Exam TAU MOED A
January 2012
Solution

Problem 1. Based on de Crippel (2011)

Consider a decision maker (DM) who has in mind two orderings on a finite set X. The first ordering, $\succ_L$, expresses his long-term goals, and the second, $\succ_S$, expresses his short-term goals.

When choosing from a set $A \subseteq X$ the DM chooses the best alternative according to his long-term preferences, unless there are "too many" alternatives that are better than this alternative according his short-term preferences. More precisely, given a choice problem $A \subseteq X$, he excludes all alternatives which are not among the k best alternatives in A according to his short-term preferences, and out of the remaining he chooses the best one according to $\succ_L$.

1) Show that the above description always defines a choice function.

Answer:

For any $A \subseteq X$, let $CS(A)$ (the consideration set from A) be the set of the best k alternatives according to $\succ_S$ in A (if $|A| < k$ let $CS(A) = A$). Clearly, $CS(A)$ is not empty. The above procedure is equivalent to choosing the best alternative from $CS(A)$ according to $\succ_L$, which is always well defined.

2) Show that it may be that the same alternative is chosen from both A and B, but is not chosen from $A \cup B$ nor from $A \cap B$

Answer:

Consider the following example: $X = \{a,b,c,d\}$, $A = \{a,b,c\}$, $B = \{a,b,d\}$, $k = 2$, $a \succ_L b \succ_L c \succ_L d$ and $d \succ_S c \succ_S b \succ_S a$.

$CS(A) = \{b,c\}$ and $CS(B) = \{b,d\}$. Thus, $C(A) = b$ and $C(B) = b$.

However, $CS(A \cap B) = CS(\{a,b\}) = \{a,b\}$ and thus, $C(A \cap B) = a$. Furthermore, $CS(A \cup B) = CS(X) = \{c,d\}$ and thus $C(A \cup B) = c$.

3) Conclude that this type of behavior conflicts with the rational man paradigm.

Answer:

Obviously, if the rational agent maximizes a preference relation, then if a is the maximizer in both A and B it is also the maximizer in $A \cup B$ and $A \cap B$.

Let N **be a set of individuals who behave according to the above procedure with** $k = 2$. **All individuals share the same long-term goals but may differ in their short-term goals**.

Consider a situation in which the N **individuals must choose together only one alternative from the set** X **and that for each alternative** $x \in X$, **there is one individual** $r(x)$ **who has the right to force** x. **An equilibrium is an alternative** y **such that no individual wants to exercise his right to force one of the alternatives that he can force**. **That is**, **for any agent** i, **the alternatives** y **is the one chosen by the agent from the set** $\{y\} \cup \{x|r(x) = i\}$.

4) **Show that if there are more alternatives than individuals then it is possible to assign the** "**forcing rights**" **such that whatever are the individuals' short-term goals and whatever are the common long-term goals**, **the only equilibrium is the top** $\succ_L$ **alternative**. **Explain why this is not necessarily correct if the number of alternatives is larger than the number of individuals**.

Answer:

If there are more individuals than alternatives, we can assign the rights such that at most one exclusive alternative is assigned to each individual.

Now, in a particular world (configuration of common long and short term preferences) let y^* be the top $\succ_L$ alternative.

First note that y^* is an equilibrium: for any agent i there is at most one other alternative y such that $r(y) = i$. Agent i can choose only from $\{y, y^*\}$, which are both in his consideration set, and thus he chooses y^*.

Consider any other $y \neq y^*$. This alternative is not an equilibrium since agent $i = r(y^*)$ faces a choice from $\{y, y^*\}$, and, choosing according to $\succ_L$, he forces y^*.

As to the last part of the question: consider a world with $X = \{x, y, z\}$ and two individuals. It must be that two alternatives, let us say, x and y, are assigned to one of the individuals, let us say 1. For the configuration $y \succ_L z \succ_L x$ and $z \succ_{i,S} x \succ_{i,S} y$ for both i we get that z is an equilibrium becasue 1 can choose from X and he chooses z.

Problem 2.

1) (**Just a warm-up**) **Give an example within the model of an "economy with houses" where the agents have strict preferences and there is a Pareto inefficient allocation for which there is no possibility for a pair of agents to conduct a mutually beneficial trade**.

Answer:

Consider the model with three houses $\{x_1, x_2, x_3\}$ and three agents with the preference relations: $x_1 \succ_1 x_2 \succ_1 x_3$, $x_2 \succ_2 x_3 \succ_2 x_1$ and $x_3 \succ_3 x_1 \succ_3 x_2$. The allocation $a(i) = x_{i+1}$ is not Pareto effiicient but there is no pair of agents who can conduct a mutually beneficial trade.

2) (**The main part of the question**): **Show that such an example is impossible in the houses economy if houses are ordered on a line and all preferences are single-peaked**.

Answer:

Let $(\succ_i)_{i=1,\ldots N}$ be a profile of single peaked preferences, and assume that the allocation $a(i)$ is inefficient. We need to show that there is at least one pair of agents that can benefit from exchanging the houses between themselves.

Without loss of generality, assume the houses are ordered such that $a(i) < a(i+1)$ for all i. Let $b(i)$ be a Pareto improvement allocation.

