# Lecture B03: Competitive Equilibrium, Efficiency and the Core

**Warning: Prelimenary!**

## The Core and its Relationship to CE

In the context of exchange economy we will define the core to be a set of allocations of the initial bundle $e = \sum e^i$ so that there is no subset of individuals $S$ which can "leave" the market with its initial bundles and allocate their own resources so that all of them will be better off. If an allocation is not in the core, it is instable in the sense that a group of agents are better off disconnecting themselves from the mechanism which allocates this allocation.

*Definition*: An allocation, $(x^i)_{i \in I}$ of $e = \sum e^i$ is in the *core* if there is no coalition $S$ and allocation $(y^i)_{i \in S}$ of $\sum_{i \in S} e^i$ among $S$'s members so that all $S$'s members are better off in $y$ than in $x$ i.e., $y^i \succ_i x^i$. for all $i \in S$.

[A discussion of the Edgeworth box; indicate the core of the economy in the diagram.]

*Proposition*: Let $(p, (x^i))$ be a competitive equilibrium, then $(x^i)_{i \in I}$ is in the core.

*Proof*:

First note that $(x^i)_{i \in I}$ is indeed an allocation of $e$.

Assume that the allocation is not in the core.

Then, there is a coalition $S$ and allocation $(y^i)_{i \in S}$ of $\sum_{i \in S} e^i$ so that $y^i \succ_i x^i$. for all $i \in S$.

It must be that $py^i > px^i = pe^i$ for all $i \in S$ and thus $p\sum_{i \in S} y^i > p\sum_{i \in S} e^i$, a contradiction to $(y^i)_{i \in S}$ of $\sum_{i \in S} e^i$.

**Convergence**

In the continuation of the discussion we demonstrate an argument that when the power of each individual is small, in the sense that every agent has "many substitutions" the core "shrinks" to the competitive allocation.

We start with a market $M$ (the set of traders is $I$). Let $mM$ be the market which $m$ times duplicates $M$, that is, it contains $m$ times the number of agents in $I$ and $m$ agents are identical (in terms of initial endowment and preferences) to one of $M$'s members. We refer to a member of $mM$ who is a duplicate of $t$ as an agent of "type $t$".

The comparison between the core of $M$ and the core of $kM$ is possible due to the following Lemma:

**Lemma (Equal treatment of the core)**: Assume that all preferences in $M$ are strictly convex, strongly monotonic and continuous. Then, if $x$ is in the core of the economy $mM$ and $i$ and $j$ are two identical agents in the economy then they consume the same bundles, that is, $x^i = x^j$.

*Proof*: Assume that $x$ is an allocation in the core in which two agents of type 1 get unequal bundles. Out of each type $t$, select one agent, $i(t)$, who is a type $t$ least well off agent according to the allocation $x$ evaluated by the preference $\succsim^t$ . Let $z^t$ be the average bundle of the type-$t$ agents get in the allocation $x$. Let $S$ be the coalition $\{i(t)\}_t$. Notice the following:

a. $(z^t)_t$ is a feasible allocation of $S$, that is $\sum_t z^t = \sum_t e^t$.

b. $z^t \succsim^t x^{i(t)}$ for all $t$, by the convexity of the preference relation.

(If for some $t$, $z^t \prec^t x^{i(t)}$ then $z^t$ is worse than all bundles held by the agents of type t, contradicting the strict convexity.)

c. $z^1 \succ x^{i(1)}$.

By the montonicity, we can reduce $z^1$ so that it is still better than $x^{i(1)}$ and distribute it among all members of $S$ to obtain a distribution of $S$'s resources which dominates $x$ for all members of $S$.

Next, notice that the core of $mM$ is a subset of the core of $M$. Moreover the core of $mM$ is a subset of the core of $mM$. Note also that any competitive price vector in $M$ is also an equilibrium price vector for $mM$.

Our goal is to show that the core of $mM$ "converges" to the set of competitive allocations of $M$ (which is the same as of $mM$). More precisely we will show that any non-competitive allocation is eliminated from the core of $mM$ for $m$ large enough. We will make do with presenting the result for a two agent economy.

