff $u(x) \geq u(y)$ iff $\alpha^\lambda u(x) \geq \alpha^\lambda u(y)$ iff $u(\alpha x) \geq u(\alpha y)$ iff $\alpha x \succsim \alpha y$. Lexicographic preferences are also homothetic.

Claim:

Any homothetic, continuous, and monotonic preference relation on the commodity bundle space can be represented by a continuous utility function that is homogeneous of degree one.

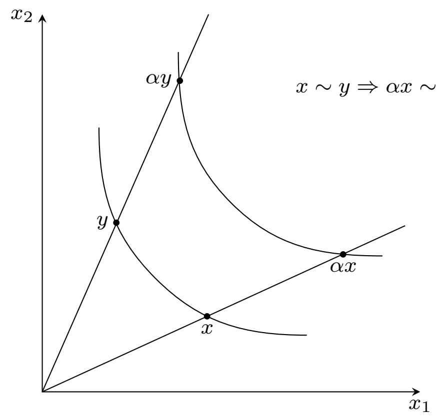

Figure 4.3
Homothetic preferences.

Proof:

We have already proved that for any bundle x there exists a unique bundle $(t(x), \ldots, t(x))$ on the main diagonal, so that $x \sim (t(x), \ldots, t(x))$, and that the function $u(x) = t(x)$ is a continuous utility representation of $\succsim$. By the assumption that the preferences are homothetic, $\alpha x \sim (\alpha t(x), \ldots, \alpha t(x))$ and thus $u(\alpha x) = \alpha t(x) = \alpha u(x)$.

Quasi-Linear Preferences

A preference is *quasi-linear* in commodity 1 (referred to as the "numeraire") if $x \succsim y$ implies $(x + \varepsilon e_1) \succsim (y + \varepsilon e_1)$ (where $e_1 = (1, 0, \ldots, 0)$ and ε is a real number (see fig. 4.4).

The indifference curves of preferences that are quasi-linear in commodity 1 are parallel to each other (relative to the first commodity axis). That is, if I is an indifference curve, then the set $I_\varepsilon = \{x |$ there exists $y \in I$ such that $x = y + (\varepsilon, 0, \ldots, 0)\}$ is an indifference curve as well.

An agent who forms his preferences by asking himself what the value of the combination of commodities $(x_2, \ldots, x_K)$ is (in terms of the first commodity) and adds that to the quantity of the first commodity is represented by the utility function $x_1 + v(x_2, \ldots, x_K)$. Such preferences are quasi-linear in commodity 1. Furthermore:

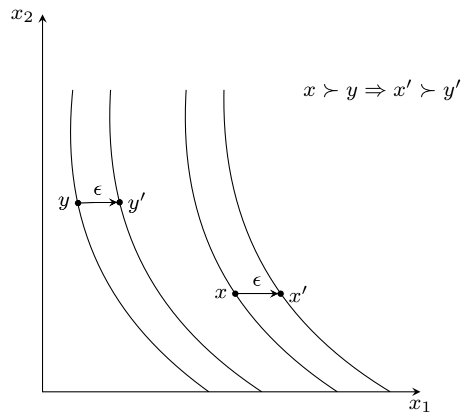

Figure 4.4
Quasi-linear (in good 1) preferences.

Claim:

Any monotonic and continuous preference relation that also satisfies strong monotonicity and quasi-linearity in commodity 1 can be represented by a utility function of the form $x_1 + v(x_2, \ldots, x_K)$.

To prove this, we need the following lemma:

Lemma:

Let $\succsim$ be a preference relation that is monotonic, continuous, quasi-linear, and strongly monotonic in commodity 1. Then, for every $(x_2, \ldots, x_K)$ there is a number $v(x_2, \ldots, x_K)$ such that $(0, x_2, \ldots, x_K) \sim (v(x_2, \ldots, x_K), 0, \ldots, 0)$.

Proof of the Lemma

The proof is given for $K = 2$. The general proof is left to the problem set.

Let $T = \{t \mid (0, t) \succ (x_1, 0) \text{ for all } x_1\}$. Assume $T \neq \emptyset$ and denote $m = \inf T$. We distinguish between two cases:

(i) $m \in T$. Then $m > 0$ and $(1, m) \succ (0, m)$. By continuity, there is an $\epsilon > 0$ such that $(1, m - \epsilon) \succ (0, m)$, and thus $(1, m - \epsilon) \succ (x_1 + 1, 0)$ for all x_1. Since $m = \inf T$, then there exists an x_1^* such that $(x_1^*, 0) \succsim (0, m - \epsilon)$, and by the quasi-linearity in commodity 1, $(x_1^* + 1, 0) \succsim (1, m - \epsilon)$, a contradiction.

(ii) $m \notin T$. Then $(x_1^*, 0) \succsim (0, m)$ for some x_1^*. By the strong monotonicity of commodity 1, $(x_1^* + 1, 0) \succ (0, m)$. By continuity, there is an $\epsilon > 0$ such that $(x_1^* + 1, 0) \succ (0, x_2)$, for any $m + \epsilon \geq x_2 > m$, contradicting $m = \inf T$.

Consequently, $T = \emptyset$, and for every x_2 there is an x_1 such that $(x_1, 0) \succsim (0, x_2) \succsim (0, 0)$, and thus by continuity $(v(x_2), 0) \sim (0, x_2)$ for some number $v(x_2)$.

Note that the Lemma doesn't hold without the quasi-linearity assumption. For example, the utility function $u(x_1, x_2) = x_2 - 1/(x_1 + 1)$ represents strongly monotonic and continuous preferences for which $m = 1$.

Proof of the Claim

By the lemma, for every $(x_2, \ldots, x_K)$ there is a number $v(x_2, \ldots, x_K)$ so that $(v(x_2, \ldots, x_K), 0, \ldots, 0) \sim (0, x_2, \ldots, x_K)$. By the quasi-linearity in commodity 1, $(x_1 + v(x_2, \ldots, x_K), 0, \ldots, 0) \sim (x_1, x_2, \ldots, x_K)$, and thus by strong monotonicity in the first commodity, the function $x_1 + v(x_2, \ldots, x_K)$ represents $\succsim$.

Thus, we used quasi-linearity for two purposes: First, we showed that for every bundle x there is a quantity of the first good $u(x)$ such that $x \sim (u(x), 0, \ldots, 0)$. By the strong monotonicity in the first commodity this allows us to view $u(x)$ as a utility function representing the consumer's preferences. Second, quasi-linearity is used to show that the function u has the structure $x_1 + v(x_2, \ldots, x_K)$.

The above claim shows that any continuous preference relation that is quasi-linear in the first commodity is consistent with the following procedure: the consumer considers the value (in terms of the first commodity) of a combination of goods $2 \ldots k$, which is independent of the quantity of the first commodity.

Quasi-concavity in all commodities is one of the ways to derive linearity:

Claim:

Any continuous preference relation $\succsim$ on $\mathbb{R}_+^K$ satisfying strong monotonicity and quasi-linearity in all commodities can be represented by a utility function of the form $\sum_{k=1}^{K} \alpha_k x_k$.

Following are two proofs for the case of $K = 2$. The general proof for any K is left for the problem set.

Proof 1:

Using the previous claim, the preference relation over the bundle space can be represented by the function $u(x_1, x_2) = x_1 + v(x_2)$ where $(0, x_2) \sim (v(x_2), 0)$. Let $(0, 1) \sim (c, 0)$.

It is sufficient to show that $v(x_2) = cx_2$.

Assume that for some x_2 we have $v(x_2) > cx_2$ (a similar argument applies in the case of $v(x_2) < cx_2$). Choose two integers S and T such that $v(x_2)/c > S/T > x_2$.

Note that if $(a, 0) \sim (0, b)$, then all points (ka, lb) for which $k + l = n$ (k, l and n are non-negative integers) reside on the same indifference curve. The proof is by induction on n. By definition, it is true for $n = 1$. The inductive assumption is that $((n-1)a, 0) \sim ((n-2)a, b) \sim \ldots \sim (a, (n-2)b) \sim (0, (n-1)b)$. By the quasi-linearity in commodity 1, $(na, 0) \sim ((n-1)a, b) \sim \ldots \sim (a, (n-1)b)$ and by the quasi-linearity in commodity 2 $(a, (n-1)b) \sim (0, nb)$ as well.

Thus, $(0, Tx_2) \sim (Tv(x_2), 0)$ and $(0, S) \sim (Sc, 0)$. However, since $S > Tx_2$, we have $(0, Tx_2) \prec (0, S)$, and since $Tv(x_2) > Sc$, we have $(Tv(x_2), 0) \succ (Sc, 0)$, a contradiction.

Proof 2:

We will now show that $v(a + b) = v(a) + v(b)$ for all a and b. By definition of v, $(0, a) \sim (v(a), 0)$ and $(0, b) \sim (v(b), 0)$. By the quasi-linearity in good 1, $(v(b), a) \sim (v(a) + v(b), 0)$ and by the quasi-linearity in good 2, $(0, a + b) \sim (v(b), a)$. Thus, $(0, a + b) \sim (v(a) + v(b), 0)$ and $v(a + b) = v(a) + v(b)$.

Let $v(1) = c$. Then, for any natural numbers m and n we have $v(m/n) = cm/n$. Since $v(0) = 0$ and v is an increasing function, it must be that $v(x) = cx$ for all x.

The equation $v(a + b) = v(a) + v(b)$ is called Cauchy's functional equation, and without further assumptions, such as monotonicity, there are nonlinear functions that satisfy it.

Differentiable Preferences

It is often assumed that utility functions are differentiable, which allows us to use standard calculus to analyze consumer behavior. In this course, I rarely use calculus with the goal of steering you away from a "mechanistic" approach to economic theory and toward thinking about decision makers' deliberation procedures. This section presents an alternative approach to differentiability.

Procedure of evaluating a move from x in the direction d: Imagine that a decision maker examines whether to move (at least a little bit) from x in the direction d. He does it having in mind a vector of subjective values of the K commodities $v(x) \in \mathbb{R}_+^K$. This vector forms the basis for calculating whether a move from x in the direction of d is an improvement or not. (When refereing to a direction of move d from x it is always assumed that the move is feasible in the sense that for $\varepsilon > 0$ small enough $x + \varepsilon d \in X$). The direction d is determined by the decision maker to be an improvement if and only if $dv(x) > 0$, that is if and only if the value of the change, as evaluated by the values at x, is positive. Given the function $v : X \to \mathbb{R}_+^K$, define $D_v(x) = \{d \mid d \cdot v(x) > 0\}$, to be the set of directions that he finds to be improvements relative to x using the above evaluation.

Improving directions of a preference relation $\succsim$ at x: We say that the vector $z \in \mathbb{R}^K$ is an *improvement* at the bundle x if $x + z \succ x$. We say that $d \in \mathbb{R}^K$ is an *improvement direction* at x if any small move from x in the direction of d is an improvement, that is, there is some $\lambda^* > 0$ such that for all $\lambda^* > \lambda > 0$ the vector λd is an improvement. Let $D_{\succsim}(x)$ be the set of all improvement directions at x.

Note that:

1. If $d \in D_{\succsim}(x)$, then $\lambda d \in D_{\succsim}(x)$ for any $\lambda > 0$.
2. If the preferences are strictly convex, then any improvement is also an improvement direction.
3. If the preferences satisfy strong monotonicity, continuity, and convexity, then any improvement is also an improvement direction. To see this, assume $x + d \succ x$. Take $\lambda^* = 1$. For any $1 > \lambda > 0$ we will now show that $x + \lambda d = \lambda(x + d) + (1 - \lambda)x \succ x$. By continuity, there is a vector $z \succ x$ with $z_k \leq (x + d)_k$ for all k and with strict inequality for every k for which $(x + d)_k > 0$. For all k, we have $(x + \lambda d)_k \geq (\lambda z + (1 - \lambda)x)_k$ and $x + \lambda d \neq \lambda z + (1 - \lambda)x$. By strong monotonicity, $x + \lambda d \succ \lambda z + (1 - \lambda)x$. Finally, by convexity, $\lambda z + (1 - \lambda)x \succsim x$. Thus, $x + \lambda d \succ x$.
4. Monotonicity implies that $d \in D_{\succsim}(x)$ for any d such that $d_k > 0$ for all k.

Differentiable Preferences: We are interested in preference relations that are consistent with the above procedure. The requirement of con-

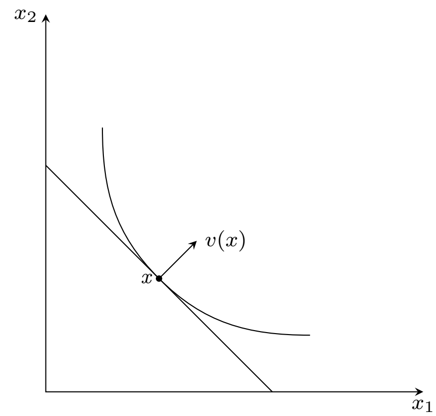

Figure 4.5
Differentiability at the bundle x.

sistency is expressed by the following (nonconventional) definition of *differentiable preferences* where the preferences are confined to those that satisfy monotonicity and convexity: We say that the preference relation $\succsim$ is *differentiable* if there is a function $v : X \to \mathbb{R}^K_+$, so that $D_{\succsim}(x) = D_v(x)$ for all x.

- The preferences represented by $2x_1 + 3x_2$ are differentiable. At each point x, $v(x) = (2, 3)$.
- The preferences represented by $\min\{x_1, \ldots, x_K\}$ are differentiable only at points where there is a unique commodity k for which $x_k < x_l$ for all $l \neq k$ (verify). For example, at $x = (5, 3, 8, 6), v(x) = (0, 1, 0, 0)$.

Differentiability of Preferences and the Differentiability of utility functions representing them: Most examples of utility functions that are used in the economic literature are differentiable, monotonic and quasi-concave. We will verify now that if preferences are represented by such a utility function the preferences are differentiable.

Claim:

Any preference relation $\succsim$, which is represented by a differentiable, strong monotonic and quasi-concave utility functions u, is differentiable.

Proof:

Given the differentiable utility function u, let $du/dx_k(x)$ be the partial derivative of u with respect to the commodity k at point x. Let $\nabla u(x)$ (the gradient) be the vector of these partial derivatives. Recall that the meaning of differentiability of u at a point x is that the rate of change of u when moving from x in any direction d is $d \cdot \nabla u(x)$. That is, $\lim_{\epsilon \to 0} \frac{u(x+\epsilon d)-u(x)}{\epsilon} = d \cdot \nabla u(x)$.

Now define the function $v(x) = \nabla u(x)$. We will show that $D_{\succsim}(x) = D_v(x)$ for all x.

$D_{\succsim}(x) \subseteq D_v(x)$: By contradiction, assume that there is a vector $d \in D_{\succsim}(x)$ such that $d \cdot v(x) \leq 0$. Without loss of generality, let $x + d \succ x$, since otherwise d can be rescaled. By continuity, there is $d' \neq d$, $d'_k \leq d_k$ for all k, such that $x + d' \succ x$. By convexity and strong monotonicity of the preferences (which follows from the quasi-concavity and strong monotonicity of u), $d' \in D(x)$. However, $d' \cdot v(x) < 0$ and thus by the differentiability of u, for any small enough δ, $u(x + \delta d') < u(x)$, a contradiction.

$D_{\succsim}(x) \supseteq D_v(x)$: This follows immediately from the differentiability of u since $dv(x) > 0$ implies $u(x + \epsilon d) > u(x)$ for any small enough ϵ. That is, $d \in D_{\succsim}(x)$.

Bibliographic Notes

The material in this lecture up to the discussion of differentiability is fairly standard and closely parallels that found in Arrow and Hahn (1971).

Problem Set 4

Problem 1. (*Easy*)
Show that a continuous preference relation $\succsim$ on $X = [0, 1]$ is strictly convex iff there exists a point x^* such that $b \succ a$ for all $a < b \leq x^*$ and for all $x^* \leq b < a$.

Problem 2. (*Difficult*)
For the case of $K = 2$, characterize the preference relations that are strictly monotonic, quasi-linear in the two commodities and homothetic (but not necessarily continuous).

Problem 3. (*Standard*)
In a world with two commodities, consider the following condition:

The preference relation $\succsim$ satisfies *convexity 4* if, for all x and $\varepsilon > 0$,

$(x_1, x_2) \sim (x_1 - \varepsilon, x_2 + \delta_1) \sim (x_1 - 2\varepsilon, x_2 + \delta_1 + \delta_2)$ implies $\delta_2 \geq \delta_1$.

Interpret convexity 4 and show that for strong monotonic and continuous preferences, it is equivalent to the convexity of the preference relation.

Problem 4. (*Standard*)
Complete the proof (for all K) of the claim that any continuous preference relation satisfying strong monotonicity and quasi-linearity in all commodities can be represented by a utility function of the form $\sum_{k=1}^{K} \alpha_k x_k$ where $\alpha_k > 0$ for all k.

Problem 5. (*Difficult*)
Complete the proof (for all K) that for any consumer preference relation $\succsim$ satisfying continuity, monotonicity, strong monotonicity with respect to commodity 1, and quasi-linearity with respect to commodity 1, there exists a number $v(x)$ such that $x \sim (v(x), 0, \ldots, 0)$ for every vector x.

Problem 6. (*Easy*)
We say that a preference relation satisfies *separability* if it can be represented by an additive utility function, that is, a function of the type $u(x) = \Sigma_k v_k(x_k)$.

a. Show that such preferences satisfy condition S: for any subset of commodities J and for any bundles a, b, c, d, we have:

$$(a_J, c_{-J}) \succsim (b_J, c_{-J}) \Leftrightarrow (a_J, d_{-J}) \succsim (b_J, d_{-J}),$$

where (x_J, y_{-J}) is the vector that takes the components of x for any $k \in J$ and takes the components of y for any $k \notin J$.

b. Show that for $K = 2$ such preferences satisfy the "Hexagon-condition": If $(a, b) \succsim (c, d)$ and $(c, e) \succsim (f, b)$, then $(a, e) \succsim (f, d)$.

c. Give an example of a continuous preference relation that satisfies condition S but does not satisfy separability.

Problem 7. (*Difficult*)

a. Check the differentiability of the lexicographic preferences in $\mathbb{R}^2$.

b. Assume that $\succsim$ is monotonic, convex, and differentiable, and in addition for every x the set $D(x)$ (the set of improving directions) is equal to the set $\{d|(x + d) \succ x\}$ (the set of improving changes). What can you say about $\succsim$?

c. Assume that $\succsim$ is a monotonic, convex, and differentiable preference relation. Let $E(x) = \{d \in \mathbb{R}^K|$ there exists $\varepsilon^* > 0$ such that $x + \varepsilon d \prec x$ for all $0 < \varepsilon \leq \varepsilon^*\}$. Show that $\{-d|\ d \in D(x)\} \subseteq E(x)$ but not necessarily $\{-d|\ d \in D(x)\} = E(x)$.

d. Consider the consumer's preferences defined by $(K = 2)$:

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 & \textit{if} \quad x_1 + x_2 \leq 1 \\ 1 + 2x_1 + x_2 & \textit{if} \quad x_1 + x_2 > 1 \end{cases}.$$

Show that these preferences are not continuous but nevertheless are differentiable according to our definition.

LECTURE 5

Demand: Consumer Choice

The consumer's choice function

We continue to have $X = \mathbb{R}^K_+$, the set of bundles with the commodities indexed $1, 2, \ldots, K$. A choice problem is now a subset of bundles. Since we are laying the foundation for "price models", we are interested in the consumer's choice in a particular class of choice problems called budget sets. A *budget set* is a set of bundles that can be represented as $B(p, w) = \{x \in \ X \ |px \leq w\}$, where p is a vector of positive numbers (interpreted as prices in terms of the wealth unit) and w is a positive number (interpreted as the consumer's wealth).

Obviously, any set $B(p, w)$ is compact (it is closed since it is defined by weak inequalities and bounded since for any $x \in B(p, w)$ and for all k, $0 \leq x_k \leq w/p_k$). It is also convex: if $x, y \in B(p, w)$, then $px \leq w$, $py \leq w$, $x_k \geq 0$, and $y_k \geq 0$ for all k. Thus, for all $\alpha \in [0, 1]$, $p[\alpha x + (1 - \alpha)y] = \alpha px + (1 - \alpha)py \leq w$ and $\alpha x_k + (1 - \alpha)y_k \geq 0$ for all k, that is, $\alpha x + (1 - \alpha)y \in B(p, w)$.

We will identify a choice function with a function $x(p, w)$ which assigns to every vector of positive numbers (of length $K + 1$) a unique bundle in $B(p, w)$. Since we assume that the behavior of the consumer is "a choice from a set" and is independent of the framing of the set and since $B(\lambda p, \lambda w) = B(p, w)$ we assume that $x(p, w) = x(\lambda p, \lambda w)$ (the function is *homogeneous of degree zero*). We refer to the function $x(p, w)$ as a *demand function.*

Note that the equality $x(p, w) = x(\lambda p, \lambda w)$ should not be interpreted as implying that "uniform inflation does not matter." We had *assumed*, rather than concluded, that the choice is made from a set independently of how the choice set is framed. Furthermore, our model of choice is static and it is assumed that the choice is not affected from previous choices. Inflation will affect behavior in a model where this strong assumption is relaxed.

The consumer's problem

We will refer to the problem of finding the $\succsim$-best bundle in $B(p, w)$ as the *consumer problem.*

Claim:

If $\succsim$ is a continuous relation, then all consumer problems have a solution.

Proof:

If $\succsim$ is continuous, then it can be represented by a continuous utility function u. By the definition of the term "utility representation", finding an $\succsim$-optimal bundle is equivalent to solving the problem $\max_{x \in B(p,w)} u(x)$. By Weierstrass theorem, since the budget set is compact and u is continuous, the problem has a solution.

To emphasize that a utility representation is not necessary for the current analysis and that we could make do with the concept of preferences, we will go through a direct proof of the previous claim that avoids using the notion of utility.

Direct Proof:

For any $x \in B(p, w)$, define the set $\mathit{Inferior}(x) = \{y \in X | x \succ y\}$. By the continuity of the preferences, every such set is open. Assume there is no solution to the consumer problem of maximizing $\succsim$ on $B(p, w)$. Then, every $z \in B(p, w)$ is a member of some set $\mathit{Inferior}(x)$, that is, the collection of sets $\{\mathit{Inferior}(x) \mid x \in B(p, w)\}$ covers $B(p, w)$. By the Heine-Borel theorem the collection (of open sets that covers a compact set) has a finite sub-collection, $\mathit{Inferior}(x^1), \ldots, \mathit{Inferior}(x^n)$, that covers $B(p, w)$. Letting x^j be the optimal bundle within the finite set $\{x^1, \ldots, x^n\}$, we obtain that x^j is an optimal bundle in $B(p, w)$, a contradiction.

Claim:

1. If $\succsim$ is convex, then the set of solutions for a choice from $B(p,w)$ (or any other convex set) is convex.
2. If $\succsim$ is strictly convex, then every consumer problem has at most one solution.

Proof:

1. Assume that both x and y maximize $\succsim$ given $B(p,w)$. By the convexity of the budget set $B(p,w)$, we have $\alpha x + (1-\alpha)y \in B(p,w)$, and by the convexity of the preferences, $\alpha x + (1-\alpha)y \succsim x \succsim z$ for all $z \in B(p,w)$. Thus, $\alpha x + (1-\alpha)y$ is also a solution to the consumer problem.
2. Assume that both x and y (where $x \neq y$) are solutions to the consumer problem $B(p,w)$. Then $x \sim y$ (both x and y are solutions to the same maximization problem) and $\alpha x + (1 - \alpha)y \in B(p,w)$ (the budget set is convex). By the strict convexity of $\succsim$, $\alpha x + (1-\alpha)y \succ x$, which is a contradiction of x being a maximal bundle in $B(p,w)$.

The Consumer Problem with Differentiable Preferences

When the preferences are differentiable, we are provided with a meaningful condition for characterizing the optimal solution: the "value per dollar" at a bundle of a commodity which is positively consumed is as large as the "value per dollar" of any other commodity.

Claim:

Assume that the consumer's preferences are differentiable and denote by $v(x^*) = (v_1(x^*), \dots, v_K(x^*))$ the vector of "subjective value numbers" at x^* (see the definition of differentiable preferences in Lecture 4). If x^* is an optimal bundle in the consumer problem and k is a consumed commodity (i.e., $x_k^* > 0$), then it must be that $v_k(x^*)/p_k \geq v_j(x^*)/p_j$ for all other j.

Proof:

Assume that x^* is a solution to the consumer problem $B(p,w)$ and that $x_k^* > 0$ and $v_j(x^*)/p_j > v_k(x^*)/p_k$ for some j (see fig. 5.1). A "move" in the direction of reducing the consumption of the k'th commodity

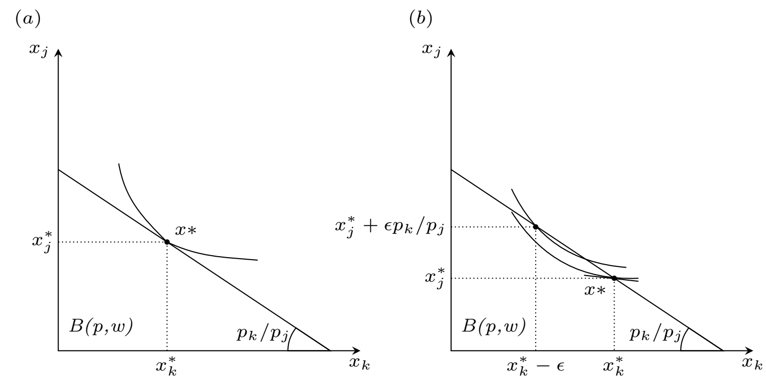

Figure 5.1
(a) x^* is a solution to the consumer problem $B(p,w)$. (b) x^* is not a solution to the consumer problem $B(p,w)$.

by 1 and increasing the consumption of the j'th commodity by p_k/p_j is an improvement direction since $v_j(x^*)p_k/p_j - v_k(x^*) > 0$. As $x_k^* > 0$, we can find $\varepsilon > 0$ small enough such that decreasing k's quantity by ε and increasing j's quantity by $\varepsilon p_k/p_j$ is a feasible improvement. This brings the consumer to a strictly better bundle inside his budget set, contradicting the assumption that x^* is a solution to the consumer problem.

Conclusion:
If x^* is a solution to the consumer problem $B(p,w)$ and both $x_k^* > 0$ and $x_j^* > 0$, then the ratio $v_k(x^*)/v_j(x^*)$ must be equal to the price ratio p_k/p_j.

The above implies the "classic" necessary conditions on the consumer's maximization when the preferences are represented by a differentiable utility function u, with positive partial derivatives, using the equality $v_k(x^*) = \partial u/\partial x_k(x^*)$.

In order to establish sufficient conditions for maximization, we require also that the preferences be convex.

Claim:

If $\succsim$ is strongly monotonic, convex, continuous, and differentiable, and if at x^*

- $px^* = w$,
- $v_k(x^*)/p_k \geq v_j(x^*)/p_j$ for any k such that $x_k^* > 0$, and for any j,

then x^* is a solution to the consumer problem.

Proof:

If x^* is not a solution, then there is a bundle y such that $py \leq w$ and $y \succ x^*$. By the comment at the end of the previous chapter $(y - x^*)$ is an improving direction. Let $\mu = v_k(x^*)/p_k$ for all k with $x_k^* > 0$. Now,

$$0 \geq p(y - x^*) = \sum p_k(y_k - x_k^*) \geq \sum v_k(x^*)(y_k - x_k^*)/\mu$$

since: (1) y is feasible and $px^* = w$, (2) for a good k with $x_k^* > 0$ we have $p_k = v_k(x^*)/\mu$, and (3) for a good k with $x_k^* = 0$, $(y_k - x_k^*) \geq 0$ and $p_k \geq v_k(x^*)/\mu$. Thus, $0 \geq v(x^*)(y - x^*)$, contradicting that $(y - x^*)$ is an improvement direction.

