**Exam TAU**
**January 2014**
**Solution**

**Problem 1.**

**We say that a binary relation $P$ over the space $X = R^n$ satisfies Property $I$ if the statement $xPy$ (the relation between $x$ and $y$) depends only on the equalities between the components of the two vectors. Formally, $P$ satisfies Property $I$ if $aPb \Leftrightarrow cPd$ for any four vectors $a$, $b$, $c$ and $d$ that satisfy (i) $a_i = a_j \Leftrightarrow c_i = c_j$, (ii) $b_i = b_j \Leftrightarrow d_i = d_j$ and (iii) $a_i = b_j \Leftrightarrow c_i = d_j$.**

**Denote $Y = \{x | \forall i \neq j, x_i \neq x_j\}$ as the set of all vectors vectors that are composed of $n$ different numbers.**

**a. Give an example (for $n = 2$) of non-degenerated preference relation on $X$ that satisfies property $I$.**

**Answer:**

Let $P$ be the preference relation represented by: $U(x_1, x_2) = \begin{cases} 1 & if \quad x_1 \neq x_2 \\ 0 & if \quad x_1 = x_2 \end{cases}$.

Clearly, it is a well defined preference relation and the relation between $U(x)$ and $U(y)$ depends only on the equalities between components of $x$ and the equalities between components of $y$ and thus the preference relation satisfies property $I$.

**Show that any preference relation satisfying property $I$:**

**b. is indifferent between the vector $(1,2,3)$ and any of the vectors $(4,2,5)$, $(2,3,1)$ and $(4,5,6)$.**

**Answer:**

Note first that for any two vectors in $Y$ (!), Property $I$ implies that if $a_i = b_j \Leftrightarrow c_i = d_j$ for any $i,j$ then $aPb \Leftrightarrow cPd$.

1. Let $x = (1,2,3)$ and $y = (4,2,5)$. Whatever can be said about $x$ and $y$ can be said about $y$ and $x$ ($x_i = y_j$ iff $y_i = x_j$) and thus $xIy$.

2. Let $x = (1,2,3), y = (2,3,1)$ and $z = (3,1,2)$. Note that $x_i = y_j \Leftrightarrow y_i = z_j \Leftrightarrow z_i = x_j$. Thus, $xPy$ implies $yPz$ and $zPx$; $yPx$ implies $zPy$ and $xPz$. Thus, in both cases, $xIy$.

3. Let $x = (1,2,3)$ and $y = (4,5,6)$. Again, $x_i = y_j$ iff $y_i = x_j$ for any $i, j$ and thus $xIy$.

**c. is indifferent between any $x,y \in Y$ satisfying $x_i \neq y_j$ for any $i,j$.**

**Answer:**

Choose $x$ and $y$ such that $x_i \neq y_j$ for any $i, j$. Assume $xPy$. As in 3 above, $x_i \neq y_j$ iff $y_i \neq x_j$ for any $i, j$. Thus, $xIy$.

**d**. **is indifferent between any** $x,y \in Y$ **where** $x$ **is a permutation of** $y$.

**Answer**:

Let $x$ and $y$ be two vectors such that $y$ is a permutation of $x$. Thus, there exists a permutation on $\{1,\dots,n\}$ such that $y_{\sigma(i)} = x_i$ for any $i$. Define a sequence of vectors $\{x^i\}$ such that $x^0 = x$ and $x^k_{\sigma(i)} = x^{k-1}_i$. Thus, any $x^k$ is a permutation of $x$, $x^0 = x$ and $x^1 = y$. There exists a minimal integer $K$ such that $x^K = x^0$.

Assume $xPy$, i.e., $x^0Px^1$. By definition, $x^0_i = x^1_j$ iff $j = \sigma(i)$. Since $x^1_i = x^2_j \Leftrightarrow j = \sigma(i)$ as well, $x^1Px^2$. Similarly $x^2Px^3$,...,$x^{K-1}Px^K = x^0$. By transitivity, $x^1Px^0$. Thus, $xPy \Rightarrow yPx$. An analgous argument applies in the case of $yPx$. Thus, $xIy$.

**e**. **Is indifferent between any** $x,y \in Y$.

**Answer**:

Choose $z \in Y$ such that all components in $z$ different from all componenets in $x$ and $y$. By (c) $xIz$ and $yIz$ and thus $xIy$.

**f**. (**much more difficult**) **Characterize the set of preference relations satisfying Property** $I$.

See: Rubinstein, Ariel (2002) Definable Preferences: Another Example (Searching for a Boyfriend in a Foreign Town) in "The Scope of Logic, Methodology and Philosophy of Science", *Proceedings of the 11th International Congress of Logic*, ed. Peter Gardenfors et al. Kluwer, vol I, 235-243. (http://arielrubinstein.tau.ac.il/papers/65.pdf).

