Lecture B-1: Economic Allocation Mechanisms: An Introduction

Warning: These lecture notes are preliminary and contain mistakes!

1. A basic setup

Let us focus on the following fundamental question: how does a group of individuals who face a shortage of resources allocate those resources among its members?

In the model we deal with a society consisting of a set of agents $I = \{1,..,n\}$ and a set of houses $H = \{1,\dots,m\}$. For simplicity assume that $m = n$ (an extension is straightforward).

Each agent can possess only one house. Each house can be occupied by at most one agent.

Agent i holds a strict ordering $\succ_i$ over the set H (I will comment on the more general case of preferences with indifferences).

We are interested in mechanisms which allocate the houses among the agents.

A *feasible allocation* a is a function from I into H such that there is no $h \in H$ for which $a(i) = a(j) = h$ for some $i \neq j$. This reflects the assumption that an agent cannot hold more than one house.

We say that an allocation a is *efficient* (or *Pareto Efficient*) if there is no other allocation b such that for all i we have $b(i) \succsim_i a(i)$ and for at least one j, $b(j) \succ_j a(j)$.

Efficiency has two interpretations. From the welfare point of view this is a minimal condition for optimality. From "stability" point of view an inefficient allocation might be unstable in the sense that a group of people could re-alloacte the houses they hold so that all be happier.

2. Mechanisms:

A *mechanism* attaches to every profile of preferences a feasible allocation. A *random mechanism* attaches to every profile a lottery over the set of feasible allocations. The following are some mechanisms (some of them involve random elements) which make sense in our setup:

(1) Maximize Points: The preferences are translated into ”points” (the more

preferred houses get more points) and a computer calculates the allocation which maximizes the sum of the ”points” over all individuals.

(2) First, look for the houses which are considered most popular by the largest number of individuals. Allocate each of them randomly to one of the individuals who likes them most. Continue the process with the remaining houses and the remaining agents.

(3) Choose randomly one of the Pareto-efficient allocations.

(4) Lottery: The houses are allocated to the individuals by a fair lottery (like has been done in Joshua’s allocation of the “nachalot" (inheritances). This random mechanism seems fair (ex-ante) but can yield unfair outcomes (ex-post) as it is invariant of the individuals’ preferences. The realized allocation might be inefficient.

(5) Sequential Dictatorship with Random Ordering: An ordering of the agents is determined by a fair lottery. Then, agents are called one after the other to choose a house from those yet unchosen. This random mechanism is sensitive to the individuals’ preferences. The procedure is fair ex-ante and yields an efficient ex-post outcome (to be shown later).

In the following mechanisms some additional information is fed in:

(6) Big Brother: A computer attaches to each individual i and each house h, a number $v_i(h)$ interpreted as the “value” of i holding h. The computer then looks for an allocation which maximizes the sum $\sum_i v_i(a(i))$ over all feasible allocations.

(7) A game: The individuals play a game which ends with an allocation or with disagreement. An example of such a game is one where in each round each individual makes a demand (for a house). If the demands are compatible (no two individuals demand the same house), the allocation is implemented; if not the game continues to the next round without a time limit.

One can see that for the $n = 2$ case, the equilibrium outcome of the game must be an agreement. However, for $n \geq 3$ there are equilibria with no agreement (all agents always demand house 1, no one can unilaterally deviate and gain). The bargaining outcome depends on additional information about the agents, such as time impatience.

(8) The houses are “auctioned” one by one. House 1 is auctioned first. The agents are required to do push ups and the one who does the most will get the house. In order to talk about an equilibrium in this model we need to enrich the

model with information about the trade-off between the individuals' preferences over houses and their readiness to do push ups.

Note, that a mechanism can be viewed as a social choice function (see Lecture 9 in the book). The social alternatives are the allocations and each individual has a preference relation on the set of allocations. Note that the assumption that an agent cares only about the house he gets makes the problem a social choice problem with *restricted* domain.

The rest of the lecture is devoted to a discussion of two models, the jungle and the market, where additional information is either a power relationship or an initial ownership. In discussing each model we will describe an equilibrium concept and we will think about it as the outcome of the mechanism.