The Demand Function of a Rational Consumer

We have arrived at a critical stage on the way to developing a market model in which we derive demand from preferences. Assume that the consumer's preferences are such that for any $B(p, w)$, the consumer's problem has a unique solution. Then we identify the demand function $x(p, w)$ with the maximizer of the consumer's preferences over $B(p, w)$. The domain of the demand function is $\mathbb{R}^{K+1}_{++}$, whereas its range is $\mathbb{R}^K_+$.

Example:

Consider a consumer in a world with two commodities having who has the following lexicographic preference relation, with the the sum of the quantities of the goods as the first priority and the quantity of commodity 1 as the second:
$x \succsim y$ if $[x_1 + x_2 > y_1 + y_2]$ or $[x_1 + x_2 = y_1 + y_2$ and $x_1 \geq y_1]$.

This preference relation is strictly convex but not continuous. It induces the following noncontinuous demand function:

$$x((p_1, p_2), w) = \begin{cases} (0, w/p_2) & if \quad p_2 < p_1 \\ (w/p_1, 0) & if \quad p_2 \geq p_1 \end{cases}.$$

We now turn to studying some properties of the demand function.

Claim (Walras's Law):
If the preferences are monotonic, then any solution x to the consumer problem $B(p, w)$ is located on its budget curve (and, thus, $px(p, w) = w$).

Proof:
If not, then $px<w$. There is an $\varepsilon > 0$ such that $p(x_1 + \varepsilon, \ldots, x_K+\varepsilon)<w$. By monotonicity, $(x_1 + \varepsilon, \ldots, x_K + \varepsilon) \succ x$, thus contradicting the assumption that x is optimal in $B(p, w)$.

Claim:
If $\succsim$ is a continuous preference relation, then the induced demand function is continuous in prices and in wealth.

Proof:
We can use the fact that the preferences have a continuous utility representation and apply a standard "maximum theorem". (Let $f(x)$ be a continuous function over X. Let A be a subset of some Euclidean space and B be a function that attaches to every a in A a compact subset of X such that its graph, $\{(a, x)| \ x \in B(a)\}$, is closed. Then the graph of the correspondence h from A into X, defined by $h(a) = \{x \in B(a) \mid f(x) \geq f(y)$ for all $y \in B(a)\}$, is closed.) Here is a direct proof that does not use the notion of a utility function:

Assume not. Then, there is a sequence of price and wealth vectors (p^n, w^n) converging to (p^*, w^*) such that $x(p^*, w^*) = x^*$, and $x(p^n, w^n)$ does not converge to x^*. Thus, we can assume that (p^n, w^n) is a sequence converging to (p^*, w^*) such that for all n the distance $d(x(p^n, w^n), x^*) > \varepsilon$ for some positive ε.

All numbers p_k^n are greater than some positive number $p_{\min}$ and all numbers w^n are less than some $w^{\max}$. Therefore, all vectors $x(p^n, w^n)$ belong to some compact set (the hypercube of bundles with no quantity above $w^{\max}/p_{\min}$), and thus, without loss of generality (choosing a subsequence if necessary), we can assume that $x(p^n, w^n) \to y^*$ for some $y^* \neq x^*$.

Since $p^n x(p^n, w^n) \leq w^n$ for all n, it must be that $p^* y^* \leq w^*$, that is, $y^* \in B(p^*, w^*)$. Since x^* is the unique solution for $B(p^*, w^*)$, we have $x^* \succ y^*$. By the continuity of the preferences, there are neighborhoods

B_{x^*} and B_{y^*} of x^* and y^* respectively in which the strict preference is preserved. For sufficiently large n, $x(p^n, w^n)$ is in B_{y^*}. Choose a bundle z^* in the neighborhood B_{x^*} so that $p^* z^* < w^*$. For all sufficiently large n, $p^n z^* < w^n$; however, $z^* \succ x(p^n, w^n)$, a contradiction.

Comment:

The above proposition applies when there is a unique bundle maximizing the consumer's preferences for every budget set. The maximum theorem applied to the case in which some budget sets have more than one maximizer states that if $\succsim$ is a continuous preference, then the set $\{(x, p, w) \mid x \succsim y \text{ for every } y \in B(p, w)\}$ is closed.

Rationalizable Demand Functions

As in the general discussion of choice, we will now examine whether choice procedures are consistent with the rational man model. In what follows various possible definitions of rationalization are considered.

One approach is to look for a preference relation (without imposing any restrictions that fit the context of the consumer) such that the chosen element from any budget set is the unique bundle maximizing the preference relation in that budget set. Thus, we say that the preferences $\succsim$ *fully rationalize* the demand function x if for any (p, w) the bundle $x(p, w)$ is the unique $\succsim$ maximal bundle within $B(p, w)$.

Alternatively, we could say that "being rationalizable" means that there are preferences such that the consumer's behavior is consistent with maximizing those preferences, that is, for any (p, w) the bundle $x(p, w)$ is a $\succsim$-maximal bundle (though not necessarily unique) within $B(p, w)$. This definition is "empty" since any demand function is consistent with maximizing the "total indifference" preference. This is why we usually say that the preferences $\succsim$ *rationalize* the demand function x if they are *monotonic*, and for any (p, w) the bundle $x(p, w)$ is a $\succsim$-maximal bundle within $B(p, w)$. Similar definitions would fit to other assumptions on the preferences.

Of course, if behavior satisfies homogeneity of degree zero and Walras's law, it is still not necessarily rationalizable in any of those senses:

Example 1:
Consider the demand function of a consumer who spends all his wealth on the "more expensive" good:

$$x((p_1, p_2), w) = \begin{cases} (0, w/p_2) & if \quad p_2 \geq p_1 \\ (w/p_1, 0) & if \quad p_2 < p_1 \end{cases}.$$

This demand function is not entirely inconceivable, and yet it is not rationalizable. To see this, assume that it is fully rationalizable or rationalizable by $\succsim$. Consider the two budget sets $B((1,2),1)$ and $B((2,1),1)$. Since $x((1,2),1) = (0,1/2)$ and $(1/2,0)$ is an internal bundle in $B((1,2),1)$, by either of the two definitions of rationalizability it must be that $(0,1/2) \succ (1/2,0)$. Similarly, $x((2,1),1) = (1/2,0)$ and $(0,1/2)$ is an internal bundle in $B((2,1),1)$. Therefore, $(0,1/2) \prec (1/2,0)$, a contradiction.

Example 2:
A consumer chooses a bundle $(z, z, \ldots, z)$ where z satisfies $z\Sigma p_k = w$.

This behavior is fully rationalized by any preferences according to which the consumer strictly prefers any bundle on the main diagonal over any bundle that is not (because, for example, he cares primarily about purchasing equal quantities from all sellers of the K goods), while on the main diagonal his preferences are according to "the more the better". These preferences rationalize his behavior in the first sense but are not monotonic. This demand function is also fully rationalized by the monotonic preferences represented by the utility function $u(x_1, \ldots, x_K) = min\{x_1, \ldots, x_K\}$.

Example 3:
Consider a consumer who spends α_k of his wealth on commodity k (where $\alpha_k \geq 0$ and $\Sigma_{k=1}^K \alpha_k = 1$). This rule of behavior is not formulated as a maximization of some preference relation. It can, however, be fully rationalized by the preference relation represented by the Cobb-Douglas utility function $u(x) = \Pi_{k=1}^K x_k^{\alpha_k}$, a differentiable function with strictly positive derivatives at all interior points. A solution x^* to the consumer problem $B(p, w)$ must satisfy $x_k^* > 0$ for all k (notice that $u(x) = 0$ when $x_k = 0$ for some k). Given the differentiability of the preferences, a necessary condition for the optimality of x^* is that $v_k(x^*)/p_k = v_l(x^*)/p_l$

for all k and l where $v_k(x^*) = du/dx_k(x^*) = \alpha_k u(x^*)/x_k^*$ for all k. It follows that $p_k x_k^*/p_l x_l^* = \alpha_k/\alpha_l$ for all k and l and thus $x_k^* = \alpha_k w/p_k$ for all k.

Example 4:

Let $K = 2$. A consumer allocates his wealth between commodities 1 and 2 in the proportion p_2/p_1 (the cheaper the good, the higher the share of wealth devoted to it). Thus, $x_1 p_1/x_2 p_2 = p_2/p_1$ and $x_i(p,w) = (p_j/(p_i + p_j))w/p_i$.

To see that this demand function is fully rationalizable, note that $x_i(p,w)/x_j(p,w) = p_j^2/p_i^2$ (for all i and j) and thus for everey (p,w)we have $p_1/p_2 = \sqrt{x_2(p,w)}/\sqrt{x_1(p,w)}$. We look for a function for which the ratio of its partial derivatives is equal to $\sqrt{x_2}/\sqrt{x_1}$. The quasi-concave function $\sqrt{x_1} + \sqrt{x_2}$ satisfies the condition. Thus, for any (p,w), the bundle $x(p,w)$ is the solution to the maximization of $\sqrt{x_1} + \sqrt{x_2}$ in $B(p,w)$.

The Weak and Strong Axioms of Revealed Preferences

We now look for general conditions that will guarantee that a demand function $x(p,w)$ can be fully rationalized. A similar discussion could apply to the definition of rationalizability that requires the bundle $x(p,w)$ to maximize a monotonic preference relation over $B(p,w)$. Of course, as we have just seen, one does not necessarily need these general conditions in order to determine whether a particular demand function is rationalizable. Guessing is often an excellent strategy.

In the general discussion of choice functions, we saw that condition α is necessary and sufficient for a choice function to be derived from some preference relation under certain assumptions about the domain of the choice function. The conditions there do not apply in the current setting since the domain of the demand function does not include finite sets and the union of two budget sets which do not include one another is not a budget set.

As in Lecture 3, we will use the concept of "revealed preferences". Define $x \succ y$ if there is (p,w) so that both x and y are in $B(p,w)$ and $x = x(p,w)$. In such a case, we will say that x is *revealed to be better* than y and a preference relation $\succsim$ satisfies the *Weak Axiom of Revealed Preferences* if it is impossible that both x is revealed to be better than y and y is revealed to be better than x. In the context of the consumer

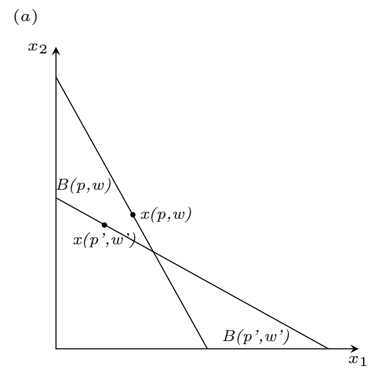

Figure 5.2
(a) Satisfies the weak axiom. (b) Does not satisfy the weak axiom.

model, the Weak Axiom can be written as: if $x(p,w) \neq x(p',w')$ and $px(p',w') \leq w$, then $p'x(p,w) > w'$ (see fig. 5.2).

The Weak Axiom states that the defined binary relation $\succ$ is asymmetric. However, the relation is not necessarily complete: there can be two bundles x and y such that for any $B(p,w)$ containing both bundles, $x(p,w)$ is neither x nor y.

Apparently, the Weak Axiom is not a sufficient condition for extending the binary relation $\succ$, as defined above, into a complete and transitive relation (an example with three goods from Hicks (1956) is discussed in Mas-Colell et al. (1995)). A necessary and sufficient condition for a demand function x satisfying Walras's law and homogeneity of degree zero to be rationalizable is discussed next.

Strong Axiom of Revealed Preference:

The Strong Axiom is a property of the demand function, which states that the relation $\succ$, derived from the demand function as before, is acyclical. Note that its transitive closure still may not be a complete relation. But as mentioned earlier the fact that the relation $\succ$ can be extended into a full-fledged preference relation is a well-known result in Set Theory. In any case, the Strong Axiom is somewhat cumbersome, and using it to determine whether a certain demand function is rationalizable may not be a straightforward task.

Comment:

As mentioned earlier, the more standard definition of rationalizability requires finding monotonic preferences $\succsim$ such that for any (p, w), $x(p, w) \succsim y$ for all $y \in B(p, w)$. We then infer from the existence of a budget set $B(p, w)$ for which $x = x(p, w)$ and $y \in B(p, w)$ only that x is weakly preferred to y. If, however, also $py < w$, we further infer that x is strongly preferred to y.

Decreasing Demand

A theoretical model may be evaluated by the reasonableness of its implications. If we find that a model yields an absurd conclusion, we need to reconsider its assumptions. We might conclude that the assumptions are unreasonable or that we have assumed "too little".

In the context of the consumer model, we might wonder whether the intuition that demand for a good falls when its price increases is a valid implication of the assumptions. We will now see that the standard assumptions of rational consumer behavior do not guarantee that demand is decreasing. The following is an example of a preference relation that induces demand which is nondecreasing in the price of one of the commodities.

Consider the preferences represented by the following utility function:

$$u(x_1, x_2) = \begin{cases} x_1 + x_2 & if \quad x_1 + x_2 < 1 \\ x_1 + 4x_2 & if \quad x_1 + x_2 \geq 1 \end{cases}.$$

These preferences might reflect reasoning of the following type: "In the bundle x there are $x_1 + x_2$ units of vitamin A and $x_1 + 4x_2$ units of vitamin B. My first priority is to get enough vitamin A. However, once I satisfy my need for 1 unit of vitamin A, I will move on to my second priority, which is to consume as much vitamin B as possible". (See fig. 5.3.)

Consider $x((p_1, 2), 1)$. Changing p_1 is like rotating the budget lines around the pivot bundle $(0, 1/2)$. At a high price of p_1 (as long as $p_1 > 2$), the consumer demands $(0, 1/2)$. If the price is reduced to within the range $2 > p_1 \geq 1$, the consumer chooses the bundle $(1/p_1, 0)$. So far, the demand for the first commodity indeed increases when its price falls. However, in the range $1 > p_1 > 1/2$ we encounter an anomaly: the consumer buys as much as possible from the second good subject to the "constraint" that the sum of the goods is at least 1, that is, $x((p_1, 2), 1) = (1/(2 - p_1), (1 - p_1)/(2 - p_1))$.

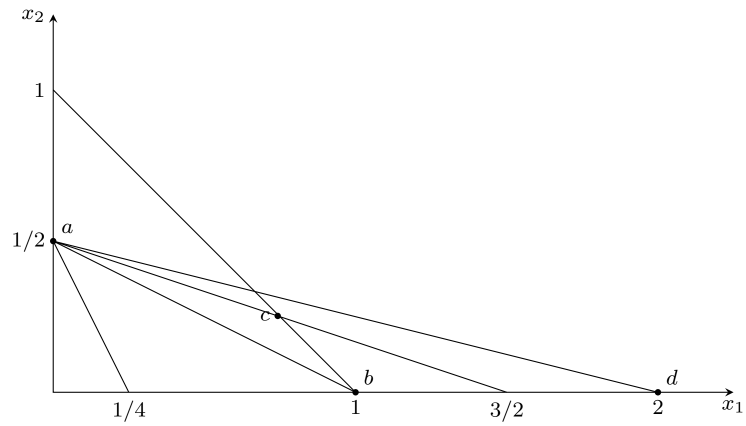

Figure 5.3
An example in which demand increases with price ($p_2 = 2$ and $w = 1$). When $2 < p_1$ the demand is at a. When $1 < p_1 < 2$ the demand is to the left of b. When $1/2 < p_1 < 1$ the demand is at a point like c. When $p_1 < 1/2$ the demand is to the right of d.

The above preference relation is monotonic but not continuous. Still we can construct a closely similar continuous preference that leads to demand which is increasing in p_1 in a similar domain. For $\delta > 0$, let $\alpha_\delta(t)$ be a continuous and increasing function on $[1-\delta, 1]$ where $\delta > 0$, so that $\alpha_\delta(t) = 0$ for all $t \leq 1-\delta$ and $\alpha_\delta(t) = 1$ for all $t \geq 1$. The utility function

$$u_\delta(x) = \alpha_\delta(x_1 + x_2)(x_1 + 4x_2) + (1 - \alpha_\delta(x_1 + x_2))(x_1 + x_2)$$

is continuous and monotonic. For δ close to 0, the function $u_\delta = u$ except in a narrow area below the set of bundles for which $x_1 + x_2 = 1$. When $p_1 = 2/3$, the demand for the first commodity is $3/4$, whereas when $p_1 = 1$, the demand is at least $1 - 2\delta$. Thus, for a small enough δ the increase in p_1 might lead to an increase in the demand.

"The Law of Demand"

We are interested in comparing demand in different environments. We have just seen that the classic assumptions about consumer behavior do not lead to a clear conclusion regarding the relation between a consumer's demand when facing $B(p, w)$ and his demand when facing $B(p + (0, \ldots, \varepsilon, \ldots, 0), w)$.

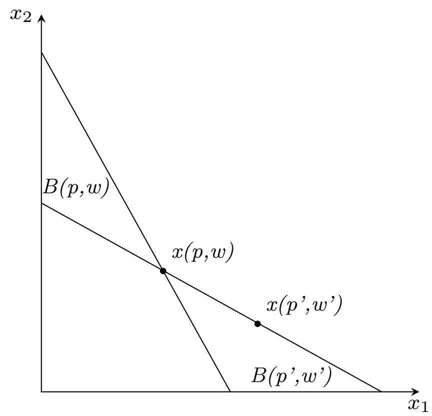

Figure 5.4
A compensated price change from (p, w) to (p', w').

A clear conclusion can be drawn when we compare the consumer's demand when facing the budget set $B(p, w)$ to his demand when facing $B(p', x(p, w)p')$. In this comparison, we imagine the price vector changing from p to an arbitrary p' and wealth changing in such a way that the consumer has exactly the resources allowing him to consume the same bundle he consumed at (p, w) (see fig. 5.4). It follows from the following claim that the compensated demand function $y(p') = x(p', p'x(p, w))$ satisfies the law of demand, that is, y_k is decreasing in p_k.

Claim:
Let x be a demand function satisfying Walras law and WA. If $w' = p'x(p, w)$, then either $x(p', w') = x(p, w)$ or $[p' - p][x(p', w') - x(p, w)] < 0$.

Proof:
Assume that $x(p', w') \neq x(p, w)$. By Walras's law and the assumption that $w' = p'x(p, w)$:

$$\begin{aligned}&[p' - p][x(p', w') - x(p, w)]\\ &= p'x(p', w') - p'x(p, w) - px(p', w') + px(p, w)\\ &= w' - w' - px(p', w') + w = w - px(p', w')\end{aligned}$$

By WA, the right-hand side of the equation is less than 0.

Bibliographic Notes

The material in this lecture is fairly standard (see, for example, Arrow and Hahn (1971), Varian (1984) and the books mentioned in the preface).

Problem Set 5

Problem 1. (*Easy*)
Show that if a consumer has a homothetic preference relation, then his demand function is homogeneous of degree one in w.

Problem 2. (*Easy*)
Consider a consumer in a world with $K = 2$ who has a preference relation that is monotonic, continuous, strictly convex, and quasi-linear in the first commodity. How does the demand for the first commodity change with w?

Problem 3. (*Moderately Difficult*)
Define a Demand Correspondence, $X(p, w) : \mathbb{R}^{K+1}_{++} \to \mathbb{R}^K_+$, to be the set of all solutions to the consumer's problem in $B(p, w)$.

a. Calculate $X(p, w)$ for the case of $K = 2$ and preferences represented by $x_1 + x_2$.
b. Let $\succsim$ be a continuous preference relation (not necessarily convex). Show that $X(p, w)$ is upper semi-continuous.
(A correspondence $F : A \to B$ is said to be upper semi-continuous if for every converging sequence $a^n \in A$ with $\lim a^n \in A$, and for every converging sequence $b^n \in B$ such that $\lim b^n$ exists and $b^n \in F(a^n)$ then $\lim b^n \in F(\lim a^n)$.)

Problem 4. (*Moderately difficult*)
Determine whether the following consumer behavior patterns are fully rationalized (assume $K = 2$):

a. The consumer consumes up to the quantity 1 of commodity 1 and spends his remaining wealth on commodity 2.
b. The consumer chooses the bundle (x_1, x_2) which satisfies $x_1/x_2 = p_1/p_2$ and costs w. (Does the utility function $u(x) = x_1^2 + x_2^2$ rationalize the consumer's behavior?)

Problem 5. (*Moderately difficult*)
Consider a consumer who has in mind a preference relation that satisfies continuity, monotonicity, and strict convexity; for simplicity, assume it is represented by a utility function u.

The consumer maximizes utility up to utility level u^0. If the budget set allows him to obtain this level of utility, he chooses the bundle in the budget set with the highest quantity of commodity 1 subject to the constraint that his utility is at least u^0.

The consumer maximizes utility up to utility level u^0. If the budget set allows him to obtain this level of utility, he chooses the bundle in the budget set with the highest quantity of commodity 1 subject to the constraint that his utility is at least u^0.

a. Formulate the consumer's problem.
b. Show that the consumer's procedure yields a unique bundle.
c. Is this demand procedure rationalizable?
d. Does the demand function satisfy Walras's law?
e. Show that in the domain of (p, w) for which there is a feasible bundle yielding utility of at least u^0 the consumer's demand function for commodity 1 is decreasing in p_1 and increasing in w.
f. Is the demand function continuous?

Problem 6. (*Moderately difficult*)
It is a common practice in economics to view aggregate demand as being derived from the behavior of a "representative consumer". Construct an example of two consumer preference relations that induce average behavior that is not consistent with maximization by a "representative consumer". That is, describe two rational "consumers", 1 and 2, who choose the bundles x^1 and x^2 from the budget set A and the bundles y^1 and y^2 from the budget set B so that the choice of the bundle $(x^1 + x^2)/2$ from A and of the bundle $(y^1 + y^2)/2$ from B is inconsistent with the model of the rational consumer.

Problem 7. (*Standard*)
A commodity k is *Giffen* if the demand for the $k'th$ good is increasing in p_k. A commodity k is *inferior* if the demand for the commodity decreases with wealth. Show that if there is a vector (p, w) such that the demand for the $k'th$ commodity is rising after its price has increased, then there is a vector (p', w') such that demand for the $k'th$ commodity falls when the income increases (being Giffen implies inferior).

Problem 8. (*Standard*)
A consumer with concave preferences

Consider a consumer world with two goods. Assume the consumer has a preference relation which is strongly monotone, continuous and satisfies strict "concavity": If $a \succsim b$ then $a \succ \delta\, a + (1-\delta)b$ for all $1 > \delta > 0$.

a. Show that for every non-negative number t there is an s such that $(0,t) \sim (s,0)$ (it will follow analogously that for every s there is an t such that $(0,t) \sim (s,0)$).

b. Fix w and p_2. Draw (and explain) the consumer's demand correspondence for good 1 as a function of p_1 (assuming he maximizes his preferences).

Problem 9. (*Standard*)
The decision to save is often a "residual" decision. To model it consider a world with K goods and denote "saving" as the K'th good. The consumer has in mind a continuous and strictly concave function v (not necessarily increasing!) defined over R_+^{K-1} where $v(x_1,..,x_{k-1})$ is the value he attaches to the combination of the first $K-1$ goods. The consumer also has in mind an aspiration level v^*. Given a price vector $p = (p_1,..,p_{K-1})$ and wealth level w he maximizes the function v over $\{(x_1,...,x_{K-1}) \mid px \leq w$ and $v(x_1,...,x_{K-1}) \leq v^*\}$. If he is left with some wealth he spends it on the K'th good (savings).

a. Show that this behavior can be (fully) rationalized by a preference relation over all vectors $(x_1,..,x_K) \in \Re_+^K$.
b. Can it be (fully) rationalized by a continuous preference relation (assume $K = 2$)?
c. Can it be (fully) rationalized by a differentiable preference relation (assume $K = 2$)?

LECTURE 6

More Economic Agents: a Consumer Choosing Budget Sets, a Dual Consumer and a Producer

A Consumer Choosing Between Budget Sets

An agent Choosing Between Choice Sets

Let X be a set of alternatives and D a of non-empty subsets of X (choice problems). We are interested in the decision maker's preference relation over D. A statement of the form "I prefer A over B" is interpreted as the expression of an individual's preference for the choice problem A over the choice problem B.

Assuming that the decision maker has a preference relation $\succsim$ defined over X, one approach to building a preference relation over D is as follows: When assessing a choice problem in D, the decision maker asks himself which alternative he would choose from this set. He prefers a set A over a set B if the alternative he would choose from A is preferable (according to his basic preference $\succsim$ defined over X) over what he would choose from B. This leads to the following definition of $\succsim^*$, a relation which we will refer to as the *indirect preferences* induced from $\succsim$:

$$A \succsim^* B \text{ if } C_{\succsim}(A) \succsim C_{\succsim}(B).$$

Obviously, $\succsim^*$ is a preference relation. If u represents $\succsim$ and the choice function is well defined, then $v(A) = u(C_{\succsim}(A))$ represents $\succsim^*$. We will refer to v as the *indirect utility function.*

The notion of indirect preferences ignores many considerations that might be taken into account when comparing choice sets. For example:
a. "I prefer $A - \{b\}$ to A even though I intend to choose a in any case since I am afraid to make a mistake by choosing b",
b. "I will choose a from A and from $A - \{b\}$; however, since I don't want to have to reject b, I prefer $A - \{b\}$ to A"or

c. "I prefer $A - \{b\}$ to A because I would choose b from A and I want to commit myself to not making that choice."

Note that in some cases (depending on the set D) one can reconstruct the choice function $C_{\succsim}(A)$ from the indirect preferences $\succsim^*$. For example, if $a \in A$ and $A \succ^* A - \{a\}$, then one can conclude that $C_{\succsim}(A) = a$.

Choice Between Budget Sets

We now turn to discuss a consumer who is choosing between budget sets. The indirect preferences on budget sets are relevant in decision situations, such as choosing a place to live or comparing different tax systems (which affect wealth and prices).

A budget set is characterized by the $K+1$ parameters (p, w). We assume that he has a preference relation $\succsim$ on the set of bundles satisfying the classical assumptions (monotonicity, continuity and convexity) and that demand, $x(p, w)$, is always well-defined. This leads to the following definition of the indirect preferences $\succsim^*$ on the set $\mathbb{R}^{K+1}_{++}$:

$(p, w) \succsim^* (p', w')$ if $x(p, w) \succsim x(p', w')$.

In this context, the indirect preference relation excludes from the consumer's deliberation considerations such as "I prefer to live in an area where alcohol is very expensive even though I don't drink".

Following are some properties of indirect preferences:

1. *Invariance to presentation:* $(\lambda p, \lambda w) \sim^* (p, w)$ for all $p, w, \lambda > 0$. This follows from $x(\lambda p, \lambda w) = x(p, w)$.
2. *Monotonicity:* The indirect preferences are weakly decreasing in p_k and strictly increasing in w. Shrinking the choice set is never beneficial under this approach and additional wealth makes it possible to consume bundles containing more of all commodities.
3. *Continuity:* If $(p, w) \succ^* (p', w')$, then $y = x(p, w) \succ x(p', w') = y'$. By continuity, there are neighborhoods B_y and $B_{y'}$ around y and y' respectively, such that for any $z \in B_y$ and $z' \in B_{y'}$ we have $z \succ z'$. By continuity of the demand function, there is a neighborhood around (p, w) in which demand is within B_y and there is a neighborhood around (p', w') in which demand is within $B_{y'}$. For any two budget sets in these two neighborhoods, $\succ^*$ is preserved.
4. *"Concavity":* If $(p^1, w^1) \succsim^* (p^2, w^2)$, then for all $1 \geq \lambda \geq 0$ we have $(p^1, w^1) \succsim^* (\lambda p^1 + (1-\lambda)p^2, \lambda w^1 + (1-\lambda)w^2)$ (see fig. 6.1). Let $z = x(\lambda p^1 + (1-\lambda)p^2, \lambda w^1 + (1-\lambda)w^2)$. By definition, $(\lambda p^1 + (1-\lambda)p^2)z \leq \lambda w^1 + (1-\lambda)w^2$. Therefore $p^1 z \leq w^1$ or $p^2 z \leq w^2$. Thus,

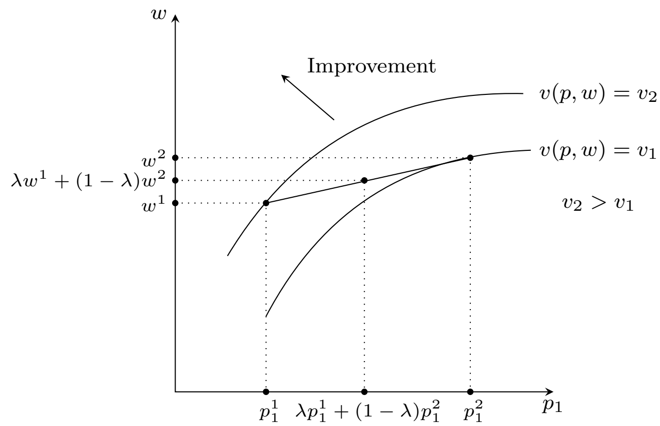

Figure 6.1
The indirect utility function is quasi-convex.