**Problem 2**

**Consider an economy with two commodities and two agents**, **in which agent** $i$ **owns an initial bundle consisting only of commodity** $i$. **Agent** $2$ **maximizes a preference relation** (**satisfying strong monotonicity**, **strict convexity and continuity**). **In contrast Agent** $1$ **has a preference relation in mind** (**satisfying all of the above properties**) **and given a price vector he is interested in selling as much as he can of commodity** $1$, **provided that his final bundle is no worse than his initial one** (**according to his preferences**).

**a**. **Define an appropriate equilibrium concept for this economy**.

**Answer**:

Let $e = (e^1,e^2)$ be the initial allocation. Let $\succeq^i$ denote agent $i's$ preferences.

An equilibrium is a pair $((a^i)_{i=1,2},(p_k)_{k=1,2})$ where:

$a^i = (a^i_1,a^i_2)$ is a bundle and

$p = (p_1,p_2)$ is the price vector,

such that:

1. $a$ is feasible: $\sum a^i = \sum e^i$.
2. $a$ minimizes $x_1$ in $\{x \mid x \succeq^1 e^1$ and $p \cdot e^1 \geq p \cdot x\}$.
and $a^2$ is $\succeq^2$ -optimal in $\{x \mid p \cdot e^2 \geq p \cdot x\}$.

**Assume that the agent's indifference curves through the initial endowment cross in exactly one other allocation in the Edgworth box**.

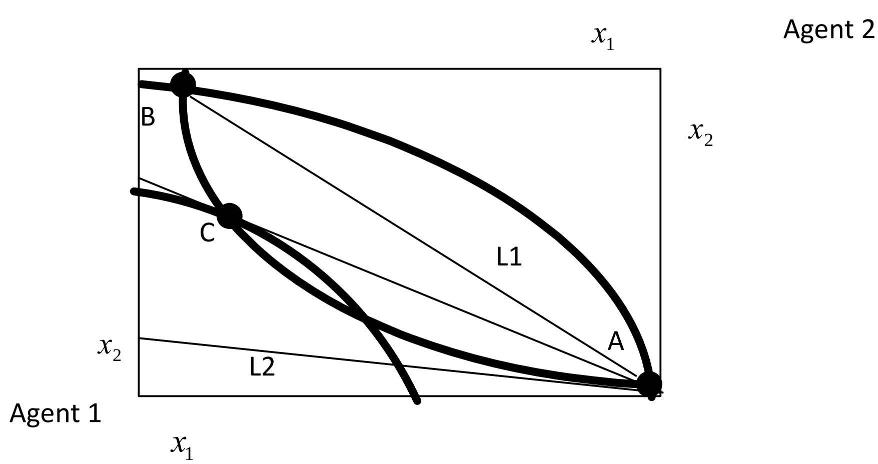
**b**. **Prove the existence of an equilibrium without using any general theorems**,

Denote $A$ to be the initial endowment and $B$ to be the second point at which the indifference curves through $A$ meet again.

If there is a point $C$ on agent $1$'s indifference curve such that agent $2$'s indifference curve through that point is tangent to the line that connects $A$ and $C$ (see graph below), then this point is an equilibrium allocation and the price vector is such that the budget line connects these two points. Thus, if agent $1$ sells any more of commodity $1$ then he crosses his indifference curve that passes through $A$ and therefore obtains a worse bundle, while agent $2$ maximizes his preferences at $C$ given the price vector.

In order to prove existence, first note that the demand of each agent is continuous: agent $1$ chooses $A$ on any budget line with slope less than $L2$ and chooses the point where the budget line crosses his indifference curve through $A$ for any other budget line. Agent $2$ has continuous and the classical demand is continuous.

Given $L1$ (the line that connects $A$ and $C$), agent $1$ chooses $B$ and agent $2$ chooses a point to the right of $B$ resulting in excess supply of good $1$. Given $L2$, agent $1$ chooses $A$ and agent $2$ chooses a point to the left of $A$, resulting in excess demand for good $1$. By the continuity of the demand function, there is a point $C$ where the market for good $1$ clears.




