Lecture B-2: Competitive Equilibrium: Definition, Examples and Existence
Warning: Preliminary notes!

The General Structure of the Competitive Equilibrium Concept

The concept of competitive equilibrium applies to models which fit to an interaction of a number of agents. The abstract structure of the model is the following.

I be a set of agents.

Each i has a preference relation $\succsim^i$ on some space of actions X^i and some other characeteristics e^i.

A set $H \subseteq \Pi_{i \in I} X^i$ of comparable profile of actions interpreted as the set of action vectors which are "in harmony".

A function $B(p, e^i)$ which assigns to each element $p \in \Delta$ (a set of "prices"/"traffic signals") a subset of feasable actions, that is $B(p, e^i) \subseteq X^i$.

A competitive equilibrium is a pair p and a profile $(a^i)_{i \in I}$ such that

(i) the action a^i is i's best action in $B(p, e^i)$ and

(ii) the vector (a^i) is in H.

The solution concept is static. We do not discuss the mechanism by which prices are evolved. "Equilibrium prices miraculously appear from nowhere". No agent has the power to determine prices.

Economy with Pure Exchange

An economy is a tuple $< I, K, (\succsim^i)_{i \in I}, (e^i)_{i \in I} >$, where, I is the set of consumers, K is the set of goods, each consumer i holds an initial bundle $e^i \in R_+^{|K|}$ and $\succsim_i$ is a preference relation defined over R_+^K (later we will sometimes replace this bundle space with a more restricted space). We assume that all preference relations satisfy the classical assumptions we have made in our discussion of the consumer; in particular, we assume that preferences are strictly monotonic, continuous and convex. The model we discuss "assumes" the existence of ownership. We interpret e^i as the bundle i owns initially. The action space of each i is a subset of the set of bundles R_+^K.

Given a price vector p an agent i can choose an element in the budget set $B(p, e^i) = \{x | px \leq pe^i\}$.

Thus, prices are the terms of exchange in the market. Two related interpretations:

(i) Prices can be viewed as “prices” for exchange using money.

(ii) All consumers can exchange the commodities k and k' in fixed exchange ratios. One unit of commodity k can be exchanged for $p_k/p_{k'}$ units of k'. The terms of exchange are linear, that is, are independent of the identity of the traders. Linearity of prices means that the terms of exchange are also independent of the traded quantities. Each agent assumes that he can trade a combination of commodities for any other combination of goods which has the same value as measured by the prices where the value is calculated "linearly".

The profiles of actions which are compatible are the profiles (x^i) for which $\sum_{i \in I} x^i = \sum_{i \in I} e^i$.

Thus, applying the general idea of competitive equilibrium to this setup we reach the following definition:

Definition: A *Competitive Equilibrium* (CE) is a vector of non-negative numbers (prices) $p = (p_1, \ldots, p_K)$ and a profile of bundles, $(x^i)_{i \in I}$, such that:

(1) x^i is $\succsim_i$ optimal from among the set $B(p, e^i) = \{x | px \leq pe^i\}$

(2) $\sum_{i \in I} x^i = \sum_{i \in I} e^i$.

The basic idea of CE is that the considerations of each agent are affected by the price vector and not by the actions of the other players. The price vector is a "transmission element" between the actions of the other agents and the actions (or considerations) available to the agent.

Comments:

(1) Consumers are price takers: they take equilibrium prices as given and their actions are optimal given those prices.

(2) In equilibrium the markets for *all* goods are cleared simultaneously.

(3) If p is a CE price vector, then so is λp for any positive λ.

(4) The mechanism of trade is not specified.

(5) Missing elements: production, time and uncertainty.

Example 1 : Consider a market with two commodities. Commodity 1 is "money" and commodity 2 is an *indivisible* good. Assume that each consumer can hold only one unit of commodity 2. Thus, the consumption set of each agent is
$X^i = \{(m,q) | \, m \in \Re_+ \text{ and } q \in \{0,1\}\}$.