$z \in B(p^1, w^1)$ or $z \in B(p^2, w^2)$ and $x(p^1, w^1) \succsim z$ or $x(p^2, w^2) \succsim z$. From $x(p^1, w^1) \succsim x(p^2, w^2)$, we conclude $x(p^1, w^1) \succsim z$.

Roy's Identity

We now look at a method of recovering the consumer demand function from indirect preferences.

In the single commodity case, each $\succsim^*$-indifference curve is a ray. If we assume monotonicity of $\succsim$, the slope of an indifference curve through (p_1, w) is w/p_1, which is $x_1(p_1, w)$.

In the general K-commodity space, we can recover the demand at (p^*, w^*) by the slope of the indifference curve through (p^*, w^*). The key observation is that the set $\{(p, w) \mid px(p^*, w^*) = w\}$ is tangent to the indifference curve of the indirect preferences through (p^*, w^*). When there is a unique tangent to the indifference curve of the indirect preferences at (p^*, w^*), knowing this tangent allows us to recover $x(p^*, w^*)$.

Claim:

Assume that the demand function is derived from maximizing a preference relation $\succsim$ which satisfies monotonicity. Then:

1. The hyperplane $H = \{(p, w) \mid px(p^*, w^*) = w\}$ is tangent to the $\succsim^*$-indifference curve at (p^*, w^*).

2. Roy's identity: When the (indirect) preferences $\succsim^*$ are represented by a differentiable (indirect) utility function v,

$$-[\partial v/\partial p_k(p^*,w^*)]/[\partial v/\partial w(p^*,w^*)] = x_k(p^*,w^*).$$

Proof:

1. By the monotonicity of the preferences $(p^*,w^*) \in H$. For any $(p,w) \in H$, the bundle $x(p^*,w^*) \in B(p,w)$. Hence, $x(p,w) \succsim x(p^*,w^*)$ and thus $(p,w) \succsim^* (p^*,w^*)$.
2. $H = \{(p,w) |\ (x(p^*,w^*),-1)(p,w) = 0\}$. Since $w^* = p^* x(p^*,w^*)$, we have also:

   $$H = \{(p,w) |\ (x(p^*,w^*),-1)(p-p^*,w-w^*) = 0\}.$$

   Since v is differentiable, the unique tangent to the indifference curve through (p^*,w^*) is the hyperplane that is perpendicular to the gradient (the vector of partial derivatives):

   $$T = \{(p,w) |\ (\partial v/\partial p_1(p^*,w^*),\ldots,\partial v/\partial p_K(p^*,w^*),\\ \partial v/\partial w(p^*,w^*))(p-p^*,w-w^*) = 0\}.$$

   By part (1) $T = H$ and therefore the vector

   $$(\partial v/\partial p_1(p^*,w^*),\ldots,\partial v/\partial p_K(p^*,w^*),\partial v/\partial w(p^*,w^*))$$

   must be proportional to the vector

   $$(x_1(p^*,w^*),\ldots,x_K(p^*,w^*),-1)$$

   and Roy's identity follows.

A Dual Consumer

A "Prime and Dual" Turtle

Consider the following two sentences:

1. The maximal distance a turtle can travel in 1 day is 1 km.
2. The minimal time it takes a turtle to travel 1 km is 1 day.

In conversation, these two sentences would seem to be equivalent. In fact, this equivalence relies on two "hidden" assumptions:

a. For (1) to imply (2), we need to assume that the turtle travels a positive distance in any period of time. Contrast this with the case in which the turtle's speed is 2 km/day, but after half a day it must rest for half a day. In this case, the maximal distance it can travel in 1 day is 1 km, though it is able to travel this distance in only half a day.

b. For (2) to imply (1), we need to assume that the turtle cannot "jump"a positive distance in zero time. Contrast this with the case in which the turtle's speed is 1 km/day, but after a day of traveling it can "jump"1 km. Thus, it can travel 2 km in 1 day (if you can't imagine a jumping turtle, think about a "frequent consumer" scheme in which the consumer gains a bonus after the consumer reaches a certain number of points).

We will now show that the above hidden assumptions are sufficient for the equivalence of (1) and (2). Let $M(t)$ be the maximal distance the turtle can travel in time t and assume that M is strictly increasing and continuous. Then, the statement "the maximal distance a turtle can travel in t^* units of time is x^*" is equivalent to the statement "the minimal time it takes a turtle to travel the distance x^* is t^*".

If the maximal distance that the turtle can travel within t^* is x^* and if it covers the distance x^* in $t < t^*$ (that is $M(t) \geq x^*$), then by the strict monotonicity of M the turtle can cover a distance larger than x^* in t^*, a contradiction.

If it takes t^* for the turtle to cover the distance x^* and if it travels the distance $x > x^*$ in t^* (that is $M(t^*) > x^*$), then by the continuity of M the turtle will already be beyond the distance x^* at some $t < t^*$, a contradiction.

The Prime Consumer

Consider first a consumer who possesses a preference relation $\succsim$ (satisfying the classical assumptions of monotonicity, continuity and strict convexity) and an initial wealth w. When facing the price vector p, he can trade w for any bundle x, such that $px \leq w$. We refer to the problem of choosing a $\succsim$-best bundle from the set $\{x \mid px \leq w\}$ as the consumer's *prime problem* and denote it by $P(p, w)$. The problem has a solution and when the solution is unique, we denote it by $x(p, w)$.

The Dual Consumer

Consider first a consumer who possesses a preference relation $\succsim$ (satisfying the classical assumptions of monotonicity, continuity and strict convexity) and has in mind a bundle z. The consumer wishes to consume the cheapest bundle (given a price vector p) which for him is at least as good as z.

Note the similarity between the dual consumer's preferences and those used in the previous chapter to demonstrate that demand might not be monotonic. The dual consumer has two goals: As long as he cannot achieve a bundle as good as x^*, he maximizes his preferences over X. Once he can, he then applies the criterion of minimizing his expenses.

We refer to the problem $\min_x\{px \mid x \succsim z\}$ as the *dual problem* and denote it by $D(p,z)$. Assuming that a solution exists and is unique (which occurs, for example, when preferences are strictly convex and continuous), we denote the solution as $h(p,z)$ and refer to it as the *Hicksian demand function.* The function $e(p,z) = ph(p,z)$ is called the *expenditure function.* (Note the analogy between the expenditure function and the consumer's indirect utility function.)

Following are some properties of the Hicksian demand function and the expenditure function:

1. $h(p,z) = h(\lambda p, z)$ and $e(\lambda p, z) = \lambda e(p,z)$.
   This follows from the fact that a bundle minimizes the function λpx over a set if and only if it minimizes the function px over that same set.
2. The Hicksian demand for the k'th commodity is decreasing in p_k while $e(p,z)$ is increasing in p_k.
   Note first that $ph(p',z) \geq ph(p,z)$ for every p'. This is because $h(p',z) \succsim z$ and the bundle $h(p',z)$ is not less expensive than $h(p,z)$ for the price vector p. Thus, $(p'-p)(h(p',z) - h(p,z)) = (p'h(p',z) - p'h(p,z)) + (ph(p,z) - ph(p',z)) \leq 0$ and if $(p'-p) = (0,...,\varepsilon,...,0)$ (with $\varepsilon > 0$), we obtain $h_k(p',z) - h_k(p,z) \leq 0$.
   Furthermore, if $p'_k \geq p_k$ for all k, then $e(p',z) = p'h(p',z) \geq ph(p',z) \geq ph(p,z) = e(p,z)$.
3. $h(p,z) \sim z$. If $h(p,z) \succ z$, then by continuity there would be a cheaper bundle at least as good as z near $h(p,z)$.
4. $h(p,z)$ and $e(p,z)$ are continuous (verify!).
5. The expenditure function is concave in p:
   Let $x = h(\lambda p^1 + (1-\lambda)p^2, z)$. By definition, $x \succsim z$. Thus, $p^i x \geq p^i h(p^i, z)$ and $e(\lambda p^1 + (1-\lambda)p^2, z) = (\lambda p^1 + (1-\lambda)p^2)x \geq \lambda e(p^1,z) + (1-\lambda)e(p^2,z)$.

6. (The Dual of Roy's Identity) The hyperplane $H = \{(p, e) \mid e = ph(p^*, z)\}$ is tangent to the graph of the expenditure function at p^*. This follows from: (i) $(p^*, e(p^*, z))$ is in H and (ii) $ph(p^*, z) \geq ph(p, z)$ for all p.

Claim:

Consider a consumer who has preferences satisfying monotonicity and continuity. Then, the bundle x^* is a solution to $P(p, w^*)$ if and only if w^* is a solution to $D(p, x^*)$.

Proof:

Assume that x^* is a solution to $P(p, w^*)$ and w^* is not a solution to $D(p, x^*)$. Then, there exists a strictly cheaper bundle x for which $x \succsim x^*$. For some positive vector ε (i.e., $\varepsilon_k > 0$ for all k), it still holds that $p(x + \varepsilon) < px^*$. By monotonicity, $x + \varepsilon \succ x \succsim x^*$ and thus x^* is not a solution to $P(p, w^*)$.

Assume that w^* is a solution to $D(p, x^*)$ but x^* is not a solution to $P(p, w^*)$. Then, there exists an x such that $px \leq w^*$ and $x \succ x^*$. By continuity, for some nonnegative vector ε different than 0, $x - \varepsilon$ is a bundle such that $x - \varepsilon \succ x^*$ and $p(x - \varepsilon) < px^* \leq w^*$ and thus w^* is not a solution to $D(p, x^*)$.

A Producer

The producer is an economic agent with the ability to transform one vector of commodities into another. Note that we use the term "producer" rather than "firm" since we are not concerned with the internal organization of the producer's activity. We first specify the producer's "technology" and then discuss his preferences.

Technology

Denote the commodities, which can be either inputs or outputs in the producer's production activity, as $1, \ldots, K$. A vector z in $\mathbb{R}^K$ is interpreted as a production combination where positive components in z are outputs and negative components are inputs. A producer's choice set is called a *technology* and it reflects the production constraints.

The following restrictions are often placed on the technology space Z (fig. 6.2.):

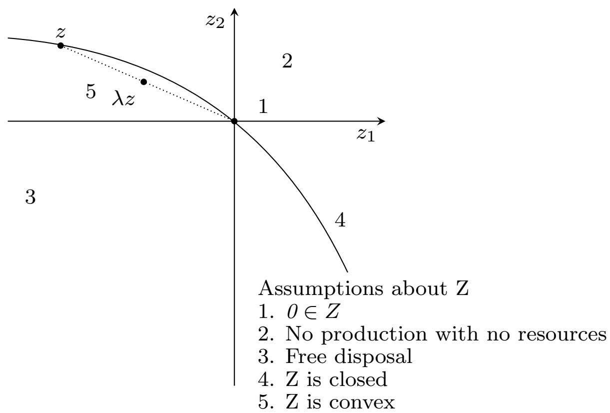

Figure 6.2
Technology.

1. $0 \in Z$ (which is interpreted to mean that the producer can remain "idle").
2. There is no $z \in Z \cap \mathbb{R}_+^K$ besides the vector 0 (i.e., there is no production with no resources).
3. *Free disposal*: If $z \in Z$ and $z' \leq z$, then $z' \in Z$ (i.e., nothing prevents the producer from being inefficient in the sense that he uses more resources than necessary to produce a particular amount of commodities).
4. Z is a closed set.
5. Z is a convex set. (This assumption embodies decreasing marginal productivity. Together with the assumption that $0 \in Z$, it implies *non-increasing returns to scale*: if $z \in Z$, then for all $\lambda < 1$, $\lambda z \in Z$.)

In some cases we will describe the producer's abilities using a *production function*. Consider, for example, the case in which commodity K is produced from commodities $1,2,\ldots,K-1$, that is, for all $z \in Z$, $z_K \geq 0$ and for all $k \neq K$, $z_k \leq 0$. The *production function* specifies, for any positive vector of inputs $v \in \mathbb{R}_+^{K-1}$, the maximum amount of commodity K that can be produced. If we start from technology Z, we can derive the production function by defining $f(v) = max\{x|\ (-v, x) \in Z\}$.

If we start from the production function f, we can derive the "technology" by defining $Z(f) = \{(-w, x)|\ x \leq y \text{ and } w \geq v \text{ for some } y = f(v)\}$. If the function f is increasing, continuous and concave and satisfies the assumption of $f(0) = 0$, then $Z(f)$ satisfies the above assumptions.

Producer Behavior

We think of the producer as an agent who has a preference relation over the space X, which contains all combinations (z, π) where $z \in Z$ and π is a number representing his profit.

For any given price vector p, the producer faces a choice set of the type $B(p) = \{(z, \pi) |\ z \in Z \text{ and } \pi = pz\}$. A rational producer maximizes a preference relation defined over X. Given a price vector p, he chooses $z \in Z$ to maximize (according to his preferences) the vector (z, pz).

Following are some examples of producer behavior which can easily be rationalized using preference relations on this space. For clarity, I focus on the case of $K = 2$ where commodity 1 is the input and commodity 2 is the output and $y = f(a)$ is the producer's production function:

1. The producer maximizes production y given the constraint $\pi \geq 0$.
2. The producer wishes to produce at least y^* units. Once he has achieved that goal, he maximizes profit.
3. The producer maximizes profit, but already employs a_1^* workers and will incur a cost c (whether in terms of money or the anguish it causes him) for each worker he fires. Thus, his utility function is given by $\pi - c \max\{0, a_1^* - a_1\}$.
4. The producer is a cooperative, which means that its profits are shared equally among its members who are both the owners and the workers. When choosing the number of members the cooperative seeks to maximize profit per member.
5. A "green producer" will have preferences over $(\pi, \mathit{pollution}(z))$ where $\mathit{pollution}(z)$ is the amount of pollution, which is dependent on z.
6. The producer maximizes his profit, π...

Another plausible behavior is to maximize the ratio of profits to costs (that is, $\frac{\pi}{p_a a}$). Note, however, that such behavior cannot be represented as the maximization of a preference relation on X since it depends on the breakdown of profit into revenues and costs and not just on profit.

While the classical assumption in economics is that a producer cares only about increasing his profit, the above examples demonstrate the richness of reasonable considerations that are ignored by making this assumption.

The Supply Function of the Profit-Maximizing Producer

We now discuss the profit-maximizing producer's behavior. The producer's problem is defined as $max_{z \in Z}\, pz$. The existence of a unique solution to the producer's problem requires some additional assumptions, e.g. that Z be *bounded from above* (i.e., there is some bound B such that $B \geq z_k$ for any $z \in Z$) and that Z be *strictly convex* (i.e., if z and z' are in Z, then the combination $\lambda z + (1-\lambda)z'$ is an internal point in Z for any $1 > \lambda > 0$).

When the producer's problem has a unique solution, we denote it by $z(p)$ and refer to the relation between p and z as the *supply function.* Note that it specifies both the producer's supply of outputs and his demand for inputs. We also define the *profit function* as $\pi(p) = max_{z \in Z} pz$.

Recall that in the discussion of the consumer, we specified his preferences and described his behavior as making a choice from a budget set determined by prices. The consumer's behavior (demand) determined the dependence of his consumption on prices. In the case of the profit-maximizing producer, we specify the technology and describe his behavior as maximizing a profit function determined by prices. The producer's behavior (supply) specifies the dependence of output and the consumption of inputs on prices.

In the case of the profit-maximizing producer, preferences are linear and the constraint is a convex set, whereas in the consumer model the constraint is a linear inequality and preferences are convex. Structure (i.e., continuity and convexity) is imposed on the profit-maximizing producer's choice set and on the consumer's preferences. Thus, the profit-maximizing producer's problem is similar to the consumer's *dual* problem (see fig. 6.3.). The former involves maximization of a linear function while the latter involves minimization.

Following are some properties of the supply and profit functions which are analogous to those of the consumer's dual problem:

1. $z(\lambda p) = z(p)$. (The producer's preference relation is identical for the price vectors p and λp.)
2. z is continuous.
3. If $z(p) \neq z(p')$, we have: $(p - p')[z(p) - z(p')] = p[z(p) - z(p')] + p'[z(p') - z(p)] > 0$. In particular, if (only) the k'th price increases, then z_k increases; that is, if k is an output ($z_k > 0$), then the supply of k increases and if k is an input ($z_k < 0$), then the demand for k decreases. Note that this result, called *the law of supply*, applies to the standard supply function (unlike the law of demand, which was applied to the compensated demand function).

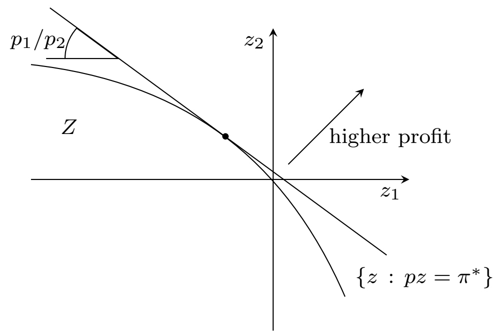

Figure 6.3
Profit maximization.

Following are some properties of the profit function:

1. $\pi(\lambda p) = \lambda\pi(p)$ (follows from $z(\lambda p) = z(p)$).
2. π is continuous (follows from the continuity of the supply function).
3. π is convex (for any p,p' and λ, if z^* maximizes $(\lambda p + (1-\lambda)p')z$, then $\pi(\lambda p + (1-\lambda)p') = \lambda pz^* + (1-\lambda)p'z^* \leq \lambda\pi(p) + (1-\lambda)\pi(p')$).
4. *Hotelling's lemma*: For any vector p^*, $\pi(p) \geq pz(p^*)$ for all p. Therefore, the hyperplane $\{(p,\pi) \mid \pi = pz(p^*)\}$ is tangent to the graph of the function π ($\{(p,\pi) \mid \pi = \pi(p)\}$) at the point $(p^*, \pi(p^*))$. The function π is differentiable (see Kreps (2013)) and $d\pi/dp_k(p^*) = z_k(p^*)$.
5. From Hotelling's lemma, it follows that if π is twice continuously differentiable, then $dz_j/dp_k(p^*) = dz_k/dp_j(p^*)$.

Comment:

When discussing the producer's behavior only in the output market (and not in the input markets) we often represent the producer in a reduced form by means of a **cost function** rather than a technology. For a producer with a technology Z, where commodities $1, .., L$ are inputs and $L+1, \ldots, K$ are outputs, define $c(p, y)$ to be the minimal cost associated with the production of the combination $y \in \mathbb{R}_+^{K-L}$ given the price vector $p \in \mathbb{R}_{++}^L$ of the input commodities $1, \ldots, L$. In other words, $c(p, y) = min_a\{pa| \ (-a, y) \in Z\}$ (see fig. 6.4.).

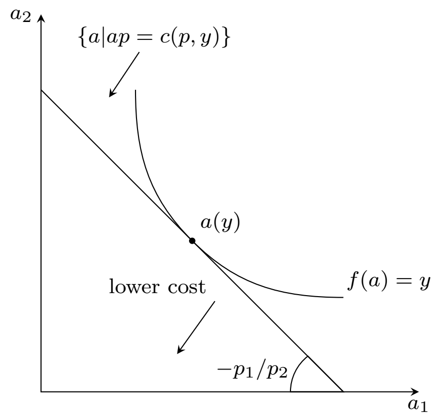

Figure 6.4
Cost Minimization.

Discussion

In the conventional economic approach, we allow the consumer to have "general" preferences but restrict the producer's goals to profit maximization. Thus, a consumer who consumes commodities in order to destroy his health is within the scope of our discussion, whereas a producer who cares about the welfare of his workers or has in mind a target other than profit maximization is not. This is an odd situation since there are various plausible alternative targets for a producer.

One could ask why a producer's objectives are usually defined so narrowly relative to a consumer's preferences. Perhaps it is simply for analytical convenience; I am certain it is not the result of an ideological conspiracy. Nonetheless, is it possible that adopting profit maximization as the "obvious" assumption regarding producer behavior leads students to view it as the exclusive normative criterion guiding a firm's behavior? As a future teacher of Economics should not you worry about it?

Bibliographic Notes

Roy and Hicks are the sources for most of the material in this lecture. Specifically, the concept of the indirect utility function is due to Roy (1942); the concept of the expenditure function is due to Hicks (1946); and the concepts of consumer surplus used in problem 6 are due to Hicks (1939). See also McKenzie (1957). For a full representation of the duality idea, see, for example, Varian (1984) and Diewert (1982).

The model of the profit-maximizing producer can be found in any microeconomics textbook. Debreu (1959) is an excellent source.

In class, I also discuss the ILJK example taken from Rubinstein (2006b)

Problem Set 6

Problem 1. (*Easy*)
Imagine that you are reading a paper in which the author uses the indirect utility function $v(p_1, p_2, w) = w/p_1 + w/p_2$. You suspect that the author's conclusions follow from v being inconsistent with the model of the rational consumer. Take the following steps to make sure that this is not the case:

a. Use Roy's Identity to derive the demand function.
b. Show that if demand is derived from a smooth utility function, then the indifference curve at the point (x_1, x_2) must have the slope $-\sqrt{x_2}/\sqrt{x_1}$.
c. Construct a utility function with the property that the ratio of the partial derivatives at the bundle (x_1, x_2) is $\sqrt{x_2}/\sqrt{x_1}$.
d. Calculate the indirect utility function derived from this utility function. Do you arrive at the original $v(p_1, p_2, w)$? If not, can the original indirect utility function still be derived from another utility function satisfying the property in (c)?

Problem 2. (*Moderately difficult*)
Show that if preferences are monotonic, continuous, and strictly convex, then the Hicksian demand function $h(p, z)$ is continuous.

Problem 3. (*Moderately difficult*)
Consider two definitions of "consumer surplus". Define:

$$CV(p, p', w) = w - e(p', z) = e(p, z) - e(p', z)$$

where $z = x(p, w)$. This is the answer to the question: What is the change in wealth that would be equivalent, from the perspective of (p, w), to the change in the price vector from p to p'? Define also:

$$EV(p, p', w) = e(p, z') - w = e(p, z') - e(p', z')$$

where $z' = x(p', w)$. This is the answer to the question: What is the change in wealth that would be equivalent, from the perspective of (p', w), to the change in the price vector from p to p'?

The following questions refers to ra two-commodity world:

a. For the case of preferences represented by the function $x_1 + x_2$, calculate the two *consumer surplus* measures.
b. Assume that the price of the second commodity is fixed and that the price vectors differ only in the price of the first commodity. Assume

further that the first good is a normal good (demand is increasing in wealth). What is the relation of the two measures to the "area below the demand function between two prices" (which is a third standard definition of consumer surplus)?

c. Explain why the two measures are identical if the individual has quasi-linear preferences in the second commodity and in a domain where the two commodities are consumed in positive quantities.

Problem 4. (*Moderately difficult*)

a. Verify that you are familiar with the envelope theorem, which states conditions under which the following is correct: Consider a maximization problem $\max_x\{u(x, \alpha_1, \ldots, \alpha_n) \mid g(x, \alpha_1, \ldots, \alpha_n) = 0\}$. Let $V(\alpha_1, \ldots, \alpha_n)$ be the value of the maximization.
Then, $\frac{\partial V}{\partial \alpha_i}(a_1, \ldots, a_n) = \frac{\partial(u-\lambda g)}{\partial \alpha_i}(x^*(a_1, \ldots, \alpha_n), a_1, \ldots, \alpha_n)$ where $x^*(a_1, \ldots, \alpha_n)$ is the solution to the maximization problem, and λ is the Lagrange multiplier associated with the solution of the maximization problem.
b. Derive Roy's Identity from the envelope theorem (hint: show that in this context $\frac{\partial V/\partial \alpha_i}{\partial V/\partial \alpha_j}(a_1, \ldots, a_n) = \frac{\partial g/\partial \alpha_i}{\partial g/\partial \alpha_j}(x^*(a_1, \ldots, \alpha_n), a_1, \ldots, \alpha_n))$.
c. What makes it easy to prove Roy's Identity without using the envelope theorem?

Problem 5. (*Easy*)
Assume that technology Z and the production function f describe the same producer who produces commodity K using inputs $1, \ldots, K-1$. Show that Z is a convex set if and only if f is a concave function.

Problem 6. (*Easy*)
Consider a producer who uses L inputs to produce $K-L$ outputs. Denote by w the price vector of the L inputs. Let $a_k(w, y)$ be the demand for the k'th input when the price vector is w and the output vector he wishes to produce is y. Show the following:

a. $C(\lambda w, y) = \lambda C(w, y)$.
b. C is nondecreasing in any input price w_k.
c. C is concave in w.
d. Shepherd's lemma: If C is differentiable, $dC/dw_k(w, y) = a_k(w, y)$ (the k'th input commodity).
e. If C is twice continuously differentiable, then for any two commodities j and k, $da_k/dw_j(w, y) = da_j/dw_k(w, y)$.

Problem 7. (*Moderately difficult*)
Consider a producer producing one commodity using L inputs, which maximizes production subject to the constraint of achieving a level of profit ρ (and does not produce at all if it cannot). Show that under reasonable assumptions:

a. The producer's problem has a unique solution for every price vector.
b. The producer's supply function satisfies monotonicity in prices.
c. The producer's supply function satisfies continuity in prices when $\rho = 0$.
d. The producer's supply function is monotonic in ρ.

Problem 8. (*Moderately difficult. Based on Radner (1993).*)
It is usually assumed that the cost function C is convex in the output vector. Some of the research on production has investigated conditions under which convexity is induced from more primitive assumptions about the production process. Convexity often fails when the product is related to the gathering of information or data processing.

Consider, for example, a firm conducting a telephone survey immediately following a TV program. Its goal is to collect information about as many viewers as possible within 4 units of time. The wage paid to each worker is w (even when he is idle). In one unit of time, a worker can talk to one respondent or be involved in the transfer of information to or from exactly one colleague. At the end of the 4 units of time, the collected information must be in the hands of one colleague (who will announce the results). Define the firm's product, calculate the cost function, and examine its convexity.

Problem 9. (*Standard*)
An event that could have occurred with probability 0.5 either did or did not occur. A firm must provide a report in the form of "the event occurred" or "the event did not occur". The report's quality (the firm's product), denoted by q, is the probability that the report is correct. Each of k experts (input) prepares an independent recommendation that is correct with probability $1 > p > 0.5$. The firm bases its report on the k recommendations in order to maximize q.

a. Calculate the production function $q = f(k)$ for $k = 1, 2, 3$.
b. We say that a " discrete" production function is concave if the sequence of marginal products is nonincreasing. Is the firm's production function concave?

Assume that the firm will get a prize of M if its report is actually correct. Assume that the wage of each worker is w.

c. Explain why it is true that if f is concave, the firm chooses k^* so that the k^*th worker is the last one for whom marginal revenue exceeds the cost of a worker.
d. Is this conclusion true in our case?