3. **The jungle**

We are about to add to the model information about the relative power of the individuals. Let S be a power relation on the set of individuals. The statement iSj means that i is stronger than j and can take from him whatever asset he holds. We assume that the relation S is an ordering and without loss of generality assume $1S2S3\ldots Sn$. (Note that the interpretation of power is not necessary physical, power can be originate by, for example, "seniority".)

An allocation is a function from I to H. We look for an allocation of the houses which will remain "stable" despite the potential forces operating in the model. We say that an allocation $a = \{a(i)\}_{i \in I}$ is a *Jungle Equilibrium* if there is no i such that there exists a house h held by one of the individuals who are weaker than him such that $h \succ_i a(i)$. In other words, an allocation is a jungle equilibrium if no agent i can improve his situation by replacing his house with a house held by a weaker agent.

Claim 1: A jungle equilibrium exists.

Proof: Consider the allocation which is obtained by "calling" the agents one by one, according to the order of power, to pick a house from those not allocated earlier in the process. Namely, agent 1 picks first and chooses his preferred house $a(1)$. Agent 2 chooses then the best house according to his preferences from among $H - \{a(1)\}$, denote it by $a(2)$ and continue on. The allocation is a jungle equilibrium since each agent i possesses a house which is the best (according to his preferences) from among $H - \{a(1), \ldots a(i-1)\}$.

Claim 2: The jungle equilibrium is unique.

Proof: Assume both a and b are two different jungle equilibria. Let i be the strongest agent for which $a(i) \neq b(i)$. Assume $a(i) \succ_i b(i)$. It must be that $a(i) = b(j)$ for some agent j weaker than i. This contradicts b being a jungle equilibrium since iSj and $b(j) \succ_i b(i)$.

Claim 3: **(The first fundamental welfare theorem) A jungle equilibrium is efficient.**

Proof: Let a be a jungle equilibrium. Consider a feasible allocation b such that $b(j) \succsim_j a(j)$ for all j with at least one strict inequality. Let j^* be the strongest agent for whom $b(j) \neq a(j)$. It must be that $b(j^*) \succ_{j^*} a(j^*)$ and that $b(j^*) = a(j)$ for some j^*Sj, a contradiction for a being a jungle equilibrium.

Claim 4: (The second fundamental welfare theorem) For any efficient allocation there is a power relation such that the allocation is a jungle equilibrium with this relation.

Proof: Let a be an efficient allocation. Define a relation iEj if $a(j) \succ_i a(i)$ interpreted as "i envies j". Given that the allocation a is efficient, the E relation is acyclic: if there was a cycle, $n_1 E n_2 E. \ldots, E n_K E n_1$ then the allocation where each i_k gets $a(i_{k+1})$ (and i_{n_K} gets $a(n_1)$) would Pareto dominate a, a contradiction to the efficiency of a. Complete the acyclic relation E to an ordering E^* and define iSj if jE^*i. The allocation a is a Jungle equilibrium of the Jungle with this power relation P since we arranged that if i envies j, then j would be stronger than him.

A discussion:

1) The uniqueness and efficiency results will not be true if preferences with indifferences.

2) The preferences don’t exhibit externalities. With externalities the efficiency

property will not necessarily hold.

3) The welfare of an individual depends only on his "consumption" (the house he possesses) and not on the budget set (considerations like "I am happy since I can consume" do not enter into the model).

4) We do not talk here about the dynamics which leads to the final allocation (an interesting issue by itself). Rather we define and analyze an equilibrium concept.

5) One can think about other notions of power where the outcome of a conflict is not deterministic and where the power depends on the house which is in dispute (namely i might be stronger than j regarding h and weaker regarding h').

4. **Markets**

Markets are based on the notion of ownership. The additional information is e, an initial allocation of the houses. Each agent i enters the model owning a house $e(i)$. The basic idea behind the notion of a market is that nobody is forced to make an exchange. Exchange is voluntary and requires the consent of all involved parties. Again, we will not talk here about a dynamics which leads to the final allocation and confine ourselves to define and analyze an equilibrium concept.

A candidate for a competitive equilibrium is a pair (a,p) which consists of:

(1) a function $a : I \to H$.

(2) a price vector $p = (p_h)$: one price for each house h.