Consumer i is characterized by a number $v^i > 0$ such that i's utility function is $u_i(m,1) = m + v^i$ and $u_i(m,0) = m$. Thus, $(m,1) \succsim^i (m',0)$ iff $m + v^i > m'$. The number v_i can be thought of as consumer i's reservation value.

Thus in this economy, the set of consumers is split into two sets, B (potential buyers) and S (potential sellers), where for any $i \in B$, $e^i = (m^i, 0)$ and for any $j \in S$, $e^j = (m^j, 1)$. Denote by $|S|$ the number of sellers in the economy (that is, the number of units of commodity 2 in the economy).

In the following we calculate the competitive equilibrium for the case in which for all $i \in B$ $m^i > v^i$ (that is, any buyer i has enough money to buy the good even for v^i). (How would you change the following so to cover also the case that not for all buyers $m^i > v^i$?). Let the price of the first good be normalize d to 1.

We can calculate the equilibrium price in the following way: Order all agents by their reservation values. Let O be a set of $|S|$ agents such that there is no $i \notin O$ and $j \in O$ such that $v^i > v^j$. In other words O includes $|S|$ agents with the highest reservation value. Let p^* be any price such that for all $i \in O$, $v^i \geq p^*$ and for all $i \notin O$, $v^i \leq p^*$. The equilibrium actions will be the following:

$i \in$	a^i
$O \cap S$	e^i
$O \cap B$	$(m^i - p^*, 1)$
$O^c \cap S$	$(m^i + p^*, 0)$
$O^c \cap B$	e^i

The equilibrium allocation may be not unique. For example if $S = \{1,2\}$ and $B = \{3\}$ with $v^1 = v^2 < v^3$ then the only equilibrium price is $p^* = v^1 = v^2$ but there are two CE allocation (one where 1 sells the good and one where 2 sells the good).

Comment: Note that the equilibrium can be calculated also using the standard supply and demand curves:

Order the buyers according to their reservation values from highest to lowest, i.e., $b_1 > \ldots > b_{|B|}$.

Order the sellers by their reservation values from lowest to highest $s_1 < \ldots < s_{|S|}$.

Let $n \leq \min\{|B|, |S|\}$ be the largest value for which $b_n \geq s_n$.

(i) $n = 0$ (that is, all sellers put a higher value on the good than any buyer).

A price vector $(1, q)$ with $s_1 \geq q \geq b_1$ combined with $x^i = e^i$ for all $i \in I$ is a CE. Verify that any price vector $(1, q)$ with q outside the range is not be a competitive price vector.

(ii) If $0 < n \leq \min\{|B|, |S|\}$, choose q to be within the intersection of $[b_{n+1}, b_n]$and $[s_n, s_{n+1}]$ (if $n = |B|$ define $b_{n+1} = 0$ and if $n = |S|$ define $s_{n+1} = \infty$) an intersection which must be non-empty). For all buyers $b_1, .., b_n$ the final bundle will be $(m_i - q, 1)$ and for all sellers $s_1, .., s_n$ the final bundle will be $(m_i + q, 0)$. For the rest, the initial and final bundles will be identical. Verify that this is a CE and that any other price vector cannot be a CE price vector.

Example 2: Assume that prior to trading in the market each consumer has to decide whether to enter the market or not (i.e., “stay at home”). Assume that entering into the market is associated with a cost $c \geq 0$ (paid to agents outside the market). In this case we can formalize the competitive equilibrium as follows:

An action for agent i is "stay home", "go and don’t trade", "go and consume a bundle"

The utility function is extended from example 1 with the deduction of c for "go and not trade" and "go and consume". We can simple write:

$B(p, (m, 0)) = \{(m, 0), (m - c, 0), (m - c - p, 1)\}$ and

$B(p, (m, 1)) = \{(m, 1), (m - c, 1), (m - c + p, 0)\}$.

For a pair containing a price vector and an action profile $((1, p^*), (a^i)_{i \in I})$ to be a CE it has to be true that:

(i) for each i the action a^i is a best from his budget set.

(ii) the number of buyers who go and trade is equal to the number of sellers who go and trade.

It is easy to see that the CE can be calculated in a similar manner to that in example 1, where the v_i of a buyer is reduced by c and the v_i of a seller is increased by c.