Problem 10. (*Moderately difficult*)

An economic agent is both a producer and a consumer. He has a_0 units of good 1. He can use some of a_0 to produce commodity 2. His production function f satisfies monotonicity, continuity, and strict concavity. His preferences satisfy monotonicity, continuity, and convexity. Given that he uses a units of commodity 1 in production, he is able to consume the bundle $(a_0 - a, f(a))$ for $a \leq a_0$. The agent has in "mind" three "centers":

- The *pricing center* declares a price vector (p_1, p_2).
- The *production center* takes the price vector as given and operates according to one of the following two rules:

  Rule 1: profit maximization, $p_2 f(a) - p_1 a$.

  Rule 2: output maximization subject to the constraint of not making any losses, that is, $p_2 f(a) - p_1 a \geq 0$.

  The output of the production center is a consumption bundle.
- The *consumption center* takes $(a_0 - a, f(a))$ as an endowment and finds the optimal consumption allocation that it can afford according to the prices declared by the pricing center.

The prices declared by the pricing center are chosen so as to create harmony between the other two centers, in the sense that the consumption center finds the outcome of the production center's activity, $(a_0 - a, f(a))$, to be optimal given the announced prices.

a. Show that under Rule 1, the economic agent consumes the bundle $(a_0 - a^*, f(a^*))$, which maximizes his preferences.
b. Characterize the economic agent's consumption using Rule 2?
c. State and prove a general conclusion about the comparison between the behavior of two individuals, one whose production center operates under Rule 1 and one whose production center operates under Rule 2.

LECTURE 7

Expected Utility

Lotteries

When thinking about decision making, we often distinguish between actions and consequences. An action is chosen and leads to a consequence. The rational man has preferences over the set of consequences and is meant to choose a feasible action that leads to the most desired consequence. In our discussion of the rational man, we have so far not distinguished between actions and consequences since it was unnecessary in modeling situations where each action deterministically leads to a particular consequence.

In this lecture we will discuss a decision maker in an environment in which the correspondence between actions and consequences is not deterministic but rather *stochastic*. The choice of an action is viewed as choosing a lottery in which the prizes are the consequences. We will be interested in preferences and choices over the set of lotteries.

Let Z be a set of consequences (prizes). In this lecture we assume that Z is a finite set. A *lottery* is a probability measure on Z, that is, a lottery p is a function that assigns a nonnegative number $p(z)$ to each prize z, where $\Sigma_{z \in Z} p(z) = 1$. The number $p(z)$ is taken to be the objective probability of obtaining the prize z given the lottery p.

Denote by $[z]$ the degenerate lottery for which $[z](z) = 1$. We will use the notation $\alpha x \oplus (1-\alpha)y$ to denote the lottery in which the prize x is realized with probability α and the prize y with probability $1-\alpha$.

Denote by $L(Z)$ the (infinite) space containing all lotteries with prizes in Z. Given the set of consequences Z, the space of lotteries $L(Z)$ can be identified with a simplex in Euclidean space: $\{x \in \mathbb{R}^Z_+ | \ \Sigma x_z = 1\}$ where $\mathbb{R}^Z_+$ is the set of functions from Z into $\mathbb{R}_+$. The extreme points of the simplex correspond to the degenerate lotteries, where one prize is received with probability 1. We will discuss preferences over $L(Z)$.

An implicit assumption in the above formulation is that the decision maker does not care about the nature of the random factors but only about the distribution of consequences. To appreciate this point, consider a case in which the probability of rain is $1/2$ and $Z = \{z_1, z_2\}$, where z_1 = "having an umbrella" and z_2 = "not having an umbrella". A "lottery" in which you have z_1 if it is raining and z_2 if it is not should not be considered equivalent to the "lottery" in which you have z_1 if it is not raining and z_2 if it is. Thus, we have to be careful not to apply the model in contexts where the attitude toward the consequence depends on the event realized in each possible contingency.

Preferences

Consider the following examples of "sound" preferences over a space $L(Z)$:

- *Preference for uniformity*: The decision maker prefers the less dispersed lottery where dispersion is measured by $\Sigma_z(p(z) - 1/|Z|)^2$.
- *Preference for greatest likelihood*: The decision maker prefers p to q if $max_z p(z)$ is greater than $max_z q(z)$.
- *The size of the support*: The decision maker evaluates each lottery by the number of prizes that can be realized with positive probability, that is, by the size of the support of the lottery, $supp(p) = \{z|p(z) > 0\}$. He prefers a lottery p over a lottery q if $|supp(p)| \leq |supp(q)|$.

These three examples are degenerate in the sense that the preferences ignore the consequences and are dependent only on the probability vectors. In the following examples, the preferences involve evaluation of the prizes as well:

- *Increasing the probability of a "good" outcome*: The set Z is partitioned into two disjoint sets G and B (good and bad). When comparing two lotteries the decision maker prefers the lottery that yields "good" prizes with higher probability.
- *The worst case*: The decision maker evaluates lotteries by the *worst possible* case. He attaches a number $v(z)$ to each prize z and $p \succsim q$ if $min\{v(z)|\ p(z) > 0\} \geq min\{v(z)|\ q(z) > 0\}$. This type of criterion is often used in computer science, where one algorithm is preferred over another if it functions better in the worst case, independently of the likelihood of the worst case.
- *Comparing the most likely prize*: The decision maker considers the prize in each lottery that is most likely (breaking ties in some

arbitrary way) and compares two lotteries according to a basic preference relation over Z.

- *Lexicographic preferences*: The prizes are ordered $z_1, \ldots, z_K$, and the lottery p is preferred to q if $(p(z_1), \ldots, p(z_K)) \geq_L (q(z_1), \ldots, q(z_K))$.
- *Expected utility*: A number $v(z)$ is attached to each prize, and a lottery p is evaluated according to its expected v, that is, according to $\Sigma_z p(z)v(z)$. Thus,

$$p \succsim q \text{ if } U(p) = \Sigma_{z \in Z} p(z)v(z) \geq U(q) = \Sigma_{z \in Z} q(z)v(z).$$

Note that the above examples can be combined in various ways to form an even richer class of examples. For example, one preference can be employed as long as it is "decisive", and a second to break ties when it is not.

The richness of examples calls for the classification of preference relations over lotteries and the study of properties that these relations satisfy. The methodology we follow is to formally state general principles (axioms) that may apply to preferences over the space of lotteries. Each axiom carries with it a consistency requirement or involves a procedural aspect of decision making. When a set of axioms characterizes a family of preferences, we will consider the set of axioms as justification for focusing on that specific family.

von Neumann and Morgenstern Axiomatization

The version of the von Neumann and Morgenstern axiomatization presented here uses the independence and continuity axioms.

The Independence Axiom

In order to state the first axiom, we require an additional concept, called *compound lotteries* (fig. 7.1): Given a K-tuple of lotteries $(p^k)_{k=1,\ldots,K}$ and a K-tuple of nonnegative numbers $(\alpha_k)_{k=1,\ldots,K}$ that sum up to 1, define $\oplus_{k=1}^K \alpha_k p^k$ to be the lottery for which $(\oplus_{k=1}^K \alpha_k p^k)(z) = \Sigma_{k=1}^K \alpha_k p^k(z)$. Verify that $\oplus_{k=1}^K \alpha_k p^k$ is indeed a lottery. When only two lotteries p^1 and p^2 are involved, we use the notation $\alpha p^1 \oplus (1-\alpha)p^2$.

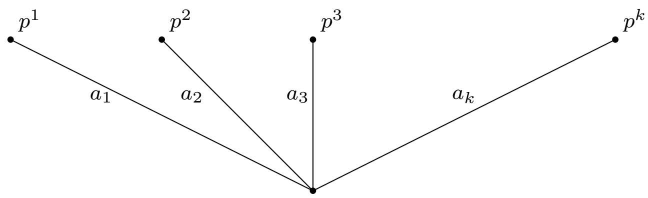

Figure 7.1
The compound lottery $\oplus_{k=1}^{K}\alpha_k p^k$.

We think of $\oplus_{k=1}^{K}\alpha_k p^k$ as a compound lottery with two stages:

Stage 1: It is randomly determined which of the lotteries $p^1, \ldots, p^K$ is realized; α_k is the probability that p^k is realized.

Stage 2: The prize received is randomly drawn from the lottery determined in stage 1.

The random factors in the two stages are taken to be independent. When we compare two compound lotteries, $\alpha p \oplus (1-\alpha)r$ and $\alpha q \oplus (1-\alpha)r$, we tend to simplify the comparison and form our preference on the basis of the comparison between p and q. This intuition is translated into the following axiom:

Independence (I):

For any $p, q, r \in L(Z)$ and any $\alpha \in (0,1)$,

$$p \succsim q \text{ iff } \alpha p \oplus (1-\alpha)r \succsim \alpha q \oplus (1-\alpha)r.$$

Note that the Independence axiom implies both convexity and concavity of the preferences. If $p \succsim q$ then by I we have $p = \alpha p \oplus (1-\alpha)p \succsim \alpha p \oplus (1-\alpha)q \succsim \alpha q \oplus (1-\alpha)q = q$. The first inequality means concavity of $\succsim$ and the second convexity of $\succsim$.

The following property follows from I:

I*:

For every $\{p^k\}_{k=1,\ldots,K}$, a vector of lotteries, q^{k^*} a lottery, and an array of nonnegative numbers $(\alpha_k)_{k=1,\ldots,K}$ such that $\alpha_{k^*} > 0$ and $\sum_k \alpha_k = 1$,

$$\oplus_{k=1}^{K}\alpha_k p^k \succsim \oplus_{k=1}^{K}\alpha_k q^k \text{ when } p^k = q^k \text{ for all } k \text{ but } k^* \text{ iff } p^{k^*} \succsim q^{k^*}.$$

To see that I^* follows from I notice that $p^{k^*} \succsim q^{k^*}$ iff

$$\oplus_{k=1,\dots,K}\alpha_k p^k = \alpha_{k^*}p^{k^*} \oplus (1-\alpha_{k^*})(\oplus_{k\neq k^*}[\alpha_k/(1-\alpha_{k^*})]p^k) \succsim$$

$$\alpha_{k^*}q^{k^*} \oplus (1-\alpha_{k^*})(\oplus_{k\neq k^*}[\alpha_k/(1-\alpha_{k^*})]p^k) = \oplus_{k=1}^{K}\alpha_k q^k.$$

Lemma (monotonicity):
Let $\succsim$ be a preference over $L(Z)$ satisfying I. Let $x, y \in Z$ such that $[x] \succ [y]$ and $1 \geq \alpha > \beta \geq 0$. Then,

$$\alpha x \oplus (1-\alpha)y \succ \beta x \oplus (1-\beta)y.$$

Proof:
If either $\alpha = 1$ or $\beta = 0$, the claim is implied by I. Otherwise, by I, $\alpha x \oplus (1-\alpha)y \succ \alpha y \oplus (1-\alpha)y = [y]$. Using I again we get: $\alpha x \oplus (1-\alpha)y \succ (\beta/\alpha)(\alpha x \oplus (1-\alpha)y) \oplus (1-\beta/\alpha)[y] = \beta x \oplus (1-\beta)y$.

The Continuity Axiom
Once again we employ a continuity assumption that is basically the same as the one employed for the consumer. Here, continuity means that the preferences are not overly sensitive to small changes in the probabilities (and could be sensitive to the changes in the prizes).

Continuity (C):
If $p \succ q$, then there are neighborhoods $B(p)$ of p and $B(q)$ of q (when the lotteries are presented as vectors in $\mathbb{R}_+^{|Z|}$), such that:

for all $p' \in B(p)$ and $q' \in B(q), p' \succ q'$.

Verify that the continuity assumption implies the following property, which sometimes is presented as an alternative definition of continuity:

C*:
If $p \succ q \succ r$, then there exists $\alpha \in (0,1)$ such that

$$q \sim \alpha p \oplus (1-\alpha)r.$$

We will now check whether some of the examples discussed earlier satisfy these two axioms.

- *Expected utility*: Note that the function $U(p)$ is linear:

$$\begin{aligned} U(\oplus_{k=1}^{K}\alpha_k p^k) &= \sum_{z\in Z}[\oplus_{k=1}^{K}\alpha_k p^k](z)v(z) = \sum_{z\in Z}\left[\sum_{k=1}^{K}\alpha_k p^k(z)\right]v(z) \\ &= \sum_{k=1}^{K}\alpha_k\left[\sum_{z\in Z}p^k(z)v(z)\right] = \sum_{k=1}^{K}\alpha_k U(p^k). \end{aligned}$$

  It follows that any such preference relation satisfies I. Since the function $U(p)$ is continuous in the probability vector, it also satisfies C.
- *Increasing the probability of a "good" consequence*: Such a preference relation satisfies the two axioms since it can be represented by the expectation of v where $v(z) = 1$ for $z \in G$ and $v(z) = 0$ for $z \in B$.
- *Preferences for the greatest likelihood*:
  This preference relation is continuous (since the function $max_{z\in Z}\{p(z)\}$ that represents it is continuous in probabilities). It does not satisfy I since, for example, although $[z_1] \sim [z_2]$, $[z_1] = 1/2[z_1] \oplus 1/2[z_1] \succ 1/2[z_1] \oplus 1/2[z_2]$.
- *Lexicographic preferences*: Satisfies I but not C (verify).
- *The worst case*: The preference relation does not satisfy C. In the two-prize case where $v(z_1) > v(z_2)$, $[z_1] \succ 1/2[z_1] \oplus 1/2[z_2]$. Viewed as points in $\mathbb{R}^2_+$, we can rewrite this as $(1, 0) \succ (1/2, 1/2)$. Any neighborhood of $(1, 0)$ contains lotteries that are not strictly preferred to $(1/2, 1/2)$, and thus C is not satisfied. The preference relation also does not satisfy I ($[z_1] \succ [z_2]$ but $1/2[z_1] \oplus 1/2[z_2] \sim [z_2]$.)

Utility Representation

By Debreu's theorem, for any relation $\succsim$ defined on the space of lotteries that satisfies C, there is a utility representation U: $L(Z) \to \mathbb{R}$, continuous in the probabilities, such that $p \succsim q$ iff $U(p) \geq U(q)$. But the theorem does not imply structure on the function U. We will use now both C and I to characterize a family of preference relations that have a representation by a utility function with the "expected utility" structure.

Theorem (vNM):

Let $\succsim$ be a preference relation over $L(Z)$ satisfying I and C. There are numbers $(v(z))_{z \in Z}$ such that:

$p \succsim q$ iff $U(p) = \Sigma_{z \in Z} p(z)v(z) \geq U(q) = \Sigma_{z \in Z} q(z)v(z)$.

Note the distinction between $U(p)$ (the utility number of the lottery p) and $v(z)$ (called a Bernoulli number or a vNM utility). The function v is a utility function representing the preferences on Z and it is the building block for the construction of $U(p)$, a utility function representing the preferences on $L(Z)$. We often refer to v as a vNM utility function representing the preferences $\succsim$ over $L(Z)$.

Proof:

Let $[M]$ and $[m]$ be a best and a worst degenerate lotteries in $L(Z)$.

Consider first the case in which $[M] \sim [m]$. By transitivity for every $z \in Z$ we have $[z] \sim [m]$. It follows from I^* that $p \sim [m]$ for any p and thus $p \sim q$ for all p,$q \in L(Z)$. Thus, any constant utility function represents $\succsim$. Choosing $v(z) = 0$ for all z, we have $\Sigma_{z \in Z} p(z)v(z) = 0$ for all $p \in L(Z)$.

Now consider the case in which $[M] \succ [m]$. By C^* and the lemma, there is a single number $v(z) \in [0, 1]$ such that $v(z)M \oplus (1-v(z))m \sim [z]$. (In particular, $v(M) = 1$ and $v(m) = 0$). By I^* we obtain that:

$p \sim (\Sigma_{z \in Z} p(z)v(z))M \oplus (1 - \Sigma_{z \in Z} p(z)v(z))m$.

And, by the lemma, $p \succsim q$ iff $\Sigma_{z \in Z} p(z)v(z) \geq \Sigma_{z \in Z} q(z)v(z)$.

The Uniqueness of vNM Utilities

The vNM utilities are unique up to positive affine transformation (namely, multiplication by a positive number and adding any scalar) but are not invariant to arbitrary monotonic transformation. Consider a preference relation $\succsim$ defined over $L(Z)$ and let $v(z)$ be the vNM utilities representing the preference relation. Of course, by defining $w(z) = \alpha v(z) + \beta$ for all z (for some $\alpha > 0$ and some β), the utility function $W(p) = \Sigma_{z \in Z} p(z)w(z)$ also represents $\succsim$.

Furthermore, assume that $W(p) = \Sigma_z p(z)w(z)$ represents the preferences $\succsim$ as well. We will show that w must be a positive affine transformation of v. It is trivially true if $\succsim$ is the total indifference. To see that this is true when $[M] \succ [m]$, let $\alpha > 0$ and β satisfy:

$w(M) = \alpha v(M) + \beta \quad \text{and} \quad w(m) = \alpha v(m) + \beta$

(the existence of $\alpha > 0$ and β is guaranteed by $v(M) > v(m)$ and $w(M) > w(m)$). For any $z \in Z$, there is $\lambda \in [0,1]$ such that $[z] \sim \lambda M \oplus (1-\lambda)m$, and therefore it must be that

$$\begin{aligned} w(z) &= \lambda w(M) + (1-\lambda)w(m) \\ &= \lambda[\alpha v(M) + \beta] + (1-\lambda)[\alpha v(m) + \beta] \\ &= \alpha[\lambda v(M) + (1-\lambda)v(m)] + \beta \\ &= \alpha v(z) + \beta. \end{aligned}$$

The Dutch Book Argument

There are those who consider expected utility maximization to be a normative principle. One of the arguments made to support this view is the following Dutch Book argument: Consider a decision maker for whom $L_1 \succ L_2$ and $\alpha L \oplus (1-\alpha)L_2 \succ \alpha L \oplus (1-\alpha)L_1$. The decision maker is vulnerable to the following trick who offers him a sequence of "exchanges":

1. Take $\alpha L \oplus (1-\alpha)L_1$ (we can describe this as a contingency with random event E, which we both agree has probability $1-\alpha$).
2. Take instead $\alpha L \oplus (1-\alpha)L_2$, which you prefer (and pay something...).
3. Let us agree to replace L_2 with L_1 in the case that E occurs (and you pay something...).
4. Note that you hold $\alpha L \oplus (1-\alpha)L_1$.
5. Start over again ...

A Discussion of the Plausibility of the vNM Theory

Many experiments have revealed systematic deviations from vNM assumptions. The most famous of them is the *Allais paradox*. One version of it (taken from Kahneman and Tversky (1979)) is the following:

Choose first between:
$L_1 = 0.25[3,000] \oplus 0.75[0]$ and $L_2 = 0.2[4,000] \oplus 0.8[0]$
and then choose between:
$L_3 = 1[3,000]$ and $L_4 = 0.8[4,000] \oplus 0.2[0]$.

Note that $L_1 = 0.25L_3 \oplus 0.75[0]$ and $L_2 = 0.25L_4 \oplus 0.75[0]$. Axiom I requires that the preference between L_1 and L_2 be in the same direction that between L_3 and L_4. However, in experiments a majority of people express the preferences $L_1 \prec L_2$, an even larger majority express the preferences $L_3 \succ L_4$. This phenomenon persists even among graduate students in economics. Among about 228 graduate students at Princeton, Tel Aviv, and New York Universities, and although they were asked to respond to the above two choice problems one after the other, 68% chose L_2 while 78% chose L_3. This means that at least 46% of the students violated property I.

The Allais example demonstrates (again) the sensitivity of preferences to the framing of the alternatives. When the lotteries L_1 and L_2 are presented as regular lotteries, most subjects prefer L_2. But, if we present L_1 and L_2 as the compound lotteries $L_1 = 0.25L_3 \oplus 0.75[0]$ and $L_2 = 0.25L_4 \oplus 0.75[0]$, most subjects prefer L_1 to L_2.

Comment:

In the proof of the vNM theorem we have seen that the independence axiom implies that if one is indifferent between z and z', one is also indifferent between z and any lottery with z and z' as its prizes. This is not plausible in cases where the fairness of the random process that selects the prizes it taken into account. For example, consider a parent who one gift and two children, M and Y. His options are to choose a lottery $L(p)$ that will award M the gift with probability p and Y with probability $1-p$. The parent does not favor one child over the other. The vNM approach "predicts" that he will be indifferent among all lotteries that determine who receives the gift, while it seems that most of us would strictly prefer $L(1/2)$.

Subjective Expected Utility (de Finetti's)

In the above discussion, a lottery was a description of the probabilities with which each of the prizes is obtained. In many contexts, an alternative induces an uncertain consequence that depends on certain events though the probabilities of those events are not given. The attitude of the decision maker to an alternative will depend on his assessment of the likelihoods of those events. In this section, we will demonstrate the basic idea of eliciting probabilities from preferences. The seminal work in this area is Savage's model. However, Savage's axiomatization is quite complicated, and we will make do here with a very simple model (due to de Finetti) that demonstrates an important aspect of the approach.

In the following, the notion of a lottery is replaced by a notion of a bet. Think about someone betting on a race with K horses (and, needless to say, the list of horses stands for any exhaustive list of exclusive events). A bet is a vector $(x_1, \ldots, x_K)$ with the interpretation that if horse k wins, the decision maker receives x_k units of money (x_k can be any real number). Let B be the set of all bets.

We will consider three properties of a preference relation on B:

- Continuity: This is the standard continuity property used on the Euclidean space.
- Weak Monotonicity: If $x_k > y_k$ for all k, then $x \succ y$.
- Additivity: If $x \succsim y$, then $x + z \succsim y + z$ for all z. (Note that this implies that if $x \succ y$, then $x + z \succ y + z$ for all z.)

A possible interpretation of the additivity property is as follows: Assume that the wealth of the decision maker has two components: One of them, z, is independent of the choice between the different bets. The other depends on the bet chosen: x or y. Additivity states that the attitude of the decision maker to the bets x and y is independent of z.

Additivity is a strong assumption since the equivalence of the preference between $x + z$ and $y + z$ and that between x and y is required to hold for *all* z including vectors where z_k (the "initial wealth" of the agent if horse k wins) depends on k. A weaker and more plausible assumption would require the equivalence to hold only in cases where z is a constant vector, but such an assumption would be too weak to drive the claim (for example, a preference relation represented by $u(x) = \max\{x_k\}$ satisfies the three axioms but cannot be represented as maximizing expected payoff for any vector of probabilities).

Claim:

A preference relation $\succsim$ satisfies Continuity, Weak Monotonicity, and Additivity if and only if there is a vector $(\pi_1, .., \pi_K)$ of non-negative numbers which sum up to 1 such that $x \succsim y$ if and only if $\sum \pi_k x_k \geq \sum \pi_k y_k$.

The number π_k in the above claim is generally interpreted as the subjective probability assigned by the agent to the event that horse k wins the race (or that state k is realized). In my opinion, this interpretation goes too far. The model does not test whether the numbers we call "subjective probabilities" are indeed probabilities in the sense that the probability assigned by the agent to the event in which one of the horses in a set H wins is $\sum_{k \in H} \pi_k$. In order for it to do so, we would have to extend the set of bets to any contract specifying a prize $x(H_l)$ for the event of one of the horses in H_l winning the race, where $\{H_l\}$ is a partition of the set $\{1, .., K\}$. In that case, we would need additional assumptions in order to guarantee that the number assigned by the agent to H_l will be exactly $\sum_{k \in H_l} \pi_k$ as required if the coefficients $\{\pi_k\}$ are indeed to be interpreted as subjective probability numbers.

Proof:

You probably have already proved this claim for $K = 2$ (see Problem Set 4 Question 7). We will prove it now for an arbitrary K, using another common technique:

A preference relation represented by $\sum \pi_k x_k$ obviously satisfies all three properties.

In the other direction, assume that $\succsim$ satisfies the three properties. First, consider the two sets $U = \{x|\ x \succsim 0\}$ and $D = \{x|\ 0 \succ x\}$. Both are nonempty. By continuity, U is closed and D is open. Note that if $x \succsim 0$ and $y \succsim 0$, then by Additivity $x + y \succsim y \succsim 0$ and if $x \prec\ 0$ and $y \prec\ 0$, then by Additivity $x + y \prec y \prec 0$. It follows that if $x \succsim 0$, then for all $\lambda = m/2^n$ (m and n natural numbers) we have $\lambda x \succsim 0$ and since the dyadic numbers are dense, Continuity implies that $\lambda x \succsim 0$ for all λ. Thus, if $x \succsim 0$ and $y \succsim 0$, then $\lambda x \succsim 0$, $(1 - \lambda)y \succsim 0$, and by Additivity also $\lambda x + (1 - \lambda)y \succsim 0$, that is, U is convex. Similarly, D is convex. By the definition of a preference relation, the sets U and D provide a partition of $\mathbb{R}^K$, that is, $U \cup D = \mathbb{R}^K$ and $U \cap D = \emptyset$.

Now use a separation theorem to conclude that there exists a non-zero vector $\pi = (\pi_1, \ldots, \pi_K)$ and a number c such that $U \subseteq \{x|\pi x \geq c\}$ and $D \subseteq \{x|\pi x \leq c\}$. Since U and D partition R^K it must be that $\{x|\pi x > c\} \subseteq\ U$ and by continuity $\{x|\pi x \geq c\} = U$ and therefore $D = \{x|\pi x < c\}$. Since $0 \in U$ it must be that $c \leq 0$. If $c < 0$ then $\pi(-\varepsilon, .., -\varepsilon) > c$ for $\varepsilon > 0$ small enough although by Weak Monotonicity $0 \succ$(-$\varepsilon, .., -\varepsilon$). Thus, $c = 0$. Also by Weak Monotonicity $\pi_k \geq 0$ for all k and since $\pi \neq 0$ we can assume that $\sum \pi_k = 1$.

Finally, $x \succsim y$ if and only if $x - y \succsim 0$ if and only if $\pi(x - y) \geq 0$ if and only if $\pi x \geq \pi y$.

Bibliographic Notes

Expected utility theory is based on von Neumann and Morgenstern (1944). Kreps (1988) has an excellent presentation of the material. For a recent survey of theories of decision making under uncertainty, see Gilboa (2009). Machina (1987) remains a recommended survey of alternative theories. Kahneman and Tversky (1979) is a must read for a psychological criticism of expected utility theory (see also Kahneman and Tversky (2000)).

Problem Set 7

Problem 1. (*Standard*)
Consider two preference relations that were described in the text: "the size of the support" and "comparing the most likely prize".

a. Check carefully whether they satisfy axioms I and C.
b. These preference relations are not immune to a certain "framing problem". Explain.

Problem 2. (*Standard. Based on Markowitz (1959).*)
One way to construct preferences over lotteries with monetary prizes is by evaluating each lottery L on the basis of two numbers: $Ex(L)$, the expectation of L, and $var(L)$, L's variance. Such a construction may or may not be consistent with vNM assumptions.

a. Show that the function $u(L) = Ex(L) - (1/4)var(L)$ induces a preference relation that is not consistent with the vNM assumptions. (For example, consider the mixtures of each of the lotteries [1] and $0.5[0] \oplus 0.5[4]$ with the lottery $0.5[0] \oplus 0.5[2]$.)
b. Show that the utility function $u(L) = Ex(L) - (Ex(L))^2 - var(L)$ is consistent with vNM assumptions.

Problem 3. (*Easy*)
A decision maker has a preference relation $\succsim$ over the space of lotteries $L(Z)$ with a set of prizes Z. On Sunday he learns that on Monday he will be told whether he has to choose between L_1 and L_2 (probability $1 > \alpha > 0$) or between L_3 and L_4 (probability $1 - \alpha$). He will make his choice at that time.

Consider two possible approaches the decision maker can take:

Approach 1: He delays his decision to Monday ("why bother with the decision now when I can make up my mind tomorrow ...").