For a pair (a,p) to be a competitive equilibrium we require that:

(1) $a(i)$ maximizes $\succsim_i$ given the "*budget set*" $\{h | p_h \leq p_{e(i)}\}$. That is, for every i, buying the house $a(i)$ is optimal given his budget set. In other words, the house $a(i)$ is the best house for agent i from the set of houses he can afford.

(2) a is feasible, that is $\{a(1),\ldots,a(n)\} = H$. In other words, the demand for each house is equal to its supply which is 1.

Prices and initial holding determine what are the trade possibilities of each agent. One interpretation of the equilibrium is as follows: each agent awakes in the market day morning and sees the price of his house. Agent i is going to a central place and in one window exchanges $e(i)$ with $p_{e(i)}$ units of an "intrinsic valueless" commodity called money. Then, he goes to the other window and buys back one house he “can afford” to buy with the money he got (probably the same house he held earlier). In equilibrium, no house is demanded by more than one agent (with the logic that it will cause an increase in the house's price until *demand is equal supply)*.

Claim 1: A competitive equilibrium exists (This proof is due to David Gale).

Proof: A "top cycle" is a sequence of agents $i_1, i_2, \ldots i_{K+1} = i_1$ such that i_k likes most $e(i_{k+1})$. To show that a top cycle exists - start arbitrarily with agent j_1 and continue by $e(j_{k+1})$ be the house which is most liked by j_k. Eventually $j_K = j_k$ for $K > k$.

Let $J_1 = \{i_k, \ldots, i_{K-1}\}$. Assign to each $j \in J_1$ the house he likes most. Continue with the rest of the agents $(I - J_1)$ and the rest of the houses (those which are not owned initially by the members of J_1). In this way we will partition I into disjoint sets $J_1, J_2 \ldots, J_L$. Attach prices p_k to all houses owned initially by the members of J_k so that $p_1 > p_2 > \ldots p_L > 0$. Verify that this price vector and the obtained assignment of houses is a competitive equilibrium.

Discussion: What makes an agent strong in the market? It is the complementarity of other people preferences with his own. Market power of an agent is an outcome of the house he held initially and the agents' preferences.

Claim 2: (The first fundamental welfare theorem) A competitive equilibrium allocation is efficient.

Proof: Let the price vector $(p_1, \ldots, p_n)$ and the allocation a be a competitive equilibrium. Assume that the allocation is not efficient. Thus, there is an allocation b such that some agents are better off and no one is worse off. In other words, for all i, $b(i) \succsim^i a(i)$ and for at least one agent the inequality is strict. Denote by J the non empty set of agents for whom $b(j) \neq a(j)$. Let $H(J)$ be the houses held by members of J (it is the same set in a and b). Since we do not allow indifferences, $b(j) \succ^j a(j)$ for all $j \in J$ and thus it must be that $p_{b(j)} > p_{a(j)}$ for all $j \in J$. Thus, $\sum_{h \in H(J)} p_h = \sum_{i \in J} p_{b(i)} > \sum_{i \in J} p_{a(i)} = \sum_{h \in H(J)} p_h$, a contradiction.

Claim 3: (The second fundamental welfare theorem) Any efficient allocation is also a competitive equilibrium allocation.

Proof: Let a be an efficient allocation. Consider the market with the initial allocation being a. By claim 1 this market has a competitive equilibrium outcome b. For every i $b(i) \succsim_i a(i)$ and since a is Pareto efficient it must be that $b(i) \sim_i a(i)$ for all i and by the strict preferences assumption $b = a$.

Claim 4: The model with strict orderings has a unique competitive equilibrium allocation.

Proof: (The following proof is based on a suggestion made by Daniel Bird and Ziv Kedem, students in Fall 2008).

Let $(e(i))_{i \in I}$ be an initial allocation of the houses. Without loss of generality let us assume that the set of houses (H) is the same as the set of agents (I) and that $e(i) = i$.

L1: Any competitive equilibrium $((a(i))_{i \in I}, (p(i)_{i \in I})$ induces a partition of I to trade cycles, each consists of a sequence of agents $i_1,..,i_K$ such that $a(i_1) = i_2$, $a(i_2) = i_3$,....,$a(i_K) = i_1$ (that is, i_1 buys i_2's house, i_2 buys i_3's house and so on). Furthermore, since $p(i_1) \geq p(i_2)$,$p(i_2) \geq p(i_3)$,..., $p(i_K) \geq p(i_1)$, the prices attached to the houses of all members of a trade cycle is the same.