Let k be the first agent that obtains a lower house in the Pareto improvement. That is, the lowest i such that $b(i) < a(i)$. Clearly, $k > 1$.

Consider the agents lower than k who change house in the Pareto omprovment (there must be such). All of them must move "up". Let l be the highest agent from among those. Clearly, $b(l) \geq a(k)$ and $a(l) \geq b(k)$. Thus, by the single peakness $a(l) \succ_k a(k)$ and $a(k) \succ_l a(l)$ and k and l have a mutually beneficial trade.

Problem 3

A decision maker has in mind a function CE, with the interpretation that for every lottery p, $CE(p)$ **is the certainly equivalence of** p. **Following are two procedures for deriving the function.**

Procedure 1: **The decision maker has in mind an increasing vNM utility function** u **and his answer satisfies** $Eu(p) = u(CE(p))$.

Procedure 2: **The decision maker has in mind two increasing, continuous and concave functions** g (**for gains**) **and** l (**for losses**) **which satisfy** $g(0) = l(0) = 0$.

$CE(p)$ **is a number** x **which equalizes the expected "loss" with the expected "gain", that is satisfies** $\sum_{y<x} p(y)l(x-y) = \sum_{y>x} p(y)g(y-x)$.

1) Explain why pD_1q **implies under the two procedures that** $CE(p) \geq CE(q)$.

Answer:

Procedure 1: If pD_1q then for any utility function u it holds that $Eu(p) \geq Eu(q)$. Therefore, $u(CE(p)) \geq u(CE(q))$, and by the monotonicity of u, $CE(p) \geq CE(q)$.

Procedure 2: Let $x^* = CE(p)$, that is, $\sum_{y>x} p(y)g(y-x^*) - \sum_{y<x} p(y)l(x^*-y) = 0$.

Given this x^*, define an increasing vNM utility function $u(y) = \begin{cases} g(y-x^*) \text{ if } y \geq x^* \\ -l(x^*-y) \text{ if } y < x^* \end{cases}$.

If pD_1q we can conclude that $Eu(p) \geq Eu(q)$. Moreover, because $Eu(p) = 0$ then $Eu(q) \leq 0$, which implies $\sum_{y>x} q(y)g(y-x^*) - \sum_{y<x} q(y)l(x^*-y) \leq 0$. Given q, the expression $\sum_{y>x} q(y)g(y-x) - \sum_{y<x} q(y)l(x-y)$ is decreasing in x, therefore $CE(q) \leq x^*$.

2) Explain why the first procedure allows behavior which is not possible under procedure 2.

Answer:

Let p be a lottery and let $p+k$ be a lottery in which all prizes are increased by k (Formally, $p(x) = (p+k)(x+k)$ for any x in the support of p). Clearly, in procedure 2, if $x = CE(p)$ then $x+k = CE(p+k)$. This is not necessarily the case for the first procedure.

For example, let $u(x) = \sqrt{x}$ and let p be lottery yielding 0 and 1 with equal probabilities. $Eu(p) = \frac{1}{2}\sqrt{0} + \frac{1}{2}\sqrt{1} = \frac{1}{2}$. Therefore, $\sqrt{CE(p)} = \frac{1}{2}$, or $CE(p) = \frac{1}{4}$.

Now, consider the lottery yielding 1 and 2 with equal probabilities. $Eu(p+1) = \frac{1}{2}\sqrt{1} + \frac{1}{2}\sqrt{2} = 1.207$. Therefore, $\sqrt{CE(p)} = 1.207$, or $CE(p) = 1.46 \neq 1.25$

3) (For thinking at home) Can any individual who operates by Procedure 2 be described as

working through procedure 1?

Answer:

Not necessarily.

Assume $l(x) = 2x$ and $g(x) = x$. That is, the losses are twice more significant than the gains.

Let p be a lottery yielding $-\frac{1}{2}$ and 1 with equal probabilities. By procedure 2, $CE(p) = 0$.

Let q be a lottery yielding 0 and 3 with equal probabilities. By procedure 2, $CE(q) = 1$.

Assume there is a utility function $u(x)$ such that yields the same CE for these two lotteries when using procedure 1. That is:

By lottery p, $\frac{1}{2}u(-\frac{1}{2}) + \frac{1}{2}u(1) = u(0)$, and by lottery q, $\frac{1}{2}u(0) + \frac{1}{2}u(3) = u(1)$.

Substituting $u(1)$ we get $\frac{1}{2}u(-\frac{1}{2}) + \frac{1}{4}u(0) + \frac{1}{4}u(3) = u(0)$, which implies that 0 is the certainly equivalent of the lottery $r = \frac{1}{2}[-\frac{1}{2}] \oplus \frac{1}{4}[0] \oplus \frac{1}{4}[3]$.

However, by procedure 2 we can see that 0 is not the $CE(r)$ since $\sum_{y<0} r(y)l(0-y) = \frac{1}{2} \cdot 2 \cdot \frac{1}{2} \neq \frac{1}{4} \cdot 1 \cdot 3 = \sum_{y>0} r(y)g(y-0)$.