Approach 2: He makes a contingent decision on Sunday regarding what he will do on Monday, that is, he decides what to do if he faces the choice between L_1 and L_2 and what to do if he faces the choice between L_3 and L_4 ("On Monday morning I will be too busy ...").

a. Formulate Approach 2 as a choice between lotteries.
b. Show that if the preferences of the decision maker satisfy the independence axiom, then his choice under Approach 2 will always be the same as under Approach 1.

Problem 4. (*Standard*)
A decision maker is to choose an action from a set A. The set of consequences is Z. For every action $a \in A$, the consequence z^* is realized with probability α, and any $z \in Z - \{z^*\}$ is realized with probability $r(a, z) = (1 - \alpha)q(a, z)$.

a. Assume that after making his choice he is told that z^* will not occur and is given a chance to change his decision. Show that if the decision maker obeys the Bayesian updating rule and follows vNM axioms, he will not change his decision.
b. Give an example where a decision maker who follows non-expected utility preference is not time-consistent.
c. Give an example where a decision maker who does not obey a Bayesian updating rule is not time-consistent.

Problem 5. (*Standard*)
Assume there is a finite number of income levels. An income distribution specifies the proportion of individuals at each level. Thus, an income distribution has the same mathematical structure as a lottery. Consider the binary relation "one distribution is more egalitarian than another".

a. Why is the von Neumann–Morgenstern independence axiom inappropriate for characterizing this type of relation?
b. Suggest and formulate a property that is appropriate, in your opinion, as an axiom for this relation. Give two examples of preference relations that satisfy this property.

Problem 6. (*Difficult. Based on Miyamoto, Wakker, Bleichrodt, and Peters (1998).*)
A decision maker faces a trade-off between longevity and quality of life. His preference relation ranks lotteries on the set of all certain outcomes of the form (q, t) defined as "a life of quality q and length t" (where q and t are nonnegative numbers). Assume that the preference relation satisfies von Neumann–Morgenstern assumptions and that it also satisfies the following:

1. There is indifference between any two lotteries $[(q, 0)]$ and $[(q', 0)]$.
2. Risk neutrality with respect to life duration: An uncertain lifetime of expected duration T is equally preferred to a certain lifetime duration T when q is held fixed.
3. Regardless of the quality of life, the longer the longer one lives the better.

a. Show that the preference relation derived from maximizing the expectation of the function $v(q)t$, where $v(q) > 0$ for all q, satisfies the assumptions.

b. Show that all preference relations satisfying the above assumptions can be represented by an expected utility function of the form $v(q)t$, where v is a positive function.

Problem 7. (*Food for thought*)
Consider a decision maker who systematically calculates that $2+3=6$. Construct a "money pump" argument against him. Discuss the argument.

Problem 8. (*Standard*)
Let A be a finite set of activities. A day's schedule is a finite list of the type: "from 0 to t_1 do activity a_1 then for a length of time t_2 do activity a_2, and so on". The individual can split his day to as many parts as he wishes. He holds preferences over schedules of a unit of time (a day). Define the mixture $\alpha s_1 \oplus (1-\alpha)s_2$ where s_1 and s_2 are schedules and $\alpha \in [0,1]$ to be the schedule in which: between time 0 to α the schedule s_1 is squeezed proportionally while from α to the end of the day the schedule s_2 is squeezed proportionally.

a. Formulate the concept of a schedule.
b. State **two** properties of the preference relation which are not associated with continuity and are necessary for the preferences to be represented by a function of the type
U(s)=$\sum_{a\in A} v(a) \times$ [the total time spent on activity a in schedule s]
c. Suggest an example of a reasonable preference relation that cannot be represented by this functional form.

LECTURE 8

Risk Aversion

Lotteries with Monetary Prizes

We proceed to a discussion of a decision maker's "expected utility preferences" for the case that the space of prizes Z is a set of real numbers and $a \in Z$ is interpreted as a "the sum of money $\$a$" In Lecture 7 we assumed the set Z is finite; here, in contrast, we apply the expected utility approach to an infinite set. for simplicity, we will still consider only lotteries with finite support. In other words, in this lecture, a lottery p is a real function on Z such that $p(z) \geq 0$ for all $z \in Z$, and there is a finite set Y such that $\sum_{z \in Y} p(z) = 1$. It is possible to extend the axiomatization presented in Lecture 7 to cover this case but we do not do it here.

We make special assumptions that fit the interpretation of the members of Z as sums of money. Recall that$[x]$ denotes the lottery which yields the prize x with certainty. We will assume that $\succsim$ satisfies *monotonicity:* if $a > b$ then $[a] \succ [b]$.

From hereon the discussion focuses on preference relations over the space of lotteries for which there is a continuous function u, such that the preference relation over lotteries is represented by the function $Eu(p) = \sum_{z \in Z} p(z)u(z)$. The function Eu assigns to the lottery p the expectation of the random variable that receives the value $u(x)$ with a probability $p(x)$.

The following argument, called the *St. Petersburg Paradox*, is sometimes presented as a justification for assuming that vNM utility functions are bounded. Assume that a decision maker has an unbounded vNM utility function u. Consider playing the following "trick" on him:

1. Assume he possesses wealth x_0.
2. Offer him a lottery that will reduce his wealth to 0 with probability $1/2$ and will increase his wealth to x_1 with probability $1/2$ so that $u(x_0) < [u(0) + u(x_1)]/2$. By the unboundedness of u, there exists such an x_1.
3. If he loses, you are happy. If he is lucky, a moment before you give him x_1, offer him a lottery that will give him x_2 with probability $1/2$ and 0 otherwise, where x_2 is such that $u(x_1) < [u(0) + u(x_2)]/2$.
4. And so on ...

Our (poor) decision maker will find himself with wealth 0 with probability 1!

First-Order Stochastic Dominance

We say that p *first-order stochastically dominates* q (written as pD_1q) if $p \succsim q$ for any $\succsim$ on $L(Z)$ satisfying vNM assumptions as well as monotonicity in money. That is, pD_1q if $Eu(p) \geq Eu(q)$ for all increasing u. This is the simplest example of questions of the type: "Given a set Λ of preference relations on $L(Z)$, for what pairs $p, q \in L(Z)$ is $p \succsim q$ for all $\succsim$ in Λ ?" In the problem set you will find another example of this kind of question.

Obviously, pD_1q if the entire support of p is to the right of the entire support of q. But we are looking for a more interesting condition on a pair of lotteries p and q, one that will not only be sufficient but also necessary for p to first-order stochastically dominate q.

For any lottery p and a number x, define $G(p,x) = \sum_{z \geq x} p(z)$ (the probability that the lottery p yields a prize at least as large as x). Denote by $F(p,x)$ the cumulative distribution function of p, that is, $F(p,x) = \sum_{z \leq x} p(z)$.

Claim:

pD_1q iff for all x, $G(p,x) \geq G(q,x)$ (alternatively, pD_1q iff for all x, $F(p,x) \leq F(q,x)$). (See fig. 8.1.)

Proof:

Let $x_0 < x_1 < x_2 < \ldots < x_K$ be the prizes in the union of the supports of p and q. First, note the following alternative expression for $Eu(p)$:

$$Eu(p) = \sum_{k \geq 0} p(x_k)u(x_k) = u(x_0) + \sum_{k \geq 1} G(p, x_k)(u(x_k) - u(x_{k-1})).$$

The intuition for this equality is simple: $u(x_0)$ is added to the calculation of $Eu(p)$ with a weight of 1. The positive difference $u(x_1) - u(x_0)$ is added with weight $G(p, x_1)$ and so on,

Now, if $G(p, x_k) \geq G(q, x_k)$ for all k, then for all increasing u,

$$Eu(p) = u(x_0) + \sum_{k \geq 1} G(p, x_k)(u(x_k) - u(x_{k-1})) \geq$$
$$u(x_0) + \sum_{k \geq 1} G(q, x_k)(u(x_k) - u(x_{k-1})) = Eu(q).$$

Conversely, if there exists k^* for which $G(p, x_{k^*}) < G(q, x_{k^*})$, then we can find an increasing function u so that $Eu(p) < Eu(q)$, by setting $u(x_{k^*}) - u(x_{k^*-1}) = 1$ and the other increments to be very small.

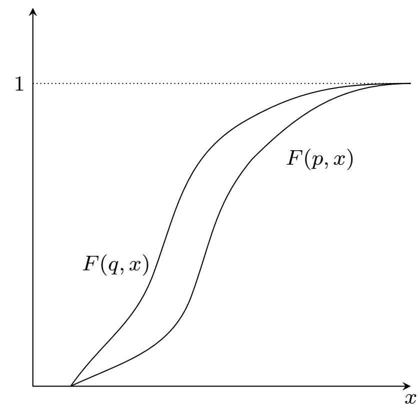

Figure 8.1
p first-order stochastically dominates q.

Risk Aversion

Let $E(p)$ be the expectation of the lottery p, that is, $E(p) = \sum_{z \in Z} p(z)z$. We say that $\succsim$ is *risk averse* if for any lottery p, $[E(p)] \succsim p$. (This is a limited definition as it applies only to the case that the prizes are one dimensional.)

We will see now that for a decision maker with preferences $\succsim$ obeying the vNM axioms, risk aversion is equivalent to the concavity of the vNM utility function representing $\succsim$.

First recall some basic properties of concave functions:

1. An increasing (concave) function must be continuous (but not necessarily differentiable).
2. The *Jensen Inequality*: For any finite sequence $(\alpha_k)_{k=1,\ldots,K}$ of positive numbers that sum up to 1, $u(\sum_{k=1}^{K} \alpha_k x_k) \geq \sum_{k=1}^{K} \alpha_k u(x_k)$.
3. The *Three Strings Lemma*: For any $a < b < c$, we have $[u(c) - u(b)]/(c - b) \leq [u(c) - u(a)]/(c - a) \leq [u(b) - u(a)]/(b - a)$.
4. If u is twice differentiable, then for any $a < c$, $u'(a) \geq u'(c)$, and thus $u''(x) \leq 0$ for all x.

Claim:

Let $\succsim$ be a preference on $L(Z)$ represented by the vNM utility function u. The preference relation $\succsim$ is risk averse iff u is concave.

Proof:

Assume that u is concave. By the Jensen Inequality, for any lottery p, $u(E(p)) \geq Eu(p)$ and thus $[E(p)] \succsim p$.

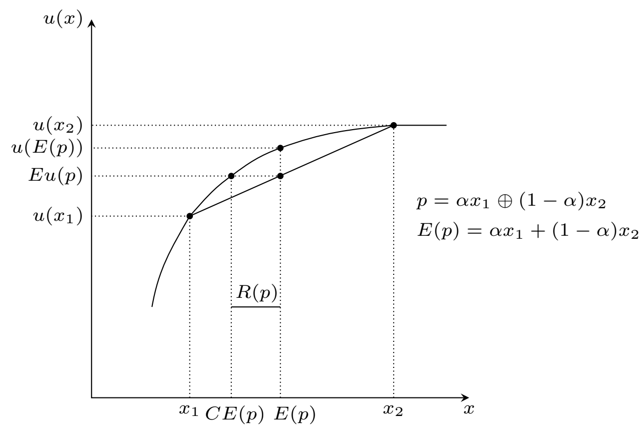

Figure 8.2
CE and risk premium.

Assume that $\succsim$ is risk averse and that Eu represents $\succsim$. For all $\alpha \in (0, 1)$ and for all $x, y \in Z$, we have by risk aversion $[\alpha x + (1-\alpha)y] \succsim \alpha x \oplus (1-\alpha)y$ and thus $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$, that is, u is concave.

Certainty Equivalence and the Risk Premium

Given a preference relation $\succsim$ over the space $L(Z)$, the *certainty equivalence* of a lottery p, denoted by $CE(p)$, is a prize satisfying $[CE(p)] \sim p$. (Verify that the existence of $CE(p)$ is guaranteed by assuming that $\succsim$ is monotonic (in the sense that for any p there are $M > m$ such that $[M] \succ p \succ [m]$) and is continuous (in the sense that the sets $\{c \in \mathbb{R} \mid [c] \succ p\}$ and $\{c \in \mathbb{R} \mid p \succ [c]\}$ are open). The *risk premium* of p is the difference $R(P) = E(p) - CE(p)$. By definition, the preferences are risk averse if and only if $R(p) \geq 0$ for all p. (See fig. 8.2.)

The "More Risk Averse" Relation

We wish to understand the natural statement: "A is more war averse than B". One possible meaning is that whenever A is ready to go to war, B is as well. Another possible meaning is that when facing the threat of war, A is ready to agree to a less attractive compromise than B is. Note that the assumption that A and B share the same concepts of "war" and "peace" and rank the attractiveness of compromises identically is implicit in these interpretations. The following two definitions of "one decision maker *is more risk averse* than another" are analogous to these two interpretations.

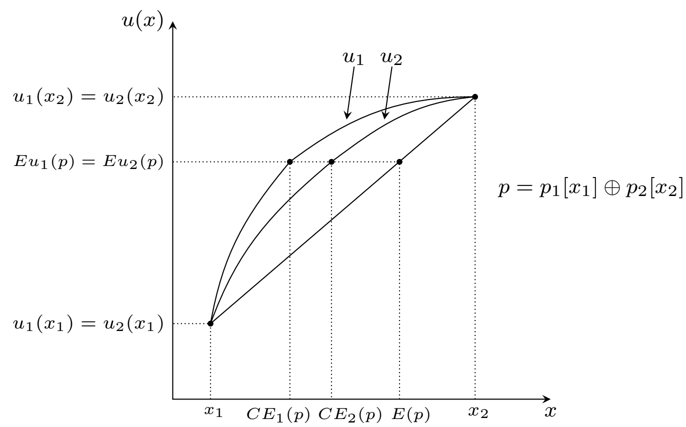

Figure 8.3
1 is more risk averse than 2.

1. The preference relation $\succsim_1$ *is more risk averse than* $\succsim_2$ if, for any lottery p and degenerate lottery c, $p \succsim_1 c$ implies that $p \succsim_2 c$.

In the case that the prizes are monetary and preferences on the prizes are monotonic, we have a second definition:

2. The preference relation $\succsim_1$ *is more risk averse than* $\succsim_2$ if $CE_2(p) \geq CE_1(p)$ for all p (see Figure 8.3).

In the case that the preferences satisfy vNM assumptions, we arrive at a third definition:

3. Assume Eu_1 and Eu_2 represent $\succsim_1$ and $\succsim_2$, respectively. The preference relation $\succsim_1$ is more risk averse than $\succsim_2$ if the function $\varphi = u_1 \cdot \mathrm{u}_2^{-1}$ defined by $u_1(t) = \varphi(u_2(t))$, is concave.

Note that definition (1) is meaningful in any space of prizes (not only those in which consequences are numerical) and for a general set of preferences (and not only those satisfying vNM assumptions).

Claim:

If both $\succsim_1$ and $\succsim_2$ are preference relations on $L(Z)$ represented by increasing and continuous vNM utility functions, then the three definitions are equivalent.

Proof:

If (2) then (1): Consider a lottery p and a number c such that $p \succsim_1 [c]$. By transitivity $[CE_1(p)] \succsim_1 [c]$ and by the monotonicity of $\succsim_1$ we have $CE_1(p) \geq c$, which implies by (2) that $CE_2(p) \geq c$ and thus $p \sim_2 [CE_2(p)] \succsim_2 [c]$ and by transitivity of $\succsim_2$, $p \succsim_2 [c]$.

If (3) then (2): By definition, $Eu_i(p) = u_i(CE_i(p))$. Thus, $CE_i(p) = u_i^{-1}(Eu_i(p))$. If $\varphi = u_1 u_2^{-1}$ is concave, then by the Jensen Inequality:
$u_1(CE_2(p)) = u_1(u_2^{-1}(Eu_2(p)) = \varphi(\sum_x p(x)u_2(x)) \geq$
$\sum_x p(x)\varphi(u_2(x)) = \sum_x p(x)u_1(x) = E(u_1(p)) = u_1(CE_1(p))$.
Since u_1 is increasing, $CE_2(p) \geq CE_1(p)$.

If (1) then (3). Consider three numbers $u_2(x) < u_2(y) < u_2(z)$ in the range of u_2 and let $\lambda \in (0,1)$ satisfy $u_2(y) = \lambda u_2(x) + (1-\lambda)u_2(z)$. For proving the concavity of φ we need to show that $\varphi(u_2(y)) \geq \lambda\varphi(u_2(x)) + (1-\lambda)\varphi(u_2(z))$, that is, $u_1(y) \geq \lambda u_1(x) + (1-\lambda)u_1(z)$. If $u_1(y) < \lambda u_1(x) + (1-\lambda)u_1(z)$, then for some $\mu > \lambda$ we have $u_1(y) < \mu u_1(x) + (1-\mu)u_1(z)$ and $u_2(y) > \mu u_2(x) + (1-\mu)u_2(z)$, that is, $y \prec_1 \mu x \oplus (1-\mu)z$ and $y \succ_2 \mu x \oplus (1-\mu)z$, which contradicts (1).

Note that the first two definitions are equivalent under the more general conditions that the preference relations share the preferences over the certain prizes and the certainty equivalences are well-defined for all lotteries.

The Coefficient of Absolute Risk Aversion

The following is another definition of the relation "more risk averse" applied to the case in which vNM utility functions are twice differentiable:

4. Let Eu_1 and Eu_2 be utility functions representing $\succsim_1$ and $\succsim_2$, respectively where u_1 and u_2 are twice differentiable functions. The preference relation $\succsim_1$ *is more risk averse than* $\succsim_2$ if $r_1(x) \geq r_2(x)$ for all x, where $r_i(x) = -u_i''(x)/u_i'(x)$.

The number $r(x) = -u''(x)/u'(x)$ is called the *coefficient of absolute risk aversion* of u at x. We will see that a higher coefficient of absolute risk aversion means a more risk-averse decision maker.

To see that (3) and (4) are equivalent, note the following chain of equivalences:

- (3) (i.e., $u_1 u_2^{-1}$ is concave) is satisfied iff
- the function $d/dt[u_1(u_2^{-1}(t))]$ is nonincreasing in t iff
- $u_1'(u_2^{-1}(t))/u_2'(u_2^{-1}(t))$ is nonincreasing in t (since $(\varphi^{-1})'(t) = 1/\varphi'(\varphi^{-1}(t))$) iff

- $u_1'(x)/u_2'(x)$ is nonincreasing in x (since $u_2^{-1}(t)$ is increasing in t) iff
- $log\,[(u_1'/u_2')(x)] = log\,u_1'(x) - log\,u_2'(x)$ is nonincreasing in x iff
- the derivative of $log\,u_1'(x) - log\,u_2'(x)$ is nonpositive iff
- $r_2(x) - r_1(x) \le 0$ for all x where $r_i(x) = -u_i''(x)/u_i'(x)$ iff
- (4) is satisfied.

For a better understanding of the coefficient of absolute risk aversion, let us look at the preferences on the restricted domain of lotteries of the type $(x_1, x_2) = px_1 \oplus (1-p)x_2$, where the probability p is fixed. Denote by u a continuously differentiable vNM utility function that represents a risk-averse preference.

Let $x_2 = \psi(x_1)$ be the function describing the indifference curve through (t, t), the point representing $[t]$. Thus, $\psi(t) = t$.

It follows from risk aversion that all lotteries with expectation t, that is, all lotteries on the line $\{(x_1, x_2)|\ px_1 + (1-p)x_2 = t\}$, are not above the indifference curve through (t, t). Thus, $\psi'(t) = -p/(1-p)$.

By definition of u as a vNM utility function representing the preferences over the space of lotteries, we have $pu(x_1) + (1-p)u(\psi(x_1)) = u(t)$. Taking the derivative with respect to x_1, we obtain $pu'(x_1) + (1-p)u'(\psi(x_1))\psi'(x_1) = 0$. Taking the derivative with respect to x_1 again, we obtain

$$pu''(x_1) + (1-p)u''(\psi(x_1))[\psi'(x_1)]^2 + (1-p)u'(\psi(x_1))\psi''(x_1) = 0.$$

At $x_1 = t$ we have

$$pu''(t) + u''(t)p^2/(1-p) + (1-p)u'(t)\psi''(t) = 0.$$

Therefore,

$$\psi''(t) = -u''(t)/u'(t)[p/(1-p)^2] = r(t)[p/(1-p)^2].$$

On this restricted space of lotteries, $\succsim_1$ is more risk averse than $\succsim_2$ in the sense of definition (1) iff the indifference curve of $\succsim_1$ through (t, t), denoted by ψ_1, is never below the indifference curve of $\succsim_2$ through (t, t), denoted by ψ_2. Combined with $\psi_1'(t) = \psi_2'(t)$, we obtain that $\psi_1''(t) \ge \psi_2''(t)$ and thus $r_2(t) \le r_1(t)$. (See fig. 8.4.)

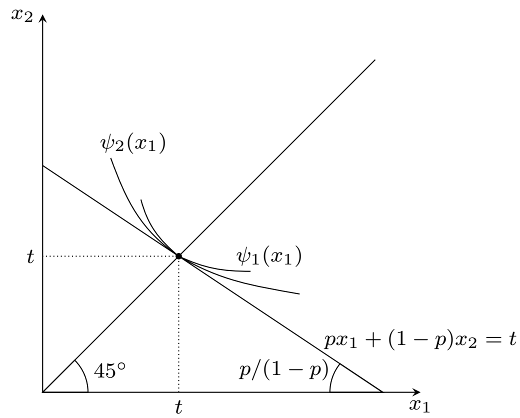

Figure 8.4
1 is more risk averse than 2.

The Doctrine of Consequentialism

Conduct the following "thought experiment":
You have \$2,000 in your bank account. You have to choose between

1. a sure loss of \$500
   and
2. a lottery in which you lose \$1,000 with probability 1/2 and lose 0 with probability 1/2.

What is your choice?

Now assume that you have \$1,000 in your account and that you have to choose between

3. a certain gain of \$500
   and
4. a lottery in which you win \$1,000 with probability 1/2 and win 0 with probability 1/2.

What is your choice?

Among the Kahneman and Tversky (1979)'s subjects, in the first case 69% preferred the lottery to the certain loss (option (2)), while in the second case 84% preferred the certain gain of \$500 (option (3)). These results indicate that about half of the population exhibit a preference for (2) over (1) and (3) over (4). Such preferences do not conflict with expected utility theory if we interpret a prize to reflect a "monetary change". However, if we assume that the decision maker identifies the prizes with final wealth levels, then we have a problem: in terms of final wealth levels, both choice problems are between a certain $1,500 and a lottery that yields $2,000 or $1,000, each with probability of 1/2. In those terms, about half of the subjects made inconsistent choices.

Nevertheless, in the economic literature it is usually assumed that a decision maker's preferences over wealth changes are induced from his preferences with regard to "final wealth levels". Formally, when starting with wealth w, denote by $\succsim_w$ the decision maker's preferences over lotteries in which the prizes are interpreted as "changes" in wealth. By the *doctrine of consequentialism* all relations $\succsim_w$ are derived from the same preference relation, $\succsim$, defined over the "final wealth levels" by $p \succsim_w q$ iff $w + p \succsim w + q$ (where $w + p$ is the lottery that awards a prize $w + x$ with probability $p(x)$). If $\succsim$ is represented by Eu, this doctrine implies that for all w, the function Ev_w represents the preferences $\succsim_w$ where $\mathrm{v}_w(x) = u(w + x)$.

Invariance to Wealth

We say that the preference relation $\succsim$ exhibits *invariance to wealth* (often called *constant absolute risk aversion*) if the induced preference relation $\succsim_w$ is independent of w, that is, $(w + p) \succsim (w + q)$ is true or false independent of w.

Claim:

Assume that Eu represents the preferences $\succsim$, which are monotonic and exhibit risk aversion and invariance to wealth. Then u must be exponential or linear.

Proof:

Let Δ be an arbitrary positive number. Verify that it is sufficient to prove the claim while confining ourselves to a $\Delta - grid$ prize space $Z = \{x \mid x = n\Delta$ for some integer $n\}$.

For any wealth level x, there is a number $q \geq 1/2$ such that $(1 - q)(x - \Delta) \oplus q(x + \Delta) \sim x$. By invariance to wealth, q is independent of x. Thus, we have $u(x + \Delta) - u(x) = [(1 - q)/q][u(x) - u(x - \Delta)]$ for all $x \in Z$. This means that the increments in the function u, when x is increased by Δ, constitute a geometric sequence with a factor of $(1 - q)/q$ (where q might depend on Δ). If $q > 1/2$ and using the formula for the sum of a geometric sequence, we conclude that on the $\Delta - grid$, $u(x) = a - b(\frac{1-q}{q})^{\frac{x}{\Delta}}$ for some a and b. If $q = 1/2$, then the function $u(x) = a + b\frac{x}{\Delta}$ for some a and b.

Note that the comparison of the lottery [0] to the simple lotteries involving a gain and loss of Δ are sufficient to characterize a unique preference relation that is consistent with: (i) the doctrine of consequentialism, (ii) the assumption that the preferences regarding lotteries over changes in wealth are independent of initial wealth and (iii) the expected utility assumptions regarding the space of lotteries in which the prizes are the final wealth levels. A number of researchers have tried to reveal the decision maker's preferences experimentally under these assumptions using the following question: "What is the probability q that would make you indifferent between a gain of \$$\Delta$ with probability q and a loss of \$$\Delta$ with probability $1-q$?" The findings have varied. Moreover, asking individuals different versions of this question can be expected to produce inconsistent answers.

Assuming that the function u is differentiable, we could prove the above claim also by looking at the preferences restricted to the space of all lotteries of the type $(x_1, x_2) = px_1 \oplus (1-p)x_2$ for some arbitrary fixed probability $p \in (0,1)$. Denote the indifference curve through (t,t) by $x_2 = \psi_t(x_1)$. Thus, $[t] \sim px_1 \oplus (1-p)\psi_t(x_1)$. Since $\succsim$ exhibits constant absolute risk aversion, it must be that $[0] \sim p(x_1 - t) \oplus (1-p)(\psi_t(x_1) - t)$ and thus $\psi_0(x_1 - t) = \psi_t(x_1) - t$ or $\psi_t(x_1) = \psi_0(x_1 - t) + t$. In other words, the indifference curve through (t,t) is the indifference curve through $(0,0)$ shifted in the direction of (t,t). Therefore, $\psi_t''(t) = \psi_0''(0)$ for all t. Since $\psi_t''(t) = -[p/(1-p)^2][u''(t)/u'(t)]$, there exists a number α such that $-u''(t)/u'(t) = \alpha$ for all t. This implies that $[log\, u'(t)]' = -\alpha$ for all t and $log\, u'(t) = -\alpha t + \beta$ for some β. It follows that $u'(t) = e^{-\alpha t + \beta}$. If $\alpha = 0$, the function $u(t)$ must be linear (implying risk neutrality). If $\alpha \neq 0$, it must be that u is an affine transformation of the function $-e^{-\alpha t}$ (with $\alpha > 0$).

Critique of the Doctrine of Consequentialism

Consider a risk-averse decision maker who likes money, obeys expected utility theory, and adheres to the doctrine of consequentialism. Rabin (2000) noted that if such a decision maker turns down the lottery $L = 1/2(-10) \oplus 1/2(+11)$, at any wealth level between \$0 and \$5,000 (a quite plausible assumption), then at the wealth level \$4,000 he must reject the lottery $1/2(-100) \oplus 1/2(+71,000)$ (a quite ridiculous conclusion).

Here is the intuition behind this observation: Since L is rejected at $w + 10$, we have that $u(w+10) \geq [u(w+21) + u(w)]/2$. Therefore, $u(w+10) - u(w) \geq u(w+21) - u(w+10)$ or

$$\frac{10}{11}\left(\frac{u(w+10) - u(w)}{10}\right) \geq \frac{u(w+21) - u(w+10)}{11}.$$

By the concavity of u the right-hand side of this equation is at least as large as the marginal utility at $w + 21$, whereas the left-hand side is at most $10/11$ of times the marginal utility at w. Thus the marginal utility at $w + 21$ is at most $10/11$ the marginal utility at w. Thus, the sequence of marginal utilities within the domain of wealth levels in which L is rejected falls at least at a geometric rate. This implies that for the lottery $1/2(-D) \oplus 1/2(+G)$ to be accepted even for a relatively low D, one would need a huge G.

