Exam TAU MOED A
January 2011
Solutionl

Problem 1.

You have read an article in a "prestigious" journal about a decision maker (DM) whose mental attitude towards elements in a finite set X is represented by a binary relation $\succ$, which is a-symmetric and transitive but not necessarily complete. The incompleteness is the result of an assumption that a DM is sometimes unable to compare between alternatives.

Another, presumingly stronger, assumption made in the article is that the DM uses the following procedure: he has n criteria in mind, each represented by an ordering (a-symmetric, transitive and complete) $\succ_i$ $(i = 1,\ldots,n)$. The DM decides that $x \succ y$ if and only if $x \succ_i y$ for every i.

a.Verify that the relation $\succ$ generated by this procedure is a-symmetric and transitive. Try to convince a reader of the paper that this is an attractive assumption by giving a "real life" example in which it is "reasonable" to assume that a DM uses such a procedure in order to compare between alternatives.

Solution:

$\succ$ is a-symmetric: If $x \succ y$ then by definition, $x \succ_i y$ for every i. Since $\succ_i$ are a-syemmtric, $y \not\succ_i x$ for all i, and by definition also $y \not\succ x$.

$\succ$ is transitive: Let $x \succ y$ and $y \succ z$. By definition, $x \succ_i y$ and $y \succ_i z$ for every i. Since $\succ_i$ are transitive, also $x \succ_i z$ for all i, and by definition $x \succ z$.

An example: A parent who considers destinations for a family vacation who ranks the different destinations according to the orderings of his children: he prefers A to B iff all his children prefer A to B.

It can be claimed that the additional assumption regarding the procedure that generates $\succ$ is not a "serious" one since given any asymmetric and transitive relation, $\succ$, one can find a set of complete orderings $\succ_1,\ldots,\succ_n$ such that $x \succ y$ iff $x \succ_i y$ for every i.

b. Demonstrate this claim for the relation on the set $X = \{a,b,c\}$ according to which only $a \succ b$ and the comparison between [b and c] and [a and c] are

not determined.

Solution:

Let $a \succ_1 b \succ_1 c$ and $c \succ_2 a \succ_2 b$. The two relations agree only on $a \succ_i b$.

c. (**Main part of the question**) **Prove this claim for the general case**.

Guidance (**for c**): **given an asymmetric and transitive relation** $\succ$ **on an arbitrary** X, **define a set of complete orderings** $\{\succ_i\}$ **and prove that** $x \succ y$ **iff for every** i, $x \succ_i y$.

Solution:

First, note that if X is a finite set and P is a asymmetric and transitive relation on X then P does not have any cycles and thus P can be extended to a complete ordering of X (see Problem Set 1).

Let Λ be the set of all complete orderings which extends $\succ$. We will see that $a \succ b$ if and only if $a \succ_i b$ for all $\succ_i \in \Lambda$:

(i) **If** $a \succ b$, then $a \succ_i b$ for all i since any $\succ_i \in \Lambda$ is an extension of $\succ$.

(ii) **If not** $a \succ b$, then let $\succ^*$ be the relation $\succ$ extended to include also $b \succ^* a$. The relation $\succ^*$ does not have cycles: if there is a cycle $x_1 \succ^* \ldots \succ^* x_n = x_1$ then

(a) if for some i we have $x_i = b \succ^* a = x_{i+1}$ then since $a = x_{i+1} \succ^* x_{i+2} \ldots \succ^* x_n = x_1 \succ^* \ldots \succ^* x_i = b$ by transitivity $a \succ b$ contradicting the assumption.

(b) otherise, by thranstivity $x_1 \succ x_2$ but also $x_2 \succ x_1$ conradicting asymettry.

Thus, $\succ^*$ can be extended to a complete ordering $\succ'$ which will be an extension of $\succ$ as well. Hence, there is an extension $\succ' \in \Lambda$ for which not $a \succ' b$.

