Exam TAU
February 2013
Solution

Problem 1.

Consider a world with two commodities in which a consumer can consume bundles that contain only one of the two commodities Facing a budget set $B(p_1,p_2,w)$, each consumer has two continuous strictly monotonic evaluation functions v_1 and v_2 and compares between $v_1(w/p_1)$ and $v_2(w/p_2)$. He spends all his resources on the good that yields a higher evaluation.

a. Is this behavior rationalizable?

Answer:

Yes, by the preferences represented by $Max\{v_1(x_1),v_2(x_2)\}$.

b. Is this behavior consistent with maximizing continuous, monotonic and convex preferences?

Answer:

Yes, by preferences with linear indifference curves (not necessarily parallel) such that:

a. any x_1 and x_2 where $v_1(x_1) = v_2(x_2)$ are on the same indifference curve. By the strict monotonicity of v_1 and v_2, these lines do not intersect.

b. in the case that there is a quantity x_i^* of commodity i where $v_i(x_i^*) > v_j(x_j)$ for any quantity of commodity j, the indifference curve through x_i^* is orthogonal to the i axis.

c. Assume that a consumer follows this procedure and sometimes purchases commodity 1 and sometimes commodity 2. Is this behavior consistent with maximizing continuous, monotonic and strict convex preferences?

Answer:

No.

Assume that this behavior is consistent with maximizing continuos. monotonic and strictly convex preferences.

First, note that there are quantities x_1^* and x_2^* such that $(x_1^*,0) \sim (0,x_2^*)$:

If the agent sometimes purchases commodity 1 and sometimes commodity 2, then there are two budget sets, (p_1',p_2',w) and (p_1'',p_2'',w''), such that $v_1(w'/p_1')\geq v_2(w'/p_2')$, and $v_1(w''/p_1'')\leq v_2(w''/p_2'')$. By the continuity of v_1 and v_2, there is α such that $v_1(\alpha w'/p_1' + (1-\alpha)w''/p_1'')=v_2(\alpha w'/p_2' + (1-\alpha)w''/p_2'')$. Thus, there exist x_1^* and x_2^* such that $v_1(x_1^*) = v_2(x_2^*)$.

If $(x_1^*,0) \succ (0,x_2^*)$ then by continuity there exists $x_1 < x_1^*$ such that also $(x_1,0) \succ (0,x_2^*)$. By the monotonicity of v_1, it holds that $v_1(x_1) < v_2(x_2^*)$ which implies that in a budget set where $p_1 x_1 = p_2 x_2^* = w$ an agent that follows this procedure is supposed to choose $(0, x_2^*)$. This is inconsistent with $(x_1, 0) \succ (0, x_2^*)$. Similarly, it cannot be that $(x_1^*, 0) \prec (0, x_2^*)$, and therefore $(x_1^*, 0) \sim (0, x_2^*)$.

Let $B(p_1, p_2, w)$ be a budget set such that $p_1 x_1^* = p_2 x_2^* = w$. The agent is indifferent between $(x_1^*, 0)$ and $(0, x_2^*)$, the two corners of the budget set, and by strict convexity he prefers any point between the two corners, $(\alpha x_1^*, (1-\alpha) x_2^*)$, to the corners themselves. This is inconsistent with the procedure, which requires choosing one of the corners.

d. Does the demand function satisfy the "law of demand" (according to which decreasing price of a commodity weakly increases the demand for it)?

Answer:

Yes. If the price of good i decreases, i.e. $p_i' < p_i$, then the consumer can buy more of good i, i.e. $w/p_i' > w/p_i$, while the amount of commodity j he can buy remains unchanged. Thus, his evaluation of commodity j ($v_j(w/p_j)$) remains constant while his evaluation of commodity i increases from $v_i(w/p_i)$ to $v_i(w/p_i')$.

If under p_i the consumer did not consume commodity i, then his demand cannot decrease and the law of demand holds.