What conclusions should we draw from this observation? This is not a refutation of expected utility theory. Rabin's argument relies on the doctrine of consequentialism, which is not a part of expected utility theory. Expected utility theory is invariant to the interpretation of the prizes. Whatever theory of decision making under uncertainty we use, the set of prizes should be the set of consequences in the mind of the decision maker. It is (at least) equally reasonable to assume that the consequences are "wealth changes" as it is to assume that the consequences are "final wealth levels".

I treat Rabin's argument as further evidence of the empirically problematic nature of the doctrine of consequentialism, according to which the decision maker makes *all* decisions with a preference relation over *the same* set of final consequences in mind. It also serves as a warning that the practice of estimating an economic agent's risk aversion parameters for small lotteries might lead to misleading conclusions if such estimates are used to characterize the decision maker's preferences regarding lotteries over large sums.

Bibliographic Notes

The measures of risk aversion are taken from Arrow (1970) and Pratt (1964). For the psychological literature discussed here, see Kahneman and Tversky (1979) and Kahneman and Tversky (2000).

The St. Petersburg Paradox was suggested by Daniel Bernoulli in 1738 (see Bernoulli (1954)). The notion of stochastic dominance was introduced into the economic literature by Rothschild and Stiglitz (1970). Rabin's argument is based on Rabin (2000).

Problem Set 8

Problem 1. (*Standard*)

a. Show that a sequence of numbers $(a_1, \ldots, a_k)$ satisfies that $\sum a_k x_k \geq 0$ for all vectors $(x_1, \ldots, x_k)$ such that $x_k > 0$ for all k iff $a_k \geq 0$ for all k.
b. Show that a sequence of numbers $(a_1, \ldots, a_k)$ satisfies that $\sum a_k x_k \geq 0$ for all vectors $(x_1, \ldots, x_k)$ such that $x_1 > x_2 > \ldots > x_K > x_{K+1} = 0$ iff $\sum_{k=1}^{l} a_k \geq 0$ for all l.

Problem 2. (*Standard. Based on Rothschild and Stiglitz (1970).*)
We say that p *second-order stochastically dominates* q and denote this by pD_2q if $p \succsim q$ for all preferences $\succsim$ satisfying the vNM assumptions, monotonicity, *and* risk aversion.

a. Explain why pD_1q implies pD_2q.
b. Let p and ε be lotteries. Define $p + \varepsilon$ to be the lottery that yields the prize t with the probability $\Sigma_{\alpha+\beta=t} p(\alpha)\varepsilon(\beta)$. Interpret $p + \varepsilon$. Show that if ε is a lottery with expectation 0, then for all p, $pD_2(p + \varepsilon)$.
c. (More difficult) Show that pD_2q if and only if for all $t < K, \Sigma_{k=0}^{t} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k] \geq 0$ where $x_0 < \ldots < x_K$ are all the prizes in the support of either p or q and $G(p, x) = \Sigma_{z \geq x} p(z)$.

Problem 3. (*Standard. Based on Slovic and Lichtenstein (1968).*)
Consider a phenomenon called *preference reversal*. Let $L_1 = 8/9[\$4] \oplus 1/9[\$0]$ and $L_2 = 1/9[\$40] \oplus 8/9[\$0]$.

Discuss the phenomenon in which many people prefer L_1 to L_2, but when asked to evaluate the certainty equivalence of these lotteries, they attach a lower value to L_1 than to L_2.

Problem 4. (*Standard*)
Consider a consumer's preference relation over K-tuples describing quantities of K uncertain assets. Denote the random return on the k'th asset by Z_k. Assume that the random variables $(Z_1, \ldots, Z_K)$ are independent and take positive values with probability 1. If the consumer buys the combination of assets $(x_1, \ldots, x_K)$ and if the vector of realized returns is $(z_1, \ldots, z_K)$, then the consumer's total wealth is $\sum_{k=1}^{K} x_k z_k$. Assume that the consumer satisfies vNM assumptions, that is, there is a function v (over the sum of his returns) so that he maximizes the expected value of v. Assume further that v is increasing and concave. The consumer preferences over the space of the lotteries induce preferences on the space of investments. Show that the induced preferences are monotonic and convex.

Problem 5. (*Standard. Based on Rubinstein (2002).*)
Adam lives in the Garden of Eden and eats only apples. Time in the garden is discrete ($t = 1, 2, \ldots$) and apples are eaten only in discrete units. Adam possesses preferences over the set of streams of apple consumption. Assume that:

a. Adam likes to eat up to 2 apples a day and cannot bear to eat 3 apples a day.
b. Adam is impatient. He would be delighted to increase his consumption on day t from 0 to 1 or from 1 to 2 apples at the expense of an apple he is promised a day later.
c. On any day that he does not have an apple, he would prefer to get 1 apple immediately in exchange for 2 apples tomorrow.
d. Adam expects to live for 120 years.

Show that if (poor) Adam is offered a stream of 2 apples starting on day 4 for the rest of his expected life, he would be willing to exchange that offer for 1 apple right away.

Problem 6. (*Moderately difficult. Based on Yaari (1987).*)
In this problem you will encounter Quiggin and Yaari's functional, one of the alternatives to expected utility theory.

Recall that expected utility can be written as $U(p) = \sum_{k=1}^{K} G_p(z_k)[u(z_k) - u(z_{k-1})]$ where $z_0 < z_1 < ... < z_K$ are the prizes in the support of p.

Let $W(p) = \sum_{k=1}^{K} f(G_p(z_k))[z_k - z_{k-1}]$, where $f : [0, 1] \to [0, 1]$ is a continuous increasing function and $G_p(z_k) = \sum_{j \geq k} p(z_j)$.

a. The literature often refers to W as the dual expected utility operator. In what sense is W dual to U?
b. Show that W induces a preference relation on $L(z)$ that may not satisfy the independence axiom.
c. What are the problems with a functional form of the type $\Sigma_z f(p(z))u(z)$? (See Handa (1977).)

Problem 7. (*The two envelopes paradox*)
Assume that a number 2^n is chosen with probability $2^n/3^{n+1}$ and the amounts of money 2^n, 2^{n+1} are put into two envelopes. One envelope is chosen randomly and given to you, and the other is given to your friend. Whatever the amount of money in your envelope, the expected amount in your friend's envelope is larger (verify). Thus, it is worthwhile for you to switch envelopes with him even without opening your envelope! Try to resolve this paradoxical conclusion.

LECTURE 9

Social Choice

Aggregation of Preference Relations

When a rational decision maker forms a preference relation, it is often on the basis of more primitive relations. For example, the choice of a PC may depend on considerations such as “size of memory”, “ranking by PC magazine”, and “price”. Each of these considerations expresses a preference relation on the set of PCs. In this lecture we look at some of the logical properties and problems that arise in the formation of preferences on the basis of more primitive preference relations.

Although the aggregation of preference relations can be thought of in the context of a single individual’s decision making, the classic context in which preference aggregation is discussed is “social choice”, where the “will of the people” is thought of as an aggregation of the preference relations held by members of society.

The foundations of social choice theory lie in the “Paradox of Voting”. Let $X = \{a, b, c\}$ be a set of alternatives. A natural criterion for the determination of collective opinion on the basis of individuals’ preference relations is the *majority rule.* However, consider a society consisting of three members ”named” 1, 2, and 3 and assume that their rankings of X are $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$, and $c \succ_3 a \succ_3 b$. According to the majority rule, $a \succ b$, $b \succ c$, and $c \succ a$, which conflicts with the transitivity of the social preferences. Note that although the majority rule does not induce a transitive social relation for *all* profiles of individuals’ preference relations, transitivity might be obtained if we restrict ourselves to a smaller domain of profiles (see problem 3 in the problem set).

The interest in social choice in economics is motivated by the recognition that explicit methods for the aggregation of preference relations are essential for *welfare economics* under the ”ideology” that what is good for a group of people depends on their preferences (and on a ”council of the wise”, for example. Social choice theory is also related to the design of *voting systems*, which are methods of making collective decisions on the basis of individuals’ preferences.

The Basic Model

A basic model of social choice consists of the following:

- X: a set of social *alternatives.*
- N: a finite set of *individuals* (denote the number of elements in N by n).
- $\succ_i$: individual i's ordering on X (an ordering is a preference relation with no indifferences, i.e., for no $x \neq y$ is $x \sim_i y$).
- *Profile*: An n-tuple of orderings $(\succ_1, \ldots, \succ_n)$ interpreted as a particular "state of society".
- *SWF* (*Social Welfare Function*): A function that assigns a single (social) preference relation (*not* necessarily an ordering) to every profile.

Note that

1. The assumption that the domain of an SWF includes only strict preferences is made only for simplicity of presentation.
2. An SWF attaches a preference relation to *every* possible profile and not just to a single profile (which exists at the moment).
3. The SWF is required to produce a *preference relation* (which is complete, reflexive and transitive). An alternative concept, called Social Choice Function, attaches a *social alternative*, interpreted as the society's choice, to every profile of preference relations.
4. An SWF aggregates only ordinal preference relations. The framework does not allow us to make a natural statement such as "the society prefers a to b since agent 1 prefers b to a but agent 2 prefers a to b much more".
5. In this model we cannot express a consideration of the type "I prefer what society prefers".
6. The elements in X are social alternatives. Thus, an individual's preferences may exhibit considerations of fairness and concern about other individuals' well-being and not only selfish aspects of the collective decision.

Examples:

Following are some examples of aggregation procedures.

1. $F(\succ_1, \ldots, \succ_n) = \succsim^*$ for some preference relation $\succsim^*$. (This is a degenerate SWF that does not account for the individuals' preferences.)
2. Define $x \to z$ if a majority of individuals prefer x to z. Order the alternatives by the number of "victories" they score, that is, x is socially preferred to y if $|\{z|x \to z\}| \geq |\{z|y \to z\}|$.
3. For $X = \{a, b\}$, $a \succsim b$ unless $2/3$ of the individuals prefer b to a.

4. "*The anti-dictator*": There is an individual i so that x is preferred to y if and only if $y \succ_i x$.
5. Let $d(\succ, \succ')$ as the number of $\{x, y\}$ for which $x \succ y$ and $y \succ' x$. The function d is a distance function between any two preferences. Choose $F(\succ_1, \ldots, \succ_n)$ to be an ordering that minimizes $\sum_i d(\succ, \succ_i)$ (ties are broken arbitrarily).
6. $F(\succ_1, \ldots, \succ_n)$ is the most frequent ordering among $(\succ_1, \ldots, \succ_n)$ (with ties broken in some predetermined way).
7. *The Borda rule*: Let $w(1) > w(2) > \ldots > w(|X|)$ be a fixed profile of weights. We say that i assigns to x the score $w(k)$ if x appears in the k'th place in $\succ_i$. Attach to x the sum of the weights assigned to x by the n individuals and rank the alternatives by those sums.

Axioms

Once again we use the axiomatization methodology. We suggest a set of (hopefully sound) axioms on social welfare functions and study their implications.

Let F be an SWF. We often use $\succsim$ as a short form for $F(\succ_1, \ldots, \succ_n)$.

Condition Par (Pareto):

For all $x, y \in X$ and for every profile $(\succ_i)_{i \in N}$, if $x \succ_i y$ for all i, then $x \succ y$.

Par requires that if all individuals prefer one alternative over the other, then the social preferences also do.

Condition IIA (Independence of Irrelevant Alternatives):

For any pair $x, y \in X$ and any two profiles $(\succ_i)_{i \in N}$ and $(\succ'_i)_{i \in N}$, if for all i, $x \succ_i y$ iff $x \succ'_i y$, then $x \succsim y$ iff $x \succsim' y$.

The IIA condition requires that if two profiles agree on the relative rankings of two particular alternatives, then the social preferences attached to the two profiles also do.

Notice that IIA allows an SWF to apply one criterion when comparing a to b and another when comparing c to d. For example, the simple social preference between a and b can be determined according to majority rule whereas that between c and d requires a $2/3$ majority.

Arrow's Impossibility Theorem

If $|X| \geq 3$, then any SWF F that satisfies conditions Par and IIA is dictatorial, that is, there is some i^* such that $F(\succ_1, \ldots, \succ_n) \equiv \succ_{i^*}$.

The four ingredients of the theorem are Par, IIA, $Transitivity$ (of the social preferences) and $|X| \geq 3$. Before presenting the proof, we show that the assumptions are *independent.* Namely, for each of the four assumptions, we present an example of a nondictatorial SWF which demonstrates that the theorem does not hold if that assumption is omitted.

- Par: An anti-dictatorial SWF satisfies IIA but not Par.
- IIA: When $|X| \geq 3$, any Borda rule (with $w(1) > w(2) > \ldots > w(|X|)$) is an SWF satisfying Par but not IIA (Problem 1).
- *Transitivity of the Social Order:* The majority rule satisfies IIA and Par for $|X| \geq 3$ but can induce a relation that is not transitive regardless of the number of individuals (verify).
- $|X| \geq 3$: For $|X| = 2$, the majority rule satisfies Par and IIA and induces a (trivial) transitive relation.

Proof of Arrow's Impossibility Theorem

Let F be an SWF that satisfies Par and IIA. Hereafter, we write $\succsim$ instead of $F(P)$ and $\succsim'$ instead of $F(P')$.

Step 1:

Let b be an alternative and m be an integer between 1 and n. Consider a profile $P = (\succ_1, \ldots, \succ_n)$ such that for all $i \leq m$, b is the best alternative according to $\succ_i$ and for all other players b is the worst. Then b is either the unique best or the unique worst alternative of $\succsim$.

Proof:

If not, then there are two other distinct alternatives a and c such that $a \succsim b \succsim c$. Consider P', a modification of P, such that for every individual where c is below a in P it will "jump" in P' to above a (and thus for $i \leq m$, the alternative c remains below b). By Par, $c \succ' a$. Since the individuals' relative rankings of a and b and of b and c are the same in P as in P' then by IIA, $a \succsim' b \succsim' c$, a contradiction.

Step 2:

Consider a profile P^0 where b is at the bottom of the rankings of all individuals. By Par, b is at the bottom of $F(P^0)$. Let P^m be a modified profile where the alterative b is upgraded to the top of the rankings for all $i \leq m$. Since by Par,

b is at the top of $F(P^n)$, there must be some m^* for which b is at the bottom of $F(P^{m^*-1})$ and at the top of $F(P^{m^*})$. By IIA, the identity of m^* does not depend on the orderings in P^0 of any two alternatives that do not involve b.

Step 3:

Let a and c be two alternatives that are not b. If P is a profile in which $a \succ_{m^*} c$, then $a \succ c$.

Proof:

Let P' be a modification of P where for all $i < m^*$ the alternative b moves to the top, for m^* it moves to between a and c and for all $i > m^*$ it moves to the bottom. Then, the profile P' relates to the pair b and a in the same way as P^{m^*-1} and thus $a \succ' b$. The profile P' relates to the pair b and c in the same way as P^{m^*} and thus $b \succ' c$. It follows that $a \succ' c$ and by IIA, also $a \succ c$.

Step 4:

Let a be an alternatives that is not b. If P is a profile in which $a \succ_{m^*} b$ (or $b \succ_{m^*} a$), then $a \succ b$ (or $b \succ a$).

Proof:

Let c be a third alternative. Let P' be a modification of P such that c moves to the top of all rankings except that of m^* in which it moves to between a and b. Then, by step 3, $a \succ' c$ and by Par, $c \succ' b$. Thus, $a \succ' b$ and by IIA also $a \succ b$.

Related Issues

Arrow's theorem was the starting point for a huge literature. We mention here three other major impossibility results.

1. *Monotonicity* is another axiom that has been widely discussed in the literature. Consider a "change" in a profile so that an alternative a, which individual i ranked below b, is now ranked by i to be above to be b. Monotonicity requires that there is no alternative c such that this change lessens the ranking of a vs. c. Muller and Satterthwaite (1977)'s theorem shows that the only SWF's satisfying Par and monotonicity are dictatorships.
2. An SWF specifies a preference relation for every profile. A *social choice function* attaches an alternative to every profile. The Gibbard- Satterthwaite theorem states that any social choice function C satisfying the condition that it is never worthwhile for an individual to misrepresent his preferences, it is never the case that $C(\succ_1, \ldots, \succ_i', \ldots, \succ_n) \succ_i$

$C(\succ_1, \ldots, \succ_i, \ldots, \succ_n)$, is a dictatorship, i.e. a function that consistently picks the dictator's most preferred alternatives.

3. A concept to SWF is the following: Let $Ch(\succ_1, \ldots, \succ_n)$ be a function that assigns a choice function to every profile of orderings on X. We say that Ch satisfies *unanimity* if for every $(\succ_1, \ldots, \succ_n)$ and for any $x, y \in A$, if $y \succ_i x$ for all i, then $x \neq Ch(\succ_1, \ldots, \succ_n)(A)$.
   We say that Ch is *invariant to the procedure* if, for every profile and for every choice set A, the following two "approaches" lead to the same outcome:

   a. Partition A into two sets A' and A''. Choose an element from A' and another from A'' and then choose one of the two.
   b. Choose an element from the unpartitioned set A.

   Dutta, Jackson, and Le Breton (2001) show that only dictatorships satisfy both unanimity and invariance to the procedure.

Bibliographic Notes

This lecture focuses mainly on Arrow's Impossibility Theorem which was proved by Arrow in his Ph.D. dissertation and published in 1951 (see the classic Arrow (1963)). Social choice theory is beautifully introduced in Sen (1970). Arrow's Impossibility theorem has many proofs. The one presented here is due to Geanakopolos (2005). Reny (2001) provides another elementary proof that demonstrates the strong logical link between Arrow's theorem and the Gibbard-Satterthwaite theorem. Problem 5 is the basis for another proof (see Kelly (1988)).

Problem Set 9

Problem 1. (*Easy*)
Prove that any Borda rule does not satisfy IIA for any case in which there are at least three alternatives and at least two individuals.

Problem 2. (*Moderately difficult. Based on May (1952).*)
Assume that the set of social alternatives, X, includes only two alternatives. Define a social welfare function to be a function that attaches a preference to any profile of preferences (allow indifference for both the SWF and the individuals' preference relations). Consider the following axioms:

- *Anonymity* If σ is a permutation of N and if $p = \{ \succsim_i \}_{i \in N}$ and $p' = \{ \succsim'_i \}_{i \in N}$ are two profiles of preferences on X so that $\succsim'_{\sigma(i)} = \succsim_i$, then $\succsim(p) = \succsim(p')$.
- *Neutrality* For any preference $\succsim_i$ define $(-\succsim_i)$ as the preference satisfying $x(-\succsim_i)y$ iff $y \succsim_i x$. Then, $\succsim(\{-\succsim_i\}_{i \in N}) = -\succsim(\{\succsim_i\}_{i \in N})$.
- *Positive Responsiveness* If the profile $\{ \succsim'_i \}_{i \in N}$ is identical to $\{ \succsim_i \}_{i \in N}$ with the exception that for one individual j either ($x \sim_j y$ and $x \succ'_j y$) or ($y \succ_j x$ and $x \sim'_j y$) and if $x \succsim y$, then $x \succ' y$.

a. Interpret the axioms.
b. Show that the majority rule satisfies all of them.
c. Prove May's theorem by which the majority rule is the only SWF satisfying the above axioms.
d. Are the above three axioms independent?

Problem 3. (*Standard*)
Let $X = [0, 1]$ and assume that each individual's preference is *single-peaked*, that is, for each i there is an alternative a_i^* such that if $a_i^* \geq b > c$ or $c > b \geq a_i^*$, then $b \succ_i c$.

Show that for any odd n, if we restrict the domain of preferences to single-peaked preferences, then the majority rule induces a "well-behaved" SWF.

Problem 4. (*Moderately difficult*)
Each of N individuals chooses a single object from among a set X, interpreted as his recommendation for the social action. We are interested in functions $\mathrm{F}{:}\mathrm{X}^N \to X$ that aggregate the individuals' recommendations (*not preferences*, just recommendations!) into a social decision.

Discuss the following axioms:

- Par: If all individuals recommend x^*, then society chooses x^*.
- I: If the same individuals support an alternative $x \in X$ in two profiles of recommendations, then x is chosen in one profile if and only if it is chosen in the other.

a. Show that if X includes at least three elements, then the only aggregation method that satisfies P and I is a dictatorship.
b. Show the necessity of the three conditions P, I, and $|X| \geq 3$ for this conclusion.

Problem 5. (*Moderately difficult*)
Some proofs of Arrow's theorem use the notions of decisive and almost decisive coalitions.

Given the SWF, we say that:

- a coalition G is *decisive* with respect to x,y if [for all $i \in G$, $x \succ_i y$] implies [$x \succ y$], and
- a coalition G is *almost decisive* with respect to x,y if [for all $i \in G$, $x \succ_i y$ and for all $j \notin G$, $y \succ_j x$] implies [$x \succ y$].

Note that if G is decisive with respect to x,y, then it is also almost decisive with respect to x,y, since "almost decisiveness" refers only to the subset of profiles in which all members of G prefer x to y and all members of $N - G$ prefer y to x.

We say that a coalition G is *decisive* if it is decisive with respect to all x, y.

Let F be an SWF satisfying Par and IIA.

a. Prove the "Field Expansion Lemma": If G is almost decisive with respect to x,y, then G is decisive with respect to x,z and with respect to y, z.
b. Conclude that if G is almost decisive with respect to x,y, then G is decisive.
c. Prove the "Group Contraction Lemma": If G is decisive and $|G| \geq 2$, then there exists $G' \subset G$ such that G' is decisive.
d. Show that there is an individual i^* such that $\{i^*\}$ is decisive.

Problem 6. (*Moderately difficult. Based on Kasher and Rubinstein (1997).*)
Who is an economist? Departments of economics are often sharply divided on this question. Investigate the approach according to which the determination of who is an economist is treated as an aggregation of the views held by department members on this question.

Let $N = \{1, \ldots, n\}$ be the set of members of a department ($n \geq 3$). Each $i \in N$ "submits" a set E_i, a *proper* subset of N ($\emptyset \subset E_i \subset N$), which is interpreted as the set of "real economists" in his view. An aggregation method F is a function that assigns a *proper* nonempty subset of N to each profile $(E_i)_{i=1,\ldots,n}$ of proper subsets of N. $F(E_1,\ldots,E_n)$ is interpreted as the set of all members of N who are considered by the group to be economists. Consider the following axioms on F:

- *Consensus:* If $j \in E_i$ for all $i \in N$, then $j \in F(E_1,\ldots E_n)$, and if $j \notin E_i$ for all $i \in N$, then $j \notin F(E_1,\ldots E_n)$.
- *Independence:* If $(E_1,\ldots,E_n)$ and $(G_1,\ldots,G_n)$ are two profiles of views so that for all $i \in N$, $[j \in E_i$ iff $j \in G_i]$, then $[j \in F(E_1,\ldots,E_n)$ iff $j \in F(G_1,\ldots,G_n)]$.

a. Interpret the two axioms.
b. Find one aggregation method that satisfies *Consensus* but not *Independence* and one that satisfies Independence but not Consensus.
c. (*Difficult*) Provide a proof similar to that of Arrow's Impossibility Theorem for the claim that the only aggregation methods that satisfy the above two axioms are those for which there is a member i^* such that $F(E_1,\ldots,E_n) \equiv E_{i^*}$.

60 Review Problems

The following is a collection of problems based on exams I have given at Tel-Aviv, Princeton and New York universities.

A. Choice

Problem A1. (*Princeton 2000. Based on Fishburn and Rubinstein* (*1982*).)
Let $X = \mathbb{R}^+ \times \{0, 1, 2, \ldots\}$, where (x, t) is interpreted as receiving \$$x$ at time t. A preference relation on X has the following properties:

- There is indifference between receiving \$0 at time 0 and receiving 0 at any other time.
- It is better to receive any positive amount of money as soon as possible.
- Money is desirable.
- The preference between (x, t) and $(y, t+1)$ is independent of t.
- Continuity.

1. Define formally the continuity assumption for this context.
2. Show that the preference relation has a utility representation.
3. Verify that the preference relation represented by the utility function $u(x)\delta^t$ (with $\delta < 1$ and u continuous, increasing and $u(0) = 0$) satisfies the above properties.
4. Formulize a concept "one preference relation is more impatient than another".
5. Discuss the claim that preferences represented by $u_1(x)\delta_1^t$ are more impatient than preferences represented by $u_2(x)\delta_2^t$ if and only if $\delta_1 < \delta_2$.

Problem A2. (*Tel Aviv 2003. Based on Gilboa and Schmeidler* (*1995*).)
An agent must decide whether to do something, Y, or not to do it, N.

A history is a sequence of results for past events in which the agent chose Y; each result is either a success S or a failure F. For example, (S, S, F, F, S) is a history with five events in which the action was carried out. Two of them (events 3 and 4) ended in failure, whereas the rest were successful.

The decision rule D is a function that assigns the decision Y or N to every possible history.

Consider the following properties of decision rules:

$A1$ After every history that contains only successes, the decision rule will dictate Y, and after every history that contains only failures, the decision rule will dictate N.

$A2$ If the decision rule dictates a certain action following some history, it will dictate the same action following any history that is derived from the first history by reordering its members. For example, $D(S, F, S, F, S) = D(S, S, F, F, S)$.

$A3$ If $D(h) = D(h')$, then this will also be the decision following the concatenation of h and h'. (Reminder: The concatenation of $h = (F, S)$ and $h' = (S, S, F)$ is (F, S, S, S, F)).

1. For every $i = 1, 2, 3$, give an example of a decision rule that does not fulfill property Ai but does fulfill the other two properties.
2. Give an example of a decision rule that fulfills all three properties.
3. (Difficult) Characterize the decision rules that fulfill the three properties.

Problem A3. (*NYU 2005*)

Let X be a finite set containing at least three elements. Let C be a choice correspondence (with domain which includes all non-empty subsets of X). Consider the following property:

If $A, B \subseteq X$, $B \subseteq A$, and $C(A) \cap B \neq \emptyset$, then $C(B) = C(A) \cap B$.

1. Show that the property is equivalent to the existence of a preference relation $\succsim$ such that $C(A) = \{x \in A | x \succcurlyeq a \text{ for all } a \in A\}$.
2. Consider a weaker property:
   If $A, B \subseteq X$, $B \subseteq A$, and $C(A) \cap B \neq \emptyset$, then $C(B) \subseteq C(A) \cap B$.
   Is this sufficient for the above equivalence?

Problem A4. (*NYU 2007. Based on Plott (1973).*)

Let X be a set and C be a choice correspondence defined on all non-empty subsets of X. We say that C satisfies Path Independence (PI) if for every two disjoint sets A and B, we have $C(A \cup B) = C(C(A) \cup C(B))$. We say that C satisfies Extension (E) if $x \in A$ and $x \in C(\{x, y\})$ for every $y \in A$ implies that $x \in C(A)$ for all sets A.

1. Interpret PI and E.
2. Show that if C satisfies both PI and E, then there exists a binary relation $\succsim$ that is complete and reflexive and satisfies $x \succ y$, and $y \succ z$ implies $x \succ z$, such that $C(A) = \{x \in A \mid \text{for no } y \in A \text{ is } y \succ x\}$.
3. Give one example of a choice correspondence satisfying PI but not E, and one satisfying E but not PI.

Problem A5. (*NYU 2008. Based on Eliaz, Richter, and Rubinstein (2011).*)
Let X be a (finite) set of alternatives. Given any choice problem A (where $|A| \geq 2$), the decision maker chooses a set $D(A) \subseteq A$ of two alternatives that he wants to examine more carefully before making the final decision.
The following are two properties of D:

$A1$: If $a \in D(A)$ and $a \in B \subset A$, then $a \in D(B)$.