Problem 2

A consumer in a two commodity world operates in the following manner:

The consumer has a preference relation $\succsim_S$ **on** $\mathbb{R}^2_+$. **His father has a preference relation** $\succsim_F$ **on the space of his son's consumption bundles**. **Both relations satisfy strong monotonicity**, **continuity and strict convexity. The father does not allow his son to purchase a bundle which is not as good**

(**from his perspective**) **as the bundle** $(M,0)$. **The son**, **when choosing from a budget set**, **maximizes his own preferences subject to the constraint imposed by his father**. **In the case that he cannot satisfy his father's wishes**, **he feels free to maximize his own preferences**.

a. **Prove that the behavior of the son is rationalizable**.

Solution:

Define $\succ$ as follows: $a \succ b$ iff (i) $a \succsim_F (M,0)$ and $(M,0) \succ_F b$, or (ii) both $a,b \succsim_F (M,0)$ and $a \succ_S b$ or (iii) both $(M,0) \succ_F a,b$ and $a \succ_S b$.

$\succ$ can easily be shown to be complete and transitive.

b. **Prove that the preferences which rationalize this kind of behavior are monotonic**.

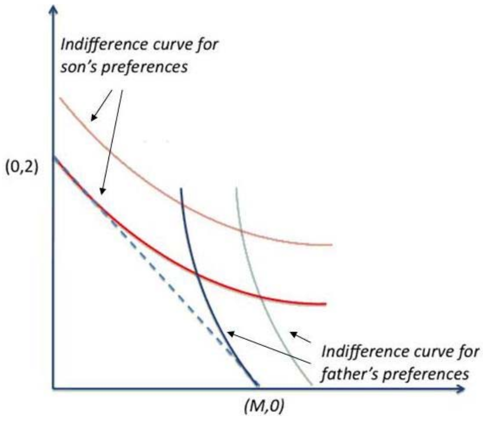
Solution:

Take any x,y st. $x_1 \geq y_1$ and $x_2 \geq y_2$. Since $\succsim_S$,$\succsim_F$ are monotonic, $x \succsim_S y$ and $x \succsim_F y$. Thus by construction of $\succ$, $x \succsim y$.

c. **Show that the preferences which rationalize this kind of behavior are not necessarily continuous nor convex** (**you can demonstrate this diagrammatically**).

Solution:

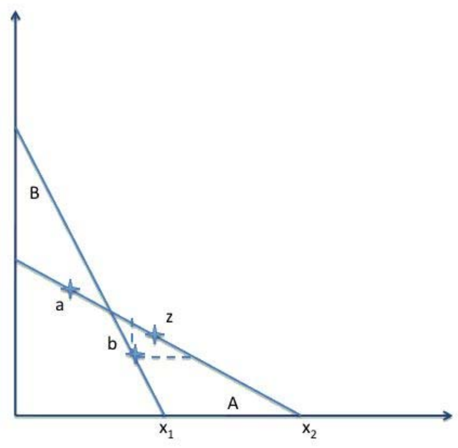
To see possible violation of continuity and convexity consider the example below.



Note that $(M,0) \succ (0,2)$ since $(0,2)$ is below the father's indifference curve passing through $(M,0)$. However we can see from the son's preferences that for any $\alpha \in (0,1)$, $(0,2) \succ \alpha(0,2) + (1-\alpha)(M,0)$ violating convexity and continuity.

d. (**Bonus**) **Assume that the father's instructions are that given the budget set** (p,w) **the son is not to purchase any bundle which is** $\succsim_F$ **-worse than** $(w/p_1, 0)$. **The son seeks to maximize his preferences subject to satisfying his father's wishes**. **Show that the son's behavior satisfies the Weak Axiom of Revealed Preferences**.

Solution:



Assume there is a violation of the WARP. Then there must be two overlapping budget sets as shown above such that a is chosen from set A and b is chosen from set B.

It must be that $a \succsim_F (x_2,0)$ and $b \succsim_F (x_1,0)$. By monotonicity, $(x_2,0) \succsim_F (x_1,0)$ and thus $a \succsim_F (x_1,0)$ Since b is chosen over a in set B, $b \succ_S a$ By monotonicity, there exists $z \in B$ st $z \succsim_S b \succ_S a$ Also by convexity $z \succsim_F x_2$, contracting a being optimal in set A.