Otherwise, the consumer buys w/p_i units of commodity i and we can conclude that $v_j(w/p_j) \le v_i(w/p_i)$. Clearly, we now have that $v_j(w/p_j) < v_i(w/p_i')$ and the consumer continues to consume from commodity i. His consumption of commodity i increases from w/p_i to w/p_i' and the law of demand again holds.

Problem 2

Society often looks for a representative agent. Assume for simplicity that the number of agents in a society is a power of 2 (1,2,4,8....). Each agent is one of a finite number of types (a member in a set T). A representative agent method (RAM) is a function F which attaches to any vector of types $(t_1, .., t_n)$ (where $n = 2^m$ and each $t_i \in T$) an element in $\{t_1, .., t_n\}$.

Make the following assumptions about F:

(1) Anonymity: For any n and for any permutation σ of $\{1, .., n\}$, **we have** $F(t_1, .., t_n) = F(t_{\sigma(1)}, .., t_{\sigma(n)})$.

(2) The "representative" is the "representative of the representatives": $F(t_1, .., t_n) = F(F(t_1, .., t_{n/2}), F(t_{n/2+1}, .., t_n))$

a. Characterize the RAMs which satisfy the two axioms.

Answer:

Claim: an RAM satisfies the two axioms iff there is an ordering of the types in T, denoted by $\succ$, such that $F(t_1,..,t_n)$ is the $\succ$-maximal type in $\{t_1,..,t_n\}$.

Proof:

$\rightarrow$

Let F be an RAM satisfying the two axioms.

Define $t_i \succ t_j$ if $F(t_i,t_j) = t_i$. The relation $\succ$ is an ordering on T and has the following characteristics:

Asymmetry: by axiom (1), $F(t_i,t_j) = F(t_j,t_i)$ and therefore if $t_i \succ t_j$, then $F(t_j,t_i) \neq t_j$, which implies that $t_j \not\succ t_i$.

Completeness: By the assumption that $F(t_i,t_j) \in \{t_i,t_j\}$, either $F(t_i,t_j) = t_i$ or $F(t_j,t_i) = t_j$. Hence, either $t_i \succ t_j$ or $t_j \succ t_i$.

Transitivity: Assume that $t_i \succ t_j$ and $t_j \succ t_h$. If not $t_i \not\succ t_h$, then $F(t_h,t_i) = t_h$. By axiom (2):

$F(t_i,t_j,t_h,t_h) = F(F(t_i,t_j),F(t_h,t_h)) \ = F(t_i,t_h) = t_h$ and

$F(t_j,t_h,t_i,t_h) = F(F(t_j,t_h),F(t_i,t_h)) \ = F(t_j,t_h) = t_j$.

However, by axiom (1) $F(t_i,t_j,t_h,t_h) = F(t_j,t_h,t_i,t_h)$, a contradiction.

Lastly, we can show that $F(t_1,..,t_n) = \ \succ$-maximal in $\{t_1,..,t_n\}$, by induction on m, where $n = 2^m$:

By definition this holds for $m = 1$. Assume that it is correct for $m = l-1$: $F(t_1,..,t_{2^{l-1}}) = \ \succ$-maximal in $\{t_1,..,t_{2^{l-1}}\}$.

Let $m = l$.

By axiom (2), $F(t_1,..,t_{2^l}) = F(F(t_1,..,t_{2^{l-1}}),F(t_{2^{l-1}+1},..,t_{2^l}))$. By assumption, $F(t_1,..,t_{2^{l-1}}) = \ \succ$-maximal in $\{t_1,..,t_{2^{l-1}}\}$ and $F(t_{2^{l-1}+1},..,t_{2^l}) = \ \succ$-maximal in $\{t_1,..,t_{2^{l-1}}\}$. Denote these two maximal types by t' and t''.

By definition, $F(t',t'')$ is the $\succ$-maximal in $\{t',t''\}$ and clearly it is also the maximal in $\{t_1,..,t_{2^l}\}$.