$A2$: If $D(A) = \{x, y\}$ and $a \in D(A - \{x\})$ for some a different than x and y, then $a \in D(A - \{y\})$.

Solve the following four exercises. A full proof is required only for the last exercise:

1. Find an example of a D function that satisfies both $A1$ and $A2$.
2. Find a function D that satisfies $A1$ and not $A2$.
3. Find a function D that satisfies $A2$ and not $A1$.
4. Show that for any function D satisfying $A1$ and $A2$ there exists an ordering $\succ$ of the elements of X such that $D(A)$ is the set of the two $\succ$-best elements in A.

Problem A6. (*Tel Aviv 2009. Inspired by Mandler, Manzini, and Mariotti (2010).*)
Consider a decision maker who is choosing an alternative from subsets of a finite set X using the following procedure:

Following a fixed list of properties (a checklist), he examines one property at a time and deletes from the set all the alternatives that do not satisfy this property. When only one alternative remains, he chooses it.

1. Show that if this procedure induces a choice function, then it is consistent with the rational man model.
2. Show that any rational decision maker can be described *as if* he follows this procedure.

Problem A7. (*Tel Aviv 2010*)
A decision maker has a preference relation over $\mathbb{R}^n_+$. A vector (x_1, x_2) is interpreted as an income combination where x_i is the dollar amount the decision maker receives at period i. Let P be the set of all preference relations satisfying:

(i) Strong Monotonicity (SM) in x_1 and x_2.

(ii) Present preference (PP): $(x_1 + \varepsilon, x_2 - \varepsilon) \succsim (x_1, x_2)$ for all $\varepsilon > 0$.

Define $(x_1, x_2)D(y_1, y_2)$ if $(x_1, x_2) \succsim (y_1, y_2)$ for all $\succsim \in P$.

1. Interpret the relation D. Is it a preference relation?
2. Is it true that $(1, 4)D(3, 3)$? What about $(3, 3)D(1, 4)$?

3. Find and prove a proposition of the following type: $(x_1, x_2)D(y_1, y_2)$ if and only if [put here a condition on (x_1, x_2) and (y_1, y_2)].

Problem A8. (*NYU 2011*)
Let X be a finite set of alternatives.

A decision maker of type 1 uses the following choice procedure. He has a subset of "satisfactory alternatives" in mind. Whenever he chooses from a set A, then (i) if there are satisfactory elements in A, he is happy to choose any satisfactory alternative which comes to his mind and (ii) If there are none, he is happy with any of the non-satisfactory alternatives.

A decision maker of type 2 has in mind a set of strict orderings. Whenever he chooses from a set A, he is happy with any alternative that is the maxima in A of at least one ordering.

1. Define formally the two types of decision makers as choice correspondences.
2. Show that any decision maker of type 1 can also be described as a decision maker of type 2.
3. Show that there is a decision maker of type 2 who cannot be described as a decision maker of type 1.

Problem A9. (*Tel Aviv 2012. Based on de Clippel* (*2011*).)
Consider a decision maker (DM) who has in mind two orderings on a finite set X. The first ordering, $\succ_L$, expresses his long-term goals, and the second, $\succ_S$ expresses his short-term goals.

When choosing from a set $A \subseteq X$ the DM chooses the best alternative according to his long-term preferences, unless there are "too many" alternatives that are better than this alternative according his short-term preferences. More precisely, given a choice problem $A \subseteq X$, he excludes all alternatives which are not among the k best alternatives in A according to his short-term preferences, and out of the remaining he chooses the best one according to $\succ_L$.

1. Show that the above description always defines a choice function.
2. Show that it may be that the same alternative is chosen from both A and B, but is not chosen from $A \cup B$ nor from $A \cap B$.
3. Conclude that this type of behavior conflicts with the rational man paradigm.

Let N be a set of individuals who behave according to the above procedure with $k = 2$. All individuals share the same long-term goals but may differ in their short-term goals.

Consider a situation in which the N individuals must choose together only one alternative from the set X and that for each alternative $x \in X$, there is one individual $r(x)$ who has the right to force x. An equilibrium is an alternative y such that no individual wants to exercise his right to force one of the alternatives that he can force. That is, for any agent i,the alternatives y is the one chosen by the agent from the set $\{y\} \cup \{x|r(x) = i\}$.

4. Show that if there are more individuals than alternatives then it is possible to assign the "forcing rights"such that whatever are the individuals' short-term goals and whatever are the common long-term goals, the only equilibrium is the top $\succ_L$ alternative. Explain why this is not necessarily correct if the number of alternatives is larger than the number of individuals.

Problem A10. (*NYU 2013*)
Consider the following procedure which yields a choice function C over subsets of a finite set X:

The decision maker has in mind a set $\{\succ_i\}_{i=1,..n}$ of orderings over X and a set of weights $\{\alpha_i\}_{i=1,..n}$. Facing a choice set $A \subseteq X$, the decision maker calculates a score for each alternative $x \in A$ by summing the weights of those orderings that rank x first from among the members of A and then chooses the alternative with the highest score.

1. Explain why a rational choice function is consistent with this procedure.
2. Give an example to show that the procedure can produce a choice function which is not rationalizable.
3. Show that for $|X| = 3$ all choice functions are consistent with the procedure.
4. Explain why it is not generally true that a choice function C which is derived from this procedure satisfies the condition that if $x = C(A) = C(B)$, then $x = C(A \cup B)$.
5. (More Difficult) Can you find a non-trivial property that is satisfied by choice functions which are derived from this procedure but not by all choice functions? Is there any choice function that cannot be explained by this procedure?

Problem A11. (*NYU 2013*)
An agent makes a binary comparison of pairs of numbers. His real interest is to maximize the sum $x_1 + x_2$. When he compares (x_1, x_2) and (y_1, y_2) he always makes the right decision if one of the pairs dominates the other. When this is not the case he might make a mistake. The technology of mistakes is characterized by a function $\alpha(G, L)$ with the interpretation that if the gain in one dimension is $G \geq 0$ and the loss in the other is $L \geq 0$, then the probability of a mistake is $\alpha(G, L)$.

1. Suggest reasonable and workable assumptions for the function α (such as $\alpha(G, L) \leq 1/2$ for all G and L).

2. Suggest a formal notion which expresses the phrase "agent 1 is more accurate than agent 2".
3. Show that according to the notion you defined in 2 the probability that three binary comparisons on the triple $(7,2)$, $(3,10)$, $(0,6)$ yields a cycle is smaller for the agent who is more accurate in his choices.
4. Show that the probability of the binary comparisons yielding a cycle on a general triple of pairs is not necessarily smaller for the agent who is more accurate.

Problem A12. (*Tel Aviv 2014*)

Consider a world in which the grand set X is the entire plane and choice sets can only be less than 180 degree closed arcs of the unit circle. Denote a choice set by $B(\alpha, \beta)$ where α and β, are the two angles that confine the arc which are numbers between 0 and 360. For example, $B(0, 90)$ is one-quarter of a circle contained in the positive quadrant.

1. Give an example of a choice function that does not satisfy the weak axiom of revealed preference.
2. Give an example of a choice function that satisfies the weak axiom of revealed preference and yet is not rationalizable.

Assume now that the choice sets are only arcs in the positive quadrant (i.e. the two angles that define the choice sets are between 0° and 90°) and that the agent maximizes a monotonic, continuous and strictly convex preference relation.

3. Show that the agent's choice function is well defined.
4. Explain how one could identify the agent's choice function from the indirect preference relation (defined over the parameters of the choice sets).

Problem A13. (*NYU 2015*)

Consider two types of decision makers: Type A has in mind the criteria $(\succ_i)_{i \in I}$ where each $\succ_i$ is an ordering of the elements in a finite set X. Whenever the agent chooses from a set $A \subseteq X$ he is satisfied with any element a such that for any other $b \in A$ there is some i (i probably depends on b) for which $a \succ_i b$.

Thus, for example, if he has one criterion in mind then the induced choice correspondence picks the unique maximal element from each set ; if he has two in mind, where one is the negation of the other, then the induced choice correspondence is $C(A) \equiv A$.

1. Show that if $a \in C(A) \cap C(B)$, then $a \in C(A \cup B)$.
2. Suggest another interesting property that the choice correspondence induced by the above procedure always satisfies.

Type B has in mind a transitive asymmetric relation $\succ$ with the interpretation that if $a \succ b$ then he will not choose b if a is available. He is described by the choice correspondence $C(A) = \{x \in A |$ there is no $y \in A$ such that $y \succ x\}$.

3. Show that every type A agent can be described as a type B agent.
4. Show that every type B agent can be described as a type A agent.

Problem A14. (*NYU 2016*)

A decision maker who compares vectors (x_1, x_2) and (y_1, y_2) in R^2_+ is implementing the following procedure, denoted by $P(v_1, v_2)$, where for $i = 1, 2$, v_i is a strictly increasing continuous function from the nonnegative numbers to the real numbers satisfying $v_i(0) = 0$:

(i) if one of the vectors dominates the other he evaluates it being superior.

(ii) if $x_1 > y_1$ and $y_2 > x_2$, he carries out a "cancellation" operation and then makes the evaluation by comparing $(x_1 - y_1, 0)$ to $(0, y_2 - x_2)$, which is accomplished by comparing $v_1(x_1 - y_1)$ with $v_2(y_2 - x_2)$ (similarly, if $x_1 < y_1$ and $x_2 > y_2$, he bases his preference on the comparison of $v_2(x_2 - y_2)$ to $v_1(y_1 - x_1)$).

1. Verify that if $v_i^*(t) = t$ (for both i), then the procedure $P(v_1^*, v_2^*)$ induces a preference relation on R^2_+.
2. Explain why $P(v_1, v_2)$ does not necessarily lead to a transitive preference relation.
3. Complete and prove the following proposition: If the procedure $P(v_1, v_2)$ induces a preference relation, then that preference relation is represented by....

Problem A15. (*NYU 2017*)

A decision maker behaves as follows when making a choice from a subset of X (a finite set):

He has in mind a set of considerations Λ, each of which is in the form of a preference relation. Each alternative x triggers an association with a subset of considerations $L(x) \subseteq \Lambda$. The decision maker holds also an ordering $\rhd$ over Λ, which expresses the priority he attributes to the different considerations. When he chooses from a set A:

(i) he thinks about the considerations in the set $L(A) = \cup_{a \in A} L(a)$,

(ii) he picks the ordering which is the $\rhd$ maximizer from $L(A)$ and

(iii) he chooses the maximal element in A according to the ordering he chose in (ii).

1. Construct an example of a choice function that is not consistent with this procedure.
2. Invent a property which all those choice functions must satisfy and prove it.

3. A group of people X chooses a leader from among the members of the group who are available to lead at that moment (a subset of X) by asking the most senior candidate to make the choice. Fit this group's procedure into the above scheme.

Problem A16. (*NYU 2018*)

A decision maker has to decide about an alternative from a set X. Occasionally, he gets recommendations in the form of "do x" (where $x \in X$). He will have to make a decision at some point of time. In order to describe his behavior we need to specify what will he do after every finite non-empty history of recommendations. Thus, a decision rule is a function from all possible non-empty finite lists of elements of X into X.

Here are some possible properties of a decision rule:

A. If $f(h) = a$, then $f(h, a) = a$.

B. If $f(h) = a$, then $f(h, b) = f(a, b)$

C. For every $a \in X$ and every history h there is h' such that $f(h, h') = a$.

D. If $h = (a_1, a_2, .., a_t)$ for $t \geq 1$, then $f(h) \in \{a_1, ..., a_t\}$.

1. Interpret each property in one sentence (or if you must two).
2. Explain why [B and D] imply A.
3. For each of the properties B , C and D, find a choice rule that satisfies only the other two properties.
4. Find a choice rule that satisfies B, C and D.

Problem A17. (*NYU 2020*)

A decision maker uses the following procedure to choose from any subset of alternatives in the finite set X: For any a and b, he has in mind a number $d(a, b) \geq 0$ such that $d(a, a) = 0$ and if $d(a, b) > 0$ then $d(b, a) = 0$. The number $d(a, b)$ in the case that $d(a, b) > 0$ is a measure by which a defeats b. From a set A, he chooses the maximizer of $\sum_{x \in A} d(a, x)$ (assume that there is always a unique maximizer).

a. Show that if the decision maker is "rational',' then he can be presented as if he activates such a procedure.

b. Give an example of a choice function that is obtained by the procedure but is not rationalizable.

c. Construct an example of a choice function that is not explicable using this procedure.

d. Suggest a property of a choice function that is necessary for a choice function to be explicable by the above procedure.

Problem A18. (*NYU 2020, based on Glazer and Rubinstein (2020)*)
An individual has in mind a directed tree $\langle V, \to \rangle$ with the root O. Each node in the tree stands for an event. The fact that $x \to y$ means that the event x can be followed by the event y. Denote by X the set of paths of the tree (a path starts from the root of the tree and reaches a terminal node). One and only one of the paths is the true one. The individual receives evidence in the form of a partial set of events (nodes) that have occurred. The evidence is not contradictory in the sense that there is at least one path that includes all the nodes. He selects a conjecture about the true path which is consistent with the evidence.

Formally, define a conjecture-builder (CB) as a function f that attaches to every set of nodes E, which contains O and is non-contradictory, a path $f(E)$ that includes all nodes in E.

a. We say that the CB is "order-based" if there is a preference relation $\succsim$ over X such that $f(E)$ is the $\succsim$ -maximal path that is consistent with E. Show that an order-based CB satisfies the following stickiness property: If the node n is on the path $f(E)$, then $f(E \cup \{n\}) = f(E)$.

b. Show that any CB that satisfies the stickiness property is order-based.

c. Consider the tree with $V = \{O, a, b, c\}$ and $O \to a$, $a \to b$, and $a \to c$. Which CB cannot be presented as order-based?

B. The Consumer and the Producer

Problem B1. (*Tel Aviv 1998*)

A consumer with wealth $w = 10$ "must" obtain a book from one of three stores. Denote the prices at each store as p_1, p_2, p_3. All prices are below w in the relevant range. The consumer has devised a strategy: he compares the prices at the first two stores and purchases the book from the first store if its price is not greater than the price at the second store. If $p_1 > p_2$, he compares the prices of the second and third stores and purchases the book from the second store if its price is not greater than the price at the third store. He uses the remainder of his wealth to purchase other goods.

1. What is this consumer's "demand function"?
2. Does this consumer satisfy "rational man" assumptions?
3. Consider the function $v(p_1, p_2, p_3) = w - p_{i^*}$, where i^* is the store from which the consumer purchases the book if the prices are (p_1, p_2, p_3). What does this function represent?
4. Explain why $v(\cdot)$ is not monotonically decreasing in p_i. Compare with the indirect utility function of the classic consumer model.

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4. Explain why $v(\cdot)$ is not monotonically decreasing in p_i. Compare with the indirect utility function of the classic consumer model.

Problem B2. (*Princeton 2001*)

1. Define a formal concept for "$\succsim_1$ and $\succsim_0$ are closer than $\succsim_2$ and $\succsim_0$".
2. Apply your definition to the class of preference relations represented by $U_1 = tU_2 + (1-t)U_0$, where the function U_i represents $\succsim_i$ $(i = 0, 1, 2)$.
3. Consider the above definition in the consumer context. Denote by $x_k^i(p, w)$ the demand function of $\succsim_i$ for good k. Show that $\succsim_1$ and $\succsim_0$ may be closer than $\succsim_2$ and $\succsim_0$, and nevertheless $|x_k^1(p,w) - x_k^0(p,w)| > |x_k^2(p,w) - x_k^0(p,w)|$ for some commodity k, price vector p and wealth level w.

Problem B3. (*Princeton 2002*)

Consider a consumer with a preference relation in a world with two goods, X (an aggregated consumption good) and M ("membership in a club", for example), which can be consumed or not. In other words, the consumption of X can be any nonnegative real number, while the consumption of M must be either 0 or 1.

Assume that the consumer's preferences are strictly monotonic and continuous and satisfy the following property:

> *Property E*: For every x, there is y such that $(y, 0) \succ (x, 1)$ (i.e., there is always some amount of the aggregated consumption good that can compensate for the loss of membership).

1. Show that any consumer's preference relation can be represented by a utility function of the type:

$$u(x,m) = \begin{cases} x & if \quad m=0 \\ x+g(x) & if \quad m=1 \end{cases}.$$

2. (Less easy) Show that the consumer's preference relation can also be represented by a utility function of the type:

$$u(x,m) = \begin{cases} f(x) & if \quad m=0 \\ f(x)+v & if \quad m=1 \end{cases}.$$

3. Explain why continuity and strong monotonicity (without property E) are not sufficient for (1).
4. Calculate the consumer's demand function.
5. Taking the utility function to be of the form described in (1), derive the consumer's indirect utility function. For the case where the function g is differentiable, verify Roy's identity with respect to commodity M.

Problem B4. (*Tel Aviv 2003*)
Consider the following consumer problem: there are two goods, 1 and 2. The consumer has a certain endowment. His preferences satisfy mono- monotonicity and continuity. Before the consumer are two "exchange functions": he can exchange x units of good 1 for $f(x)$ units of good 2, or he can exchange y units of good 2 for $g(y)$ units of good 1. Assume the consumer can make only one exchange.

1. Show that if the exchange functions are continuous, then a solution to the consumer problem exists.
2. Explain why strong convexity of the preference relation is not sufficient to guarantee a unique solution if the functions f and g are increasing and convex.
3. Interpret the statement "the function f is increasing and convex".
4. Suppose both functions f and g are differentiable and concave and that the product of their derivatives at point 0 is 1. Suppose also that the preference relation is strongly convex. Show that under these conditions, the agent will not find two different exchanges, one exchanging good 1 for good 2, and one exchanging good 2 for good 1, optimal.
5. Now assume $f(x) = ax$ and $g(y) = by$. Explain this assumption. Find a condition that will ensure it is not profitable for the consumer to make more than one exchange.

Problem B5. (*NYU 2005*)
A consumer has preferences that satisfy monotonicity, continuity, and strict convexity, in a world of K goods. The goods are split into two categories, 1 and 2, of K_1 and K_2 goods respectively ($K_1 + K_2 = K$). The consumer receives two types of money: w_i units of money of type i, which can be exchanged only for goods in the i'th category given a price vector p_i.

Define the induced preference relation over the two-dimensional space (w_1, w_2). Show that these preferences are monotonic, continuous, and convex.

Problem B6. (*NYU 2005. Inspired by Chen, Lakshminarayanan, and Santos (2005).*)
In an experiment, a monkey is given $m = 12$ coins, which he can exchange for apples or bananas. The monkey faces m consecutive choices in which he gives a coin either to an experimenter holding a apples or another experimenter holding b bananas.

1. Assume that the experiment is repeated with different values of a and b and that each time the monkey trades the first 4 coins for apples and the next 8 coins for bananas.
   Show that the monkey's behavior is consistent with the classical assumptions of consumer behavior (namely, that his behavior can be explained as the maximization of a monotonic, continuous, and convex preference relation on the space of bundles).
2. Assume that it was later observed that when the monkey holds an arbitrary number m of coins, then, irrespective of the values of a and b, he exchanges the first 4 coins for apples and the remaining $m - 4$ coins for bananas. Is this behavior consistent with the rational consumer model?

Problem B7. (*NYU 2006*)
Consider a consumer in a world of 2 commodities who has to make choices from budget sets parametrized by (p, w), with the additional constraint that the consumption of good 1 is limited by some external bound $c \geq 0$. That is, in his world, a choice problem is a set of the form $B(p, w, c) = \{x | px \leq w$ and $x_1 \leq c\}$. Denote by $x(p, w, c)$ the consumer's choice from $B(p, w, c)$.

1. Assume that $px(p, w, c) = w$ and $x_1(p, w, c) = \min\{0.5w/p_1, c\}$. Show that this behavior is consistent with the assumption that demand is derived from a maximization of some preference relation.
2. Assume that $px(p, w, c) = w$ and $x_1(p, w, c) = \min\{0.5c, w/p_1\}$. Show that this consumer's behavior is inconsistent with preference maximization.
3. Assume that the consumer chooses his demand for x by maximizing the utility function $u(x)$. Denote the indirect utility by $V(p, w, c) =$

$u(x(p,w,c))$. Assume V is "well-behaved". Outline the idea of how one can derive the demand function from the function V in case that $\partial V/\partial c(p,w,c) > 0$.

Problem B8. (*Tel Aviv 2006*)

Imagine a consumer who lives in a world with $K+1$ commodities and behaves in the following manner: The consumer is characterized by a vector D, consisting of the commodities $1,\ldots,K$. If he can purchase D, he will consume it and spend the rest of his income on commodity $K+1$. If he is unable to purchase D, he will not consume commodity $K+1$ and will purchase the bundle tD ($t \le 1$) where t is as large as he can afford.

1. Show that there exists a monotonic and convex preference relation that explains this pattern of behavior.
2. Show that there is no monotonic and continuous preference relation that explains this pattern of behavior.

Problem B9. (*NYU 2007*)

A consumer in a world of K commodities maximizes the utility function $u(x) = \sum_k x_k^2$.

1. Calculate the consumer's demand function (whenever it is uniquely defined).
2. Give another preference relation (not just a monotonic transformation of u) that induces the same demand function.
3. For the original utility function u, calculate the indirect preferences for $K = 2$. What is the relationship between the indirect preferences and the demand function? (It is sufficient to answer for the domain where $p_1 < p_2$.)
4. Are the preferences in (1) differentiable (according to the definition given in class)?

Problem B10. (*NYU 2008*)

A decision maker has a preference relation over the pairs (x_{me}, x_{him}) with the interpretation that x_{me} is an amount of money he will get and x_{him} is the amount of money another person will get. Assume that:

(i) for all (a,b) such that $a > b$, the decision maker strictly prefers (a,b) over (b,a).

(ii) if $a' > a$, then $(a',b) \succ (a,b)$.

The decision maker has to allocate M between him and another person.

1. Show that these assumptions guarantee that he will never allocate to the other person more than he gives himself.

2. Assume (i), (ii), and
   (iii) The decision maker is indifferent between (a, a) and $(a - \epsilon, a + 4\epsilon)$ for all a and $\epsilon > 0$.
   Show that nevertheless he might allocate the money equally.
3. Assume (i), (ii), (iii), and
   (iv) The decision maker's preferences are also differentiable (according to the definition given in class).
   Show that in this case, he will allocate to himself (strictly) more than to the other.

Problem B11. (*Tel Aviv 2010*)

A basketball coach considers buying players from a set A. Given a budget w and a price vector $(p_a)_{a \in A}$, the coach can purchase any set such that the total cost of the players in it is not greater than w. Discuss the rationality of each of the following choice procedures, defined for any budget level w and price vector P:

P1: The consumer has in mind a fixed list of the players in A: $a_1, \ldots, a_n$. Starting at the beginning of the list, when he arrives to the $i'th$ player, he adds him to the team if his budget allows him to after his past decisions and then continues to the next player on the list with his remaining budget. This continues until he runs out of budget or has gone through the entire list.

P2: He purchases the combination of players that minimize the excess budget he is left with.

Problem B12. (*NYU 2010*)

A consumer in a two-commodity world operates in the following manner: The consumer has a preference relation $\succsim_S$ on $\mathbb{R}^2_+$. His father has a preference relation $\succsim_F$ on the space of his son's consumption bundles. Both relations satisfy strong monotonicity, continuity, and strict convexity. The father does not allow his son to purchase a bundle that is not as good (from his perspective) as the bundle $(M, 0)$. The son, when choosing from a budget set, maximizes his own preferences subject to the constraint imposed by his father. In the case that he cannot satisfy his father's wishes, he feels free to maximize his own preferences.

1. Prove that the behavior of the son is rationalizable.
2. Prove that the preferences that rationalize this kind of behavior are monotonic.
3. Show that the preferences that rationalize this kind of behavior are not necessarily continuous or convex (you can demonstrate this diagrammatically).

4. (Bonus) Assume that the father's instructions are that given the budget set (p, w) the son is not to purchase any bundle that is $\succsim_F$-worse than $(w/p_1, 0)$. The son seeks to maximize his preferences subject to satisfying his father's wishes. Show that the son's behavior satisfies the Weak Axiom of Revealed Preferences.

Problem B13. (*NYU 2012*)
A consumer operates in a world with K commodities. He has in mind a list of consumption priorities, a sequence (k_n, q_n) where $k_n \in \{1, ..., K\}$ is a commodity and q_n is a quantity (commodities may appear more than once in the sequence). When facing a budget set (p, w) he purchases the goods according to the order of priorities in the list, until his budget is exhausted. (In the case that his money is exhausted during the n'th stage he purchases whatever proportion of the quantity q_n that he can afford).

1. How does the demand for the k'th commodity responds to the p_k, p_j $(j \neq k)$ and w?
2. Suggest an increasing utility function which rationalizes the consumer's behavior.
3. Using the utility function you suggested in (2) prove the Roy equality for this consumer at (p, w) where the consumer exhausts his entire budget while satisfying his n'th goal.

Problem B14. (*Tel Aviv 2013*)
Consider a consumer in a world with two commodities. He has two continuous strictly-increasing evaluation functions v_1 and v_2 with a range $[0, \infty)$. Facing a budget set $B(p_1, p_2, w)$, the consumer compares between $v_1(w/p_1)$ and $v_2(w/p_2)$ and spends all of his resources on the good that yields a higher evaluation (in the case of a tie he arbitrarily chooses one of the goods).

1. Show that this behavior is consistent with maximizing continuous, monotonic and convex preferences over R^2_+.
2. Show that this behavior is inconsistent with maximizing continuous, monotonic and strictly convex preferences over R^2_+.
3. Does the demand function satisfy the "law of demand" (according to which a decrease in the price of a commodity weakly increases the demand for it)?

Problem B15. (*NYU 2013*)
Imagine a consumer who operates in two stages when facing a budget set $B(p, w)$ in a world with the commodities $1, ..., K$ split into two exclusive non-empty groups A and B:

Stage 1: He allocates w between the two groups by maximizing a function v on the set of pairs (w_A, w_B).

Stage 2: He chooses an A-bundle maximizing a function u_A defined over the A-bundles given w_A, and independently chooses a B-bundle that maximizes a function u_B defined over the B-bundles given w_B.

1. Show that if the consumer is interested in choosing a bundle (over the K commodities) that in the end maximizes the (ridiculous) utility function $\Pi_{k=1,..,K} x_k^{\alpha_k}$ (where $\alpha_k > 0\ \forall k$ and $\sum_{k=1}^{K} \alpha_k = 1$), then he can attain this goal by following the procedure above with some functions (v, u_A, u_B).
2. Show that the claim in (1) is not true in general. For example, you might (but don't have to) look at the case $K = 4$, $A = \{1, 2\}$, $B = \{3, 4\}$ and the utility function $max\{x_1x_3, x_2x_4\}$. (Note that this is the max, not the min function.)
3. (More Difficult) Show that if the consumer follows the above procedure, then it might be that his overall choice cannot be rationalized. (For the first stage, you can choose a simple function like $v = min\{w_A, w_B\}$.)

Problem B16. (*NYU 2014*)
A DM needs to decide how to allocate a budget between two activities: 1 and 2. A combination of activities is a pair (a_1, a_2) where a_i is the level of activity i. The DM's problem is to choose a combination of activities given a budget w and a vector of prices for the activities (p_1, p_2).

Two consultants, A and B, are involved in the DM's process. Each consultant submits to the DM a recommendation which is the outcome of maximizing a "classical" and differentiable preference relation defined over the set of all activity combinations. Assume that whatever the "budget set" is, consultant A always recommends a (weakly) higher level of activity 1 than B does. Formally, assume that at each combination of activities (a_1, a_2) the "marginal rate of substitution" (the ratio of local values) of A is strictly larger than that of B.

The DM collects the two recommendations and then:

If both recommend that the level of a certain activity i should be higher than that of the other activity, then the DM follows the more "moderate recommendation", namely the one which is closer to the main diagonal.