L2: If $((a_i)_{i \in I}, (p_i)_{i \in I})$ is a competitive equilibrium in the market with the set of traders of houses be I and $C = \{i_1,..,i_K\}$ is a trade cycle, then $((a_i)_{i \in I-C}, (p_i)_{i \in I-C})$ is a competitive equilibrium in the market with the set of traders and houses $I - C$ (verify).

Now, let (a,p) and (b,q) be competitive equilibria. We will show that for all i $a(i) = b(i)$.

Let M be the highest number in the price vector p. Consider a trade cycle of the agents $(i_1,..,i_K)$ whose houses' price is M. All agents in this cycle achieve the house they liked most in the set of all houses. Let i_1 be the agent in this cycle who is the richest according to q. He can afford in q house i_2 and thus it must be that $a(i_1) = b(i_1) = i_2$. Now, i_2 is also a richest trader from among the traders in $C = \{i_1,..,i_K\}$ and thus he can afford i_3 in q. Therefore, $a(i_2) = b(i_2) = i_3$ and so on.. We infer that $(i_1,..,i_K)$ is a trade cycle also in (b,q).

By L2 $((a(i))_{i \in I-C}, (p(i))_{i \in I-C})$ and $((b(i))_{i \in I-C}, (q(i))_{i \in I-C})$ are competitive equilibria in the market with the set of agents and houses be $I - C$. Continuing as in the above leads to conclude that $a(i) = b(i)$ for all i.

Problem Set B-1:

Problem 1:

Show that in this model for any efficient allocation a there is an i such $a(i) \succsim_i h$ for all h.

Problem 2:

Consider a world with K commodities where an initial bundle w can be divided in any way between the n agents. Assume that each agent has a classical preference relation (satisfying continuity, strict monotonicity and strong convexity) over the set of bundles and is restricted to consuming a bundle within a set of bundles X^i which satisfies compactness, free disposal and convexity. Define an equilibrium as an allocation $(a(i))_i$ of w such that there is no i stronger than j such that there exists a bundle $x(i) \leq a(i) + a(j)$ in X^i such that $x(i) \succ_i a(i)$. (That is, i can attack only one weaker agent. After the seizure of the weaker agent's bundle ($a(j)$) he holds the bundle $a(i) + a(j)$. If $a(i) + a(j) \notin X^i$ he must dispose some of the commodities in order to possess a feasable bundle in X^i).

Show the existence of jungle equilibrium in this model.

Show that the jungle equilibrium is efficient.

Problem 3:

Show that if agents have preferences with indifferences, then there might be a jungle equilibrium which is not efficient.

Show that if agents have preferences with indifferences, then there might be a competitive equilibrium which is not efficient.

Problem 4:

Can a competitive equilibrium be always obtained by a chain of pairwise exchanges where each exchange has the property that the exchange improves the situation of the two parties?

Problem 5:

Consider a model with $n+1$ agents and n houses. The "new" agent 0 initially owns the n houses and cares only about an additional good, "money". Each agent i ($i = 1,\ldots,n$) initially holds \$$m_i$ where $m_1 > m_2 > \ldots > m_n$ and has lexicographic preferences with first priority be the house he will own and second priority is the money left in his pocket after purchasing the house.

Define a concept of equilibrium and show its existence.

Problem 6:

Show that for every jungle equilibrium $a = \{a(i)\}_{i \in I}$ there is a price vector p such that (a,p) is a competitive equilibrium satisfying that if iSj than i is wealthier than j (measured by the competitive price vector).

Problem 7:

Invent at least two comparative statics results , one for the jungle equilibrium and one for the competitive equilibrium.

Problem 8:

Consider the following dynamic process.

Starting point: start from an arbitrary assignment (an agent can be assigned to an house or to the "street")

At stage $t+1$ each agent selects the best house (from his point of view) from the houses which at stage t are deserted or held by agents which are not stronger than him. At each house which is approached by more than one agent, only the stronger stays. The rest are sent to the "street".

The process stops at T when the assignment at period T and $T-1$ are identical

Show that he process must stop and will stop at a jungle equilibrium.