(Compare with an equilibrium model which allows also an equilibrium where all agents are left out).

Example 3: Let $K = 2$. Consider the case that initially α consumers hold the bundle $(1,0)$ and β consumers hold the bundle $(0,1)$. All consumers have preferences represented by the utility function $min\{x_1, x_2\}$.

If $\alpha < \beta$, there is no equilibrium with $p_1, p_2 > 0$ because if there is a positive CE price vector then for every i the bundle x^i is on the main diagonal (equal quantities of the two commodities) and thus $\sum_{i \in I} x^i$ must be on the main diagonal whereas the total initial bundles $\sum_{i \in I} e^i = (\alpha, \beta)$ are not.

There exists an equilibrium with $p = (1,0)$, in which each “leftist" consumer consumes $(1,1)$ and each “rightist" consumer consumes $(0, 1 - \alpha/\beta)$.

If $\alpha = \beta$, then any price vector $(1,q)$ is an equilibrium price vector and the leftist consumers choose $(1/(1+q), 1/(1+q))$ and the rightist consumers choose $(q/(1+q), q/(1+q))$. In other words, the competitive equilibrium is consistent with any share of the “surplus” between the two type of agents.

Example 4: Consider a market with α consumers holding the initial bundle $(1,0)$ and having the utility function $min\{x_1, x_2\}$ and β consumers holding the initial bundle $(0,1)$ and having a utility function $x_1 x_2$. In all equilibria the bundles of all agents of the same type are the same (since given a positive price vector their choice problems are identical and have a unique solution). Denote the final bundles held by the leftists and the rightists by x^L and x^R, respectively.

Consider a candidate for competitive equilibrium $(1, p^*)$, where $p^* > 0$. It must be that $x^L = (1/(1+p^*), 1/(1+p^*))$ and $x^R = (p^*/2, 1/2)$. The “market clearing” equilibrium condition implies that $\alpha(1/(1+p^*), 1/(1+p^*)) + \beta(p^*/2, 1/2) = (\alpha, \beta)$ and thus, $p^* = (2\alpha - \beta)/\beta$. However, this is an equilibrium only if $(2\alpha - \beta) \geq 0$, i.e., $\alpha \geq \beta/2$.

When $\alpha \leq \beta$ the vector $(1,0)$ is a competitive equilibrium price vector and the leftists obtain the bundle $(1,1)$.

Example 5 (Robinson Crusoe): Consider a two-commodity market with one consumer holding the initial bundle e^1. In competitive equilibrium the chosen vector must be e^1. Thus, we look for a price vector for which the initial bundle maximizes the consumer’s preferences given the budget constraint. If the consumer’s preferences are convex, then such a price vector exists (determined by the tangent to the indifference curve which passes through e^1). The existence of such prices for general convex preferences follows from a separation theorem. For convex differential preferences, the vector $(v_k(e^1))_k$ is a CE price vector.

Example 6 (Edgeworth Box): $|I| = 2$, Edgeworth box is a helpful diagrammatic tool for discussing the situation (Discussion, Figure).

Demonstration of CE and a proof of existence (it is easy since the price vector could be thought as one dimensional).

Example 7: A Story about Two Gates

N individuals want to enter a stadium which has two gates, 1 and 2. The average waiting time at a gate depends on the number of agents who choose to enter the stadium through that gate. Let $f_i(x)$ be the average waiting time at gate i if x people choose that gate. Assume $f_i(0) = 0$ and that f_i is strictly increasing. Assume that all individuals prefer to wait in line as little as possible.

The competitive equilibrium approach states that each individual takes the best action given the constraints dictated by some "numbers" which he takes as given. The actions and numbers should be compatible. In this case let us apply the spirit of competitive equilibrium in the following way:

Define equilibrium as a pair of numbers (t_1, t_2), where t_i is interpreted as the waiting time at gate i, and $(a^j)_{j \in \{1,..,N\}}$, where a^j is j's action (an element of $\{1,2\}$) satisfying that:

1) a^j is the best action for agent j, i.e., for each j, if $a^j = 1$ it requires that $t_1 \leq t_2$ and if $a^j = 2$ it requires that $t_2 \leq t_1$

2) For both i we have $f_i(K_i) = t_i$ where K_i is the number of agents who choose gate i, i.e., $K_i = |\{j | a_j = i\}|$.