If consultant A recommends a higher level of activity 1 and B recommends a higher level of activity 2, then the DM spends his entire budget such that he consumes equal levels of the two activities (i.e., a combination on the main diagonal).

1. Assume that A aims to maximize $2a_1 + a_2$ (and in the case of indifference recommends only activity 1) and B seeks to maximize $a_1 + 2a_2$ (and in the case of indifference recommends only activity 2). Is the DM's behavior rationalizable in the sense that there exists a convex and monotonic preference relation that rationalizes the DM's behavior?
2. Extend your answer to any two consultants that satisfy the question's assumptions.

Problem B17. (*NYU 2015*)
Consider a decision maker on the space $X = [0, 1]$ where $t \in X$ is interpreted as the portion of the day he contributes to society.

1. Assume that he has a strictly convex and continuous preference relation over X. Show that he has a "single peak" preference relation, namely that there exists x^* such that for every $x^* \leq y < z$ or $z < y \leq x^*$ he strictly prefers y to z. Find a strictly convex preference relation on this space which is not continuous.
2. Assume that the domain of the decision maker's choice function contains all sets of the form $B(w, \rightarrow) = \{x \in X \mid x \geq w\}$, as well as of the form $B(w, \leftarrow) = \{x \in X \mid x \leq w\}$, where $w \in [0, 1]$. Interpret these sets. Show that the decision maker's choice function induced from a strictly convex and continuous preference relation is always well-defined and continuous in w.

Problem B18. (*NYU 2016*)
Consider an economic agent with preferences $\succsim^1$ on the set of the bundles in a K-commodity world. The agent holds a bundle x^* and can consume any part of it; however, he feels obliged to give to his friend (who holds the preference relation $\succsim^2$) a bundle which will be at least as good for his friend as a fixed bundle y^*. Assume that $x^*_k > y^*_k$ for all k. Both preference relations satisfy strong monotonicity, continuity and strict convexity.

1. State the agent's problem and explain why a solution exists and is unique.
2. Denote the bundle the agent consumes given x^* as $z(x^*)$. The agent's indirect preferences on the space of initial bundles can be defined by $a^* \succsim^* b^*$ if $z(a^*) \succsim^1 z(b^*)$. Show that the indirect preferences are strictly convex and continuous.
3. Show that if $\succsim^1$ is differentiable then so is $\succsim^*$.

Problem B19. (*NYU 2017*)
Consider a consumer in a world with n goods who makes a choice from a budget set in the following way: he consults with n "classical consumers", each of whom recommends a bundle that maximizes the consultant's preference relation (which is monotonic, continuous, convex and has the property that there is a unique solution to its maximization over any budget set). Then, he chooses the average between the consultants' recommendations.

1. Assume that each consultant has a unique good which he always recommends to purchase exclusively. Can the consumer be rationalized?
2. Prove that the demand function of the consumer satisfies both continuity and "Walras' Law".
3. 3. Construct an example of a pair of monotonic and convex preference relations (in a world with two goods and two consultants) in which the aggregated consumer cannot be fully rationalized by any preference relation (not even one that is not monotonic). For simplicity you can construct the preference relations to be non-continuous.

Problem B20. (*NYU 2018*)
Consider a consumer in a world with n goods who A residual good Assume that the decision to save is a "residual" decision. To model it consider a world with K goods and denote "saving" as the K'th good. Assume that the consumer has in mind a continuous and strictly concave function v defined over R_+^{K-1} where $v(x_1, .., x_{k-1})$ is the value he attaches to the combination of the first $K-1$ goods. This function is not necessarily increasing! The consumer also has in mind an aspiration level v^*. He is not interested in combinations of $K-1$ goods with a value which exceeds v^*. That is, given a price vector $p = (p_1, .., p_{K-1})$ and

wealth level w he maximizes the function v over $\{(x_1,.,,,x_{K-1}) \mid px \leq w$ and $v(x_1,..,x_{K-1}) \leq v^*\}$. If he is left with some wealth he spends it on the K'th good (savings).

1. Show that this behavior can be (fully) rationalized by a preference relation over all vectors $(x_1,..,x_K) \in \Re^K_+$.
2. Can it be (fully) rationalized by a continuous preference relation (focus on $K=2$)?
3. Can it be (fully) rationalized by a differentiable preference relation (focus on $K=2$)?

Problem B21. (*NYU 2019*)

In this question, you are provided with an alternative definition of convex preferences (based on Richer and Rubinstein (2019)). Let X be a set and Λ be a set of primitive preference relations on X. We say that a preference relation $\succsim$ on X is Λ-convex if the following condition is satisfied: For every two elements a and b in X, if for every primitive preference relation $\trianglerighteq \in \Lambda$ there is $y \neq a$ (although y can be equal to b) such that $a \trianglerighteq y$ and $y \succsim\ b$, then $a \succsim$ b. Thus, for example, consider an individual with a preference relation over the universities in the US and his primitive relations (members of Λ) are "more prestigious" and "cheaper tuition". Then if a is more prestigious than b and he prefers b to c and if a is cheaper than b and he also prefers b to c then for his preferences to be Λ-convex the individual must prefer a to c.

a. Assume that $X=[0,1]$ and $\Lambda = \{\leq\ ,\ \geq\}$. Show that a preference relation is Λ-convex if and only if it is convex in the standard sense.

b. Assume that $\Lambda = \{P^1, P^2\}$ where P^1, P^2 are preference relations represented by the utility functions u^1 and u^2. Let $U(x) = min\{u^1(x), u^2(x)\}$. Show that the preference relation represented by U is Λ-convex.

C. Uncertainty

Problem C1. (*Princeton 1997*)
A decision maker forms preferences over the set X of all possible distributions of a population over two categories (such as living in two locations). An element in X is a vector (x_1, x_2) where $x_i \geq 0$ and $x_1 + x_2 = 1$. The decision maker has two considerations in mind:

- He thinks that if $x \succsim y$, then for any z, the mixture of $\alpha \in [0, 1]$ of x with $(1 - \alpha)$ of z should be at least as good as the mixture of α of y with $(1 - \alpha)$ of z.
- He is indifferent between a distribution that is fully concentrated in location 1 and one that is fully concentrated in location 2.

1. Show that the only preference relation that is consistent with the two principles is the degenerate indifference relation ($x \sim y$ for any $x, y \in X$).
2. The decision maker claims that you are wrong because his preference relation is represented by a utility function $|x_1 - 1/2|$. Why is he wrong?

Problem C2. (*Tel Aviv 1999*)
Tversky and Kahneman (1986) report the following experiment: each participant receives a questionnaire asking him to make two choices, the first from $\{a, b\}$ and the second from $\{c, d\}$:

a. A sure profit of $240.
b. A lottery between a profit of $1,000 with probability 25% and 0 with probability 75%.
c. A sure loss of $750.
d. A lottery between a loss of $1,000 with probability 75% and 0 with probability 25%.

The participant will receive the sum of the outcomes of the two lotteries he chooses. 73% of the participants chose the combination a and d. Is their behavior sensible?

Problem C3. (*Princeton 2001*)
A consumer has to make a choice of a bundle *before* he is informed whether a certain event, which is expected with probability α and affects his welfare, has happened or not. He assigns a vNM utility $v(x)$ to the consumption of the bundle x when the event occurs, and a vNM utility

$v'(x)$ to the consumption of x should the event not occur. Having to choose a bundle, the consumer maximizes his expected utility $\alpha v(x) + (1-\alpha)v'(x)$. Both v and v' induce preferences on the set of bundles satisfying the standard assumptions about the consumer. Assume also that v and v' are concave.

1. Show that the consumer's preference relation is convex.
2. Find a connection between the consumer's indirect utility function and the indirect utility functions derived from v and v'.
3. A new commodity appears on the market: "A discrete piece of information that tells the consumer whether or not the event occurred". The commodity can be purchased prior to the consumption decision. Use the indirect utility functions to characterize the demand function for the new commodity.

Problem C4. (*NYU 2006*)

Consider a world with balls of K different colors. An object is called a bag and is specified by a vector $x = (x_1, .., x_K)$ (where x_k is a nonnegative integer indicating the number of balls of color k). For convenience, denote by $n(x) = \sum x_k$ the number of balls in bag x.

An individual has a preference relation over bags of balls.

1. Suggest a context where it will make sense to assume that:
   i. For any integer λ, $x \sim \lambda x$.
   ii. If $n(x) = n(y)$, then $x \succsim y$ iff $x + z \succsim y + z$.
2. Show that any preference relation that is represented by $U(x) = \sum x_k v_k / n(x)$ for some vector of numbers $(v_1, \ldots, v_k)$ satisfies the two axioms.
3. Find a preference relation that satisfies the two properties that cannot be represented in the form suggested in (2).

Problem C5. (*NYU 2007*)

Identify a professor's lifetime with the interval $[0, 1]$. There are $K + 1$ academic ranks, $0, \ldots, K$. All professors start at rank 0 and eventually reach rank K. Define a career as a sequence $t = (t_1, \ldots, t_K)$ where $t_0 = 0 \le t_1 \le t_2 \le \ldots \le t_K \le 1$ with the interpretation that t_k is the time it takes to get the k'th promotion. (Note that a professor can receive multiple promotions at the same time.) Denote by $\succsim$ the professor's preferences on the set of all possible careers.

For any $\epsilon > 0$ and for any career t such that $t_K \le 1 - \epsilon$, define $t + \epsilon$ to be the career $(t + \epsilon)_k = t_k + \epsilon$ (i.e., all promotions are delayed by ϵ).

Following are two properties of the professor's preferences:
Monotonicity: For any two careers t and s, if $t_k \leq s_k$ for all k, then $t \succsim s$, and if $t_k < s_k$ for all k, then $t \succ s$.
Invariance: For every $\epsilon > 0$ and every two careers t and s for which $t + \epsilon$ and $s + \epsilon$ are well defined, $t \succsim s$ iff $t + \epsilon \succsim s + \epsilon$.

1. Formulate the set L of careers in which a professor receives all K promotions at the same time. Show that if $\succsim$ satisfies continuity and monotonicity, then for every career t there is a career $s \in L$ such that $s \sim t$.
2. Show that any preference that is represented by the function $U(t) = -\sum \Delta_k t_k$ (for some $\Delta_k > 0$) satisfies Monotonicity, Invariance, and Continuity.
3. One professor evaluates a career by the maximum length of time one has to wait for a promotion, and the smaller this number the better. Show that these preferences cannot be represented by the utility function described in (2).

Problem C6. (*NYU 2008*)
An economic agent has to choose between projects. The outcome of each project is uncertain. It might yield a failure or one of K "types of success". Thus, each project z can be described by a vector of K non-negative numbers, $(z_1, \dots, z_K)$, where z_k stands for the probability that the project success will be of type k. Let $Z \subset \mathbb{R}^K_+$ be the set of feasible projects. Assume Z is compact and convex and satisfies "free disposal". The decision maker is an Expected Utility maximizer. Denote by u_k the vNM utility from the k'th type of success, and attach 0 to failure. Thus the decision maker chooses a project (vector) $z \in Z$ in order to maximize $\sum z_k u_k$.

1. First, formalize the decision maker's problem. Then, formalize (and prove) the claim: if the decision maker suddenly values type k success higher than before, he would choose a project assigning a higher probability to k.
2. Apparently, the decision maker realizes that there is an additional uncertainty. The world may go "one way or another". With probability α the vNM utility of the k'th type of success will be u_k and with probability $1 - \alpha$ it will be v_k. Failure remains 0 in both contingencies.

   First, formalize the decision maker's new problem. Then, formalize (and prove) the claim: Even if the decision maker would obtain the same expected utility, would he have known in advance the direction of the world, the existence of uncertainty makes him (at least weakly) less happy.

Problem C7. (*NYU 2009*)

For any nonnegative integer n and a number $p \in [0, 1]$, let (n, p) be the lottery that gets the prize $\$n$ with probability p and $\$0$ with probability $1 - p$. Let us call those lotteries *simple lotteries*. Consider preference relations on the space of simple lotteries.

We say that such a preference relation satisfies Independence if $p \succeq q$ iff $\alpha p \oplus (1-\alpha)r \succeq \alpha q \oplus (1-\alpha)r$ for any $\alpha > 0$, and any simple lotteries p, q, r for which the compound lotteries are also simple lotteries.

Consider a preference relation satisfying the Independence axiom, strictly monotonic in money and continuous in p. Show that:

1. (n, p) is monotonic in p for $n > 0$, that is, for all $p > p'$ $(n, p) \succ (n, p')$.
2. For all n there is a unique $v(n)$ such that $(1, 1) \sim (n, 1/v(n))$.
3. It can be represented with the expected utility formula: that is, there is an increasing function v such that $pv(n)$ is a utility function that represents the preference relation.

Problem C8. (*Tel Aviv 2012*)

A decision maker has in mind a function CE, with the interpretation that for every lottery p, $CE(p)$ is the certainly equivalence of p. Following are two procedures for deriving the function.

Procedure 1: The decision maker has in mind an increasing vNM utility function u and his answer satisfies $Eu(p) = u(CE(p))$.

Procedure 2: The decision maker has in mind two increasing, continuous and concave functions g (for gains) and l (for losses) which satisfy $g(0) = l(0) = 0$. $CE(p)$ is a number x which equalizes the expected "loss"with the expected "gain", that is satisfies $\sum_{y<x} p(y)l(x-y) = \sum_{y>x} p(y)g(y-x)$.

1. Explain why pD_1q implies under the two procedures that $CE(p) \geq CE(q)$.
2. Explain why the first procedure allows behavior which is not possible under procedure 2.
3. (More Difficult) Can any individual who operates by procedure 2 be described as working through procedure 1?

Problem C9. (*NYU 2012*)
Consider a decision maker in the world of lotteries, with $Z = R$ being monetary prizes. The decision maker assigns a number $v(z)$ to each amount of money z. The function v is continuous and increasing. The decision maker evaluates each lottery p according to: $U(p) = \alpha[max\{v(z)|z \in supp(p)\}] + (1-\alpha)[\min\{v(z)|z \in supp(p)\}]$.

1. Characterize the decision makers of this type who are "risk averse".
2. Show that if two decision makers of this type, with $\alpha = 1/2$, hold the functions v_1 and v_2 and $v_1 \circ v_2^{-1}$ is concave, then decision maker 1 is more risk averse than decision maker 2.
3. (More difficult) Assume that the two decision makers use $\alpha = 1/2$. Is the concavity of $v_1 \circ v_2^{-1}$ a necessary condition for decision maker 1 to be more risk averse than decision maker 2.

Problem C10. (*NYU 2014*)
Consider the following family of preference relations defined over $L(Z)$ (the set of all lotteries with prizes in some finite set Z): The DM has in mind a function which assigns to each prize $z \in Z$ a value $v(z)$. He partitions Z into two sets G and B such that if $g \in G$ and $b \in B$ then $v(g) > v(b)$. He evaluates any lottery p by

$p(Supp(p) \cap G) \max_{z \in Supp(p) \cap G} v(z) + p(Supp(p) \cap B) \min_{z \in Supp(p) \cap B} v(z)$.

These evaluations form his preferences over $L(Z)$ (where $p(A) = \sum_{z \in A} p(z)$).

1. Explain the procedure in words.
2. Show that such a preference relation satisfies neither the Independence axiom nor the Continuity axiom.
3. Show that a weaker independence property holds: If $Supp(p) = Supp(q)$ then for every $1 > \alpha > 0$ and every r, $p \succsim q$ iff $\alpha p \oplus (1-\alpha) r \succsim \alpha q \oplus (1-\alpha) r$.
4. Describe in words and then formally define a "monotonicity property" that holds.

Problem C11. (*NYU 2015*)
Define an "amount of money" to be any positive integer. Define a "wallet" to be a collection of amounts of money. Denote the wallet with K amounts of money $x_1, .., x_K$ by $[x_1, .., x_K]$. Thus, for example, the wallet [3,3,4] with a total of 10 is identical to the wallet [4,3,3] and is different than the wallet [3,4] which has a total of 7. Let X be the set of all wallets. The following are two properties of preference relations over X :

(i) *Monotonicity*:

1. Adding an amount of money to the wallet or increasing one of the amounts is weakly improving.
2. Increasing all amounts is strictly improving.

(ii) *Split aversion:* Combining two amounts of money is (at least weakly) improving (thus [7,3] is at least as good as [4,3,3]).

1. Let v be a function defined on the natural numbers satisfying (i) $v(0) = 0$, (ii) it is strictly increasing and (iii) superadditivity ($v(x + y) \geq v(x) + v(y)$ for all x, y). Show that the function $u([x_1, ..., x_K]) = \Sigma_{k=1,.,.,K} v(x_k)$ is a utility function which represents a preference relation on X that satisfies monotonicity and split aversion.
2. Give an example of a preference relation satisfying monotonicity but not split aversion and an example of a preference relation satisfying split aversion but not monotonicity.
3. Define the notion that one preference relation is more split averse than the other.
4. Find a preference relation (satisfying monotonicity and split aversion) which is less split averse than any other split averse and monotonic preference relation.
5. Show that the relation represented by the function $u([x_1, ..., x_K]) = \max\{x_1, .., x_K\}$ is more split averse than any preference relation of the type described in part (1).

Problem C12. (*NYU 2016*)
Discuss the attitude of an agent towards lotteries over a set of consequences $Z = \{a, b, c\}$ satisfying that he ranks a first and c last.

Consider any preference relation (on $L(Z)$) satisfying independence and continuity. Obviously, each preference relation can be described by a single number $v \in (0,1)$ by attaching the numbers $1,v$, 0 to the three alternatives. Denote this preference relation by $\succsim_v$.

For a set $V \subseteq (0,1)$, define a choice correspondence $C_V(A)$ as the set of all $p \in A$ satisfying that there is no $q \in A$ such that $q \succ_v p$ for all $v \in V$.

Define the binary relation pD^*q if $p(a) \geq q(a)$ and $p(a)+p(b) \geq q(a)+q(b)$ with at least one strict inequality. Consider the choice correspondence C defined by $p \in C(A)$ if there is no $q \in A$ such that qD^*p. Show that $C = C_V$ for some set V.

Problem C13. (*NYU 2017.*)

Let Z be a finite set of prizes. Let X be the set of all lotteries with one or two prizes in the support. Denote by $p(x)x \oplus p(y)y$ the lottery that gets x with probability $p(x)$ and y with probability $p(y)$. Assume a preference relation $\succsim$ on X only (and not on the set of all lotteries). The relation satisfies Continuity and Independence.

Show that the preference relation can be represented as an expected utility maximizer although the domain is smaller than what we assumed in class. But before that:

1. Formulate the Independence axiom for this domain.
2. Explain why the claim does not follow immediately from the proof given in class.
3. Explain why Continuity* and Monotonicity do hold (don't prove it again, just explain). *Continuity**: $\forall x,y,z \in Z$ s.t. $x \succ z \succ y \exists \alpha \in (0,1)$ s.t. $z \sim \alpha x \oplus (1-\alpha)y$. *Monotonicity*: Let $x,y \in Z$ s.t. $x \succ y$ and $1 \geq \alpha > \beta \geq 0$. Then $\alpha x \oplus (1-\alpha)y \succ \beta x \oplus (1-\beta)y$.
4. Let M be a $\succsim$-best prize and m the $\succsim$-worst prize. Define $v(x)$ as the probability satisfying that $x \sim v(x)M \oplus (1-v(x))m$. Show that if $(1-\alpha)x \oplus \alpha M \sim y$ then $v(y) = \alpha + (1-\alpha)v(x)$.
5. (More difficult) Complete the proof.

Problem C14. (*NYU 2020.*)

A middleman is operating in a world with K divisible goods. He faces two sets of prices p and q where p_k is the price he has to pay for one unit of good k and and q_k is the price he receives when he sells one unit of good k. He starts the day in village A where he can buy goods at the prices in p, and travels to B where he sells the goods at the prices in q. He begins the day with a limited amount of money M and he aims to have as much money as possible at the end of the day.

(a) Formulate the middleman problem. Derive his demand function (as a function of p, q, M). Drive also an indirect utility function for the middleman.

(c) Assume that the middleman faces uncertainty: with probability α he expects to be robbed on the way from A and B and that all his goods will be stolen (though not his money – which he leaves at A). Assume he behaves consistently with expected utility theory and is risk averse. Draw a diagram in which the horizontal axis is the money he will have at the end of the day if he is not robbed and the vertical axis is the money he will have at the end of the day if he is robbed. Explain (diagramatically) why it is possible that he will actively trade only one of the goods and will not use the entire amount of money.

D. Social Choice

Problem D1. (*Princeton 2000*)
Consider the following social choice problem: a group has n members who must choose from a set containing 3 elements $\{A, B, L\}$, where A and B are prizes and L is the lottery that yields each of the prizes A and B with equal probability. Each member has a strict preference over the three alternatives that satisfies vNM assumptions. Show that there is a nondictatorial social welfare function that satisfies the independence of irrelevant alternatives axiom (even the strict version I^*) and the Pareto axiom (Par). Reconcile this fact with Arrow's Impossibility Theorem.

Problem D2. (*NYU 2009*)
We will say that a choice function C is consistent with the *majority vetoes a dictator procedure* if there are three preference relations $\succ_1$, $\succ_2$, and $\succ_3$ such that $c(A)$ is the $\succ_1$ maximum unless both $\succ_2$ and $\succ_3$ agree on another alternative being the maximum in A.

1. Show that such a choice function might not be rationalizable.
2. Show that such a choice function satisfies the following property: if $c(A) = a$, $c(A - \{b\}) = c$ for b and c different from a, then $c(B) = c$ for any B that contains c and is a subset of $A - \{b\}$.
3. Show that not all choice functions could be explained by the majority vetoes a dictator procedure.

Problem D3. (*Tel Aviv 2009. Inspired by Miller (2007).*)
Lately we have been using the term a "reasonable reaction" quite frequently. In this problem we assume that this term is defined according to the opinions of the individuals in the society with regard to the question: "What is a reasonable reaction?"

Assume that in a certain situation, the possible set of reactions is X and the set of individuals in the society is N.

A "reasonability perception" is a nonempty set of possible reactions that are perceived as reasonable.

The social reasonability perception is determined by a function f that attaches a reasonability perception (a nonempty subset of X) to any profile of the individuals' reasonability perception (a vector of nonempty subsets of X).

1. Formalize the following proposition:
   Assume that the number of reactions in X is larger than the number

of individuals in the society and that f satisfies the following four properties:

a. If in a certain profile all the individuals do not perceive a certain reaction as reasonable, then neither does the society.
b. All the individuals have the same status.
c. All the reactions have the same status.
d. Consider two profiles that are different only in one individual's reasonability perception. Any reaction that f determines to be reasonable in the first profile, and regarding which the individual did not change his opinion from reasonable to unreasonable in the second profile, remains reasonable.

Then f determines that a reaction is socially reasonable if and only if at least one of the individuals perceives it as reasonable.

2. Show that all four properties are necessary for the proposition.
3. Prove the proposition.

Problem D4. (*Tel Aviv 2010*)

Let $\succsim$ be a preference relation on $\mathbb{R}^n$ satisfying the following properties:

Weak Pareto (WP): If $x_i \geq y_i$ for all i, then $x = (x_1, \ldots, x_n) \succsim y = (y_1, \ldots, y_n)$, and if $x_i > y_i$ for all i, then $(x_1, \ldots, x_n) \succ (y_1, \ldots, y_n)$.

Independence (IIA): Let $a, b, c, d \in \mathbb{R}^n$ be vectors such that in any coordinate $a_i > b_i$, $a_i = b_i$, or $a_i < b_i$ if and only if $c_i > d_i$, $c_i = d_i$, or $c_i < d$, accordingly. Then, $a \succsim b$ iff $c \succsim d$.

1. Find a preference relation different from those represented by $u_i(x_1, \ldots, x_n) = x_i$ which satisfies the two properties.
2. Show, for $n = 2$, that there is an i such that $a_i > b_i$ implies $a \succ b$.
3. Provide a "social choice" interpretation for the result in (2). Explain how it differs from Arrow's Impossibility Theorem.
4. Expand (2) for any n.

Problem D5. (*NYU 2012. Based on Rubinstein (1980).*)

An individual is comparing pairs of alternatives within a finite set X ($|X| \geq 3$). His comparison yields unambiguous results, such that either x is evaluated to be better than y (denoted $x \rightarrow y$) or y is evaluated to be better than x ($y \rightarrow x$). A ranking method assigns to each such relation $\rightarrow$ (namely, complete, irreflexive and antisymmetric relation, but not necessarily transitive) a preference relation $\succsim(\rightarrow)$ over X. Consider the following axioms with respect to ranking methods:

(*i*) *Neutrality*: "the names of the alternatives are immaterial". (Formally, let σ be a permutation of X and let $\sigma(\to)$ be the relation defined by $\sigma(x)\sigma(\to)\sigma(y)$ iff $x \to y$. Then, $x \succsim (\to)y$ iff $\sigma(x) \succsim (\sigma(\to))\sigma(y)$.)

(*ii*) *Monotonicity*: if $x \succsim (\to)y$, then $x \succ (\to')y$ where $\to'$, differs from $\to$ only in the existence of one alternative z such that $z \to x$ and $x \to' z$.

(*iii*) *Independence*: The ranking between any two alternatives depends only on the results of comparisons that involve at least one of the two alternatives.

1. Define $N_{\to}(x) = |\{z|x \to z\}|$ (the number of alternatives beaten by x). Explain why the scoring method defined by $x \succsim (\to)y$ if $N_{\to}(x) \geq N_{\to}(y)$ satisfies the three axioms.
2. For each of the properties, give an example of a ranking method which satisfies the other two properties but not that one.
3. Prove that the above scoring method is the only one that satisfies the three properties.

Problem D6. (*Tel Aviv 2013*)

Society often looks for a representative agent. Assume for simplicity that the number of agents in a society is a power of 2 (1,2,4,8....). Each agent is one of a finite number of types (a member in a set T). A representative agent method (RAM) is a function F which attaches to any vector of types $(t_1, .., t_n)$ (where $n = 2^m$ and each $t_i \in T$) an element in $\{t_1, .., t_n\}$. Make the following assumptions about F:

(*i*) Anonymity: For any n and for any permutation σ of $\{1, .., n\}$, we have $F(t_1, .., t_n) = F(t_{\sigma(1)}, .., t_{\sigma(n)})$.

(*ii*) The "representative" is the "representative of the representatives": $F(t_1, .., t_n) = F(F(t_1, .., t_{n/2}), F(t_{n/2+1}, .., t_n))$.

1. Characterize the RAMs which satisfy the two axioms.
2. Suggest an RAM that satisfies (*i*) but not (*ii*) and an RAM that satisfies (*ii*) but not (*i*).

Problem D7. (*Tel Aviv 2014*)

We say that a binary relation P over the space $X = R^n$ satisfies Property I if the statement xPy (the relation between x and y) depends only on the equalities between the components of the two vectors. Formally, P satisfies Property I if $aPb \iff cPd$ for any four vectors a, b, c and d

that satisfy (i) $a_i = a_j \iff c_i = c_j$, (ii) $b_i = b_j \iff d_i = d_j$ and (iii) $a_i = b_j \iff c_i = d_j$.

Denote $Y = \{x | \forall i \neq j,\ x_i \neq x_j\}$ as the set of all vectors vectors that are composed of n different numbers.

1. Give an example (for $n = 2$) of non-degenerated preference relation on X that satisfies property I.

Show that any preference relation satisfying property I:

2. is indifferent between the vector $(1, 2, 3)$ and any of the vectors $(4, 2, 5)$, $(2, 3, 1)$ and $(4, 5, 6)$.
3. is indifferent between any $x, y \in Y$ satisfying $x_i \neq y_j$ for any i,j.
4. is indifferent between any $x, y \in Y$ where x is a permutation of y.
5. is indifferent between any $x, y \in Y$.
6. (much more difficult) Characterize the set of preference relations satisfying Property I.

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