Problem 3.

Consider an economy with two commodities, **in which each agent owns an initial bundle <u>consisting of only one of the two goods</u>**. **Each agent has a preference relation satisfying strong monotonicity**, **continuity and strict convexity**. **Given a price vector**, **each agent is interested in selling as much as he can from the commodity he possesses**, **provided that his final bundle is no worse than his initial one** (**according to his preferences**).

a. **Define an appropriate equilibrium concept for this economy**.

Demonstrate this equilibrium in an Edgeworth box (**that is**, **in an economy with two agents**).

Solution:

Let e be the initial allocation - e^i is the bundle held by agent i. For some agents $e^i = (e_1^i, 0)$ and for the others $e^i = (0, e_2^i)$.

Let $\succ^i$ denote agent $i's$ preferences.

An equilibrium is a pair $((a^i),(p_k))$ where:

a is a pair of bundles - agent $i's$ bundle is $a^i = (a_1^i, a_2^i)$.

$p = (p_1, p_2)$ is the price vector of the two commodities.

such that:

1. a is feasible: $\sum a^i = \sum e^i$.

2. If $e^i = (e_1^i, 0)$ then $a_1^i = \min x_1$ s.t.$(x_1, x_2) \succeq^i e^i$ and $p \cdot e^i \geq p \cdot (x_1, x_2)$.

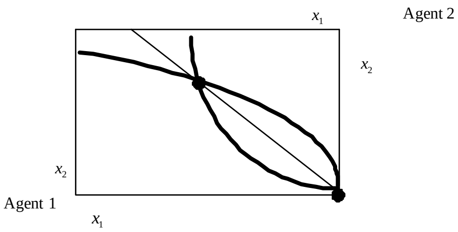
If $e^i = (0, e_2^i)$ then $a_2^i = \min x_2$ s.t.$(x_1, x_2) \succeq^i e^i$ and $p \cdot e^i \geq p \cdot (x_1, x_2)$.

b. **Show** (**graphically**) **that in a world with two agents**, **who initially hold two different commodities**, **such an equilibrium always exists**.

Solution:

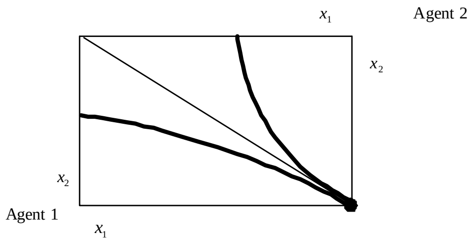
Assume WOLG that agent 1 holds only commodity 1 and agent 2 holds only commodity 2. Consider the following 3 cases:

1. The indifference curves through the initial bundle cross again in the box. In this case, the equilibrium allocation is the second crossing, and the price vector is such that the budget line passes through the two crossing points.



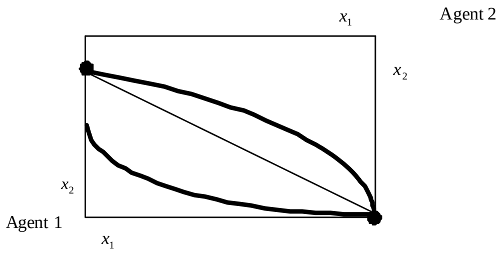
2. The indifference curves through the initial bundle do not cross again in the box and at the initial bundle the slope of agent 1 curve is higher than agent 2's.

Any prices such that the budget line does not cross the curves is an equilibrium, and the final allocation is identical to the initial one.



3. The indifference curves through the initial bundle do not cross again in the box,.and at the initial bundle the slope of agent 1 curve is lower than agent 2's.

If agent 2's (1's) curve crosses commodity 1 (2) axis, the equilibrium allocation is this crossing. The prices are set such that the budget line connects the initial and final allocations.



Otherwise, it is the bundle opposite to the initial one. The prices are set such that the budget line connects the initial and final allocations.