Analysis: There is no equilibrium with $K_1 = 0$ because in this case $t_1 = 0$, $K_2 = N$ and $t_2 > 0$. However, it is not optimal for an agent to choose 2 while $t_1 < t_2$. Similarly there is no equilibrium with $K_2 = 0$. It follows that in equilibrium both $K_1 > 0$ and $K_2 > 0$; both actions are optimal and thus it must be that in equilibrium $t_1 = t_2$. To summarize, an equilibrium is characterized by $f_1(K_1) = f_2(K_2)$ and $K_1 + K_2 = N$.

Exercise: In what sense can you say that the equilibrium is "efficient".

Note the difference between this type of analysis and an analysis of the same situation as a game. Here an agent does not take into account the influence of his own choice on the waiting time as opposed to the model of a strategic game in which he does.

Fixed Points Theorems: Reminder

Brouwer's fixed point theorem

Let X be a compact convex subset of $\Re^n$ and let $f: X \to X$ be a function such that:

▲f is continuous (that is the graph of f is closed, i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n = f(x_n)$ for all n, $x_n \to x$, and $y_n \to y$, we have $y = f(x)$).

Then f has a *fixed point* x^* at which $f(x^*) = x^*$.

Note that each of the following conditions is necessary for Brower's theorem to hold.

(i) X is closed: Consider the function $f(x) = x^2$ on $(0,1)$.

(ii) X is bounded: Consider the function $f(x) = x + 1$ on $\Re$.

(iii) X is convex: Let X be a circle and consider any non-degenerate rotation of X.

(iv) f has a closed graph: Let $x = [0,1]$ and let $f(x) = 1 - x$ for all $x \neq 1/2$ and $f(1/2) = 0$.

Kakutani's fixed point theorem

Let X be a compact convex subset of $\Re^n$ and let $f: X \to X$ be a set-valued function (that is $f(x) \subseteq X$ and $f(x) \neq \emptyset$ for any $x \in X$) satisfying:

▲for all $x \in X$ the set $f(x)$ is convex

▲the graph of f is closed (i.e. for all sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n \in f(x_n)$ for all n, $x_n \to x$, and $y_n \to y$, we have $y \in f(x)$).

Then f has a *fixed point* x^* at which $x^* \in f(x^*)$.

Note that the condition that the set $f(x)$ is always convex is necessary for Kakutani's theorem to hold.

Consider $X = [0,1]$ and let $f(x) = \{1\}$ for all $x < 1/2$, $f(x) = \{0\}$ for all $x > 1/2$ and $f(1/2) = \{0,1\}$.

Existence

Efforts have been made to prove the existence of equilibria for a wide class of markets. We are interested in such theorems in order to:

(1) Guarantee that the concepts we talk about are not empty.

(2) Test the consistency of the models we build.

Here we will prove a very basic existence theorem of competitive equilibrium. It is presented here in order to give the flavor of the existence theorems.

Let Δ be the set of all price vectors, i.e., $\{p \in \Re_+^K | \sum_k p_k = 1\}$. Define $x^i(p,w)$ to be i's demand given the price vector p and income w. We assume that the demand is well defined for every price vector p including the vectors in which some prices are zero. We also assume, that the demand functions x^i are continuous even at the boundaries of Δ.

Let $z^i(p) = x^i(p,pe^i) - e^i$ be i's *excess demand*.

Let $z(p) = \sum_i z^i(p)$ be the total excess demand.

The function z is continuous. It also satisfies Walras' Law: $pz(p) = 0$ since $pz(p) = \sum_{i \in I} pz_i(p) = \sum_{i \in I}(px^i(p,pe^i) - pe^i) = 0$.

Proposition (Existence): Assume that for each agent i the function $z^i(p)$

(1) is well defined for all p

(2) is continuous.