**Theorem**: Let $M$ be a two agent market. Assume that the preferences of the agents are strictly convex and strongly monotonic. Assume that $y$ is not competitive allocation. Then there is $k^*$ large enough such that $y$ is not in the core of $kM$ for any $k \geq k^*$.

**Proof**: If $y$ is not competitive then on the line combining $e$ and $y$ there are partitions such that for at least one type, let us say 1, it is strictly preferred to $y^1$. There are several possible configurations:

Case (i): Assume first that there is a bundle $z^1$ which is a convex combination of $y^1$ and $e^1$ such that $z^1 \succ^1 y^1$.

By the continuity there are integers $m$ and $n$, $m < n$ such that
$g^1 = (m/n)e^1 + (1 - m/n)y^1$ and $g^1 \succ^1 y^1$.

Let $k > n$ and let $S$ be a coalition which includes $n$ consumers of type 1 and $n - m$ of type 2. Allocate $g^1$ to the agents in $S$ of type 1 and $e - y^1$ to type 2 agents.

This allocation is $S$-feasible since
$ng^1 + (n - m)(e - y^1) = me^1 + (n - m)(e^1 + e^2) = ne^1 + (n - m)e^2$.

The consumers of type 1 are strictly better and the consumers of type 2 are indifferent relative to the $y$.

By continuity we can modify the allocation so that *all* members of $S$ will be strictly better off.

(explanation, in this case type 1 guys trade too much and they do better by a coalition with a fewer type 2 so that they ca trade less)

Case (ii): $e^1$ is a convex combination of $y^1$ and $z^1$ such that $z^1 \succ^1 y^1$. By the convexity there is a $w^1$ which is a convex combination of $y^1$ and $e^1$ such that $w^1 \succ^1 y^1$ and we are back in case (i).

Case (iii): If $y^1$ is a convex combination of $z^1$ and and $e^1$ and $z^1 \succ^1 y^1$ then by continuity there ate integers $m$ and $n$ such that $y^1 = (m/n)z^1 + (1 - m/n)e^1$ and $z^1 \succ^1 y^1$.

Consider a coalition of $m$ type 1 and $n$ type 2. Allocate to the type 1 agents the bundle $z^1$ and to the type 2 the bundle $y^2$.

$$mz^1 + ny^2 = mz^1 + n(e^1 + e^2 - y^1) = mz^1 + ne^2 + n(m/n)(e^1 - z^1) = me^1 + ne^2$$

and the continuation as in case (i).

Thus, for $k$ large enough the allocation $y$ is not in the core.

**Problem set B03**

1. Consider an economy with a set $S$ of owners of one unit of an indivisible good and a set, $B$, of agents who do not own the good. Assume that agent $i$ owns $m_i$ money and his preference is represented by the utility function $u_i(m,x) = m + v_i x \quad (x \in \{0,1\}$ and $m$ is a positive number). Calculate the set of efficient allocations of the $S$ units of the good. Demonstrate the welfare theorem for this model.

2. Consider a two-agent market, $M$, where agent 1 has the initial bundle $(1,0)$ and the utility function $(x_1 + x_2)$ and agent 2 has the initial bundle $(0,1)$ and the utility function $min\{x_1, x_2\}$.

a. Calculate the core of this market and of $kM$.

b. Calculate the competitive allocations.

3. (Based Question 5 on Kreps page 227; read pages 202-205 before solving the problem!).

a. Consider a market with two goods and two consumers. Agent 1's utility function is $w(x_1^1) + w(x_1^2) + x_2^1$ where $x^i$ is the consumption bundle of agent $i$. Agent 2's utility function is $w(x_1^1) + w(x_1^2) + x_2^2$. Assume that $w$ is increasing, strictly concave and even differentiable (if you like).

Note that each agent cares about the quantity of consumption of the other agent as well. Suppose that the social endowment is allocated initially equally among the two agents. What will be the competitive equilibrium? Characterize the set of efficient allocations of the social endowment. Is the competitive equilibrium efficient?

b. Repeat on (a) assuming the existence of a third agent whose utility function is $w(x_1^3) + x_2^3$.