$\leftarrow$

1. Trivial

2. The $\succ$-maximal type in $\{t_1,..,t_n\}$ is either in $\{t_1,..,t_{n/2}\}$ or in $\{t_{n/2+1},..,t_n\}$. In either case, it is the $\succ$-maximal in its set and therefore it is chosen by F. Thus, this type is also in $\{F(t_1,..,t_{n/2}),F(t_{n/2+1},..,t_n)\}$ and it will be chosen from $(t_1,..,t_n)$ by F.

b Suggest an RAM that satisfies (1) but not (2) and an RAM that satisfies (2) but not (1).

Answer:

(1) but not (2): choosing the second-best type according to some ordering $\succ$.on T.

(2) but not (1): choosing the type of the first agent: $F(t_1,..,t_n) = t_1$.

Problem 3

Consider the housing model we talked about in class (where the number of houses is equal to the number of individuals).

a. We will say that an allocation $a = (a(i))_{i \in I}$ is an equilibrium if there are "choice sets" $(S(i))_{i \in I}$ such that:

(i) $a(i)$ is the i-best in $S(i)$

(ii) for any two agents i and j either $S(i) \subset S(j)$ or $S(j) \subset S(i)$.

Show that a is an equilibrium if and only if a is Pareto efficient.

Answer:

$\rightarrow$

Let a be an equilibrium according to the above definition. By (ii), we can order the agents such that $S(1) \subset S(2) \subset \ldots \subset S(n)$.

Consider a feasible allocation b such that $b(i) \succsim_i a(i)$ for all $i \in I$ for at least one agent j, such that $b(j) \succ_j a(j)$.

Let i^* be the highest $i \in I$ such that $b(i) \neq a(i)$. It must be that $b(i^*) \succ_{i^*} a(i^*)$ and that $b(i^*) = a(j)$ for some $j < i^*$. However, $a(j) \in S(j) \subset S(i^*)$, which contradicts $a(i^*)$ being the i-best in $S(i^*)$.

$\leftarrow$

Let a be a Pareto-efficient allocation. Construct the sets $S(i)$ by the following steps:

Step 1:

There is at least one agent such that $a(i) \succ_i a(j)$ for all $j \neq i$: otherwise we obtain a contradiction to a being efficient (see problem 1 in $B-1$).

Denote this agent by 1.

Define $S(1) = \cup_{j \in I} a(j)$. Clearly, $a(1)$ is the 1-best in $S(1)$

Repeat this procedure with the remaining agents:

At step l, there is at least one agent in $I \setminus \{1, \ldots, l-1\}$ such that $a(i) \succ_i a(j)$ for all $j \in I \setminus \{1, \ldots, l-1, i\}$.

Denote this agent by l.

Define $S(l) = \cup_{j \in I \setminus \{1,\ldots,l-1\}} a(j)$. Clearly, $a(l)$ is the l-best in $S(l)$

Lastly, note that by construction, for any two agents i and j, either $S(i) \subset S(j)$ or $S(j) \subset S(i)$.

b. We will say that an allocation $a = (a(i))_{i \in I}$ is a 2-equilibrium if there are "choice sets" $(S(i))_{i \in I}$ such that

(i) $a(i)$ is the i-best in $S(i)$; and

(ii) $S(i)$ contains two elements.

Show that unless one of the alternatives is the worst according to all preferences, then a 2-efficient equilibrium always exists.

Answer:

Claim: If none of the alternatives is the worst according to all agents' preferences, then there is an allocation such that no agent receives his worst alternative.

Proof: Let a be an allocation with the minimal number of agents who receive their worst alternative. Assume, for the purpose of contradiction, that this number is positive and let i be an agent who receives his worst alternative. Since no alternative is worst according to all agents, there is an individual j who does not consider $a(i)$ to be the worst alternative.

Let b be an allocation such that $b(i) = a(j)$, $b(j) = a(i)$, and $b(h) = a(h)$ for all $h \neq i,j$. The number of individuals who receive their worst alternative in b is smaller than in a, in contradiction to a having the minimal number of agents who receive their worst alternative.

Now, let a be an allocation in which no agent receives his worst alternative. For each agent, i, define S_i to be a set containing $a(i)$ and i's worst alternative. Clearly, $a(i)$ is best according to i in S_i.