(3) satisfies Walras' law and

(4) satisfies that, for any vector p with $p_k = 0$, there is a surplus of demand, that is $p_k = 0$ implies $z_k^i(p) > 0$.

Then a competitive equilibrium exists.

Proof: The set Δ is compact and convex. We construct a function $g : \Delta \to \Delta$ which has a fixed point and for which we can show that any fixed point is a competitive price vector. Ideally, the function will describe a dynamic process: if price vector "today" is p, then the price vector "tomorrow" will be $g(p)$. The function g which we construct has attractive features but it is far from describing a reasonable dynamics. It is defined here only for the sake of the proof.

Let $g : \Delta \to \Delta$, be defined by $g_k(p) = [p_k + max\{0, z_k(p)\}]/[1 + \sum_j max\{0, z_j(p)\}]$

Note that $g(p) \in \Delta$. When there is an excess of supply for k ($z_k(p) < 0$), $g_k(p) \leq p_k$. However, when $z_k(p) \geq 0$ the price $g_k(p)$ is not necessarily above p_k ($g_k(p) \geq p_k$ for at least one commodity k).

The function g satisfies the conditions of Brouwer's fixed point theorem and thus g has a fixed point p^*.

We will now show that p^* is a CE price. By the assumption that if $p_k^* = 0$ then $z_k^i(p^*) > 0$, the fixed point must be a strictly positive vector.

It is sufficient to point out that for all k, $z_k(p^*) = 0$ since then the pair p^* and the profile $(x_i(p^*, p^*e^i))_{i\in I}$ constitute a CE.

Since $\sum_k p_k z_k(p^*) = 0$ and all prices are positive, it is sufficient to show that for all $k, z_k(p^*) \leq 0$ since then $z_k(p^*) = 0$ for all k.

From the equation $g_k(p^*) = p_k^*$ we obtain: $max\{0, z_k(p^*)\} = p_k^* \sum_j max\{0, z_j(p^*)\}$.

Multiply the two sides of the equation by $z_k(p^*)$ and sum over all k we obtain: $\sum_k z_k(p^*) max\{0, z_k(p^*)\} = \sum_k z_k(p^*) p_k^* \sum_j max\{0, z_j(p^*)\}$.

By Walras law ($\sum_k p_k z_k(p^*) = 0$) and therefore $\sum_k [z_k(p^*) max\{0, z_k(p^*)\}] = 0$ which implies that for all k, $z_k(p *) \leq 0$.

The proof so far does not provide exactly what we are looking for. The main problems are:

(i) The demand for the case that a price is 0 might be not well-defined. (ii) The demand might be multi-valued.

Some work is needed to deal with those problems. In particular we need (i) to find a subsapce of prices which will not include the boundaries and on which the function will get values inside the set. (ii) to replace the proof which is using the Brower's theorem with a proof which will use Kakutani's theorem.

The First Welfare Theorem

We say that an allocation of a bundle e, $(x^i)_{i\in I}$, is *efficient* if there is no other allocation $(y^i)_{i\in I}$ of e such that for all i, $y_i \succsim x_i$ and for at least one agent i the inequality is strict.

The First Welfare Theorem connects the concept of Competitive Equilibrium to the concept of efficient allocation. Any outcome of competitive equilibrium is necessarily efficient.

Proposition: (First Fundamental Welfare Theorem): A CE allocation is always an efficient allocation of the social endowment.

Proof: Let $(p, (x^i)_{i\in I})$ be a CE. Let $(y^i)_{i\in I}$ be an allocation of $e = \sum_{i\in I} e^i$ so that $y_i \succsim x_i$

for all i with strict inequality for some j. It must be that $py^j > pe^j$ and for all other i, $py^i \geq pe^i$ (otherwise there would be a feasible bundle superior to x^i). Summing over all consumers, we have $p\sum_{i\in I} y^i > p\sum_{i\in I} x^i$ which is a contradiction to the equilibrium requirement that $\sum_{i\in I} x^i = e$.

The Second Welfare Theorem

Proposition: (Second Fundamental Welfare Theorem): An efficient allocation of the social endowment is a CE allocation for some economy.

Proof (without using the existence theorem): Let $(x^i)_{i\in I}$ be an efficient allocation. Consider the economy with $e^i = x^i$ for all i.

Let $x^* = \sum_i x^i$. Consider the set $Y = \{y|$ there is an allocation (y^i) of y such that $y^i \succsim_i x^i$ and for at least one i also $y^i \succ_i x^i\}$. The set Y is not empty, is convex (why?) and $x^* \notin Y$. By the separation theorem, there is a non zero vector p such that $py \geq px^*$ for all $y \in Y$. For all k we have $p_k \geq 0$ since $(x^* + (0,\ldots 1,\ldots.0) \in Y)$.

Assume that there is an agent j such that $y^j \succ_j x^j$ and $py^j \leq px^j$. By continuity there is also z^j such that $z^j \succ_j x^j$ and $pz^j < px^j$. The bundle $x^* + (z^j - x^j) \in Y$ but $p(x^* + (z^j - x^j)) < px^*$. A contradiction.

Problem set B-02

1. Assume that the terms of trade between a and b are characterized by p_{ab} which expresses the number of b that can be traded for one unit of a.

Show that unless $p_{a,c} = p_{a,b}p_{b,c}$ agents can make "arbitrage trades" to obtain goods without giving anything in return.

Show that if $p_{a,c} = p_{a,b}p_{b,c}$ for any three goods a, b and c, then there is a price vector $(p_{1,\ldots},p_K)$ such that $p_{a.b} = p_a/p_b$.

2. Define a market with $m+n$ consumers and two commodities in which all consumers have the utility function $u(x_1,x_2) = x_1x_2$.

m consumers have the initial bundle $(2,1)$ and the remaining n consumers have the initial bundle $(1,5)$. Calculate the CE prices as a function of m and n.

3. Prove the existence of CE for the Edgeworth box case using diagrams which depict the excess demand for commodity 1 as a function of the price vector $(1,p)$.

4. Construct an example of a market in which one consumer has convex preferences and one has non-convex preferences in which there is no CE.

5. Define a market with $a+b$ consumers who live for two periods. The first a consumers have one unit of food in the first period and 0 in the second. The remaining b consumers have 1 unit of food in the second period and 0 in the first. All consumers are time indifferent (i.e., they are indifferent between consuming x units of food in the first period and y in the second or consuming y units in the first and x in the second). The goods are perfectly divisible. Assume that the preferences are monotonic, continuous, strictly convex and differentiable. Calculate the relative CE prices for the case of $a = b$. What can you say about the CE when $a > b$?

6. Formulate and prove the statement: In a world with K commodities, if the markets for $K-1$ goods clear then so does the K-th market.

7. Consider a society of N agents. Each agent has to choose a bar, either 1 or 2. When allocating their unit of time between the bars, each agent takes into account the densities n_1 and n_2 (where $n_1 + n_2 = N$) in the bars, but ignores his own influence on n_1 and n_2. In other words the agents relate to n_1 and n_2 like agents relate to the prices in a competitive market.

Agent i chooses an allocation of time (x_1^i, x_2^i) (a pair of non-negative numbers which sum up to 1) so as to maximize the function $U^i(x_1^i, x_2^i) = x_1^i u_1^i(n_1) + x_2^i u_2^i(n_2)$ where $u_h^i(n_h)$ is a continuous strictly decreasing function.

(a) Characterize the behavior of agent i as a function of n_1 and n_2.

Define an equilibrium as a pair of numbers n_1 and n_2 and a profile of allocations $\{(x_1^i, x_2^i)\}_{i=1,\ldots,N}$ such that:

(1) For every bar h, $\sum_i x_h^i = n_h$ (i.e., n_h is the correct density)

(2) For any i, (x_1^i, x_2^i) maximizes i's utility given n_1 and n_2.

(b) Show that if $u_h^i = u$ for all i and for all h, then in equilibrium $n_1 = n_2 = N/2$.

(c) Prove the existence of an equilibrium (for the general case).