**c**. **Show that the equilibrium is Pareto-efficient**.

**Answer**:

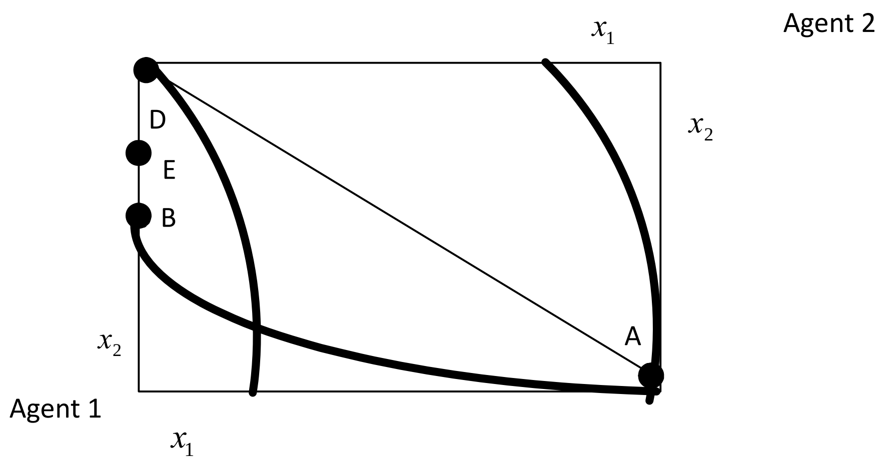
For any point on or below agent $2$'s indifference curve through $A$, either: (i) agent $1$ sells less of commodity $1$ or (ii) he obtains a strictly worse bundle than $A$. Thus, we cannot weakly improve agent $2$'s situation without hurting agent $1$.

**d**. **Show that without the above assumption**, **an equilibrium may not be Pareto-efficient**.

**Answer**:

Consider the following case:

The agents' preferences are such that (i) agent $1$ prefers the upper left corner of the box, i.e. $D$ to $A$, and (ii) the slope of agent $2$'s indifference curve through $D$ is greater than that of the line that connects $A$ and $D$.




In this case, $D$, is an equilibrium allocation and the price vector is such that the budget line connects $A$ and $D$. Thus, agent $1$ sells his entire endowment of commodity $1$ without obtaining a worse bundle and agent $2$ maximizes his preferences given his budget constraint. However, this is not a Pareto-efficient allocation since agent $1$ is willing to be at $E$ (where he still obtains a bundle as good as the initial one after selling all of his units of commodity $1$), which would strictly improve agent $2$'s situation.

**Problem 3**

**Consider a world in which the grand set $X$ is the entire plane and choice sets can only be less than 180 degree closed arcs of the unit circle**. **Denote a choice set by $B(\alpha,\beta)$ where $\alpha$ and $\beta$, are the two angles that confine the arc which are numbers between $0$ and $360$**. **For example**, **$B(0,90)$ is one-quarter of a circle contained in the positive quadrant**.

**a**. **Give an example of a choice function that does not satisfy the weak axiom of revealed preference**.

**Answer**:

Assume that the agent always chooses the middle of the arc. Then, WA is violated. For example, the agent chooses the middle bundle $x$ from the quarter of a circle $B(0,60)$, which lies on the $30^\circ$ ray, while from the smaller arc $B(0,40)$, which still contains $x$, he chooses the bundle that lies on the 20-degree ray.

**b**. **Give an example of a choice function that satisfies the weak axiom of revealed preference and yet is not rationalizable**.

**Answer**:

Assume that the agent always chooses the leftmost bundle on the arc (looking from the origin). The bundles chosen in $B(0,120)$, $B(120,240)$ and $B(240,0)$ create a cycle.

However, WA is never violated: Assume that the agent chooses $x$ while $y$ is another point on the arc. Since each arc is less than $180^\circ$ and $x$ is the leftmost point on the arc, we can conclude that $y$ is less than $180^\circ$ right of $x$. Therefore, $x$ is more than $180^\circ$ right of $y$. Thus, if $y$ is chosen as the leftmost point on a given arc, then $x$ cannot be on that arc as well. Thus, if $x$ is revealed to be preferred to $y$, then $y$ is never revealed to be preferred to $x$.

**Assume now that the choice sets are only arcs in the positive quadrant** (**i**.**e**. **the two angles that define the choice sets are between** $0^\circ$ **and** $90^\circ$) **and that the agent maximizes a monotonic**, **continuous and strictly convex preference relation**.

**c**. **Show that the agent's choice function is well defined**.

The choice set is compact and therefore the continuity of the preferences implies that a solution to the agent's problem in $B(\alpha,\beta)$ does exist.

Assume, in contradiction, that the two alternatives $x$ and $y$ are solutions. Then, $x \sim y$ and by the strict convexity of the preferences, $\alpha x + (1-\alpha)y \succ y$. However, $\alpha x + (1-\alpha)y$ is strictly below the budget set and by the monotonicity of the preferences there exists a bundle on $B(\alpha,\beta)$ that is strictly preferred to $\alpha x + (1-\alpha)y$. By transitivity, this bundle is also preferred to $y$. A contradiction.

**d**. **Explain how one could identify the agent's choice function from the indirect preference relation** (**defined over the parameters of the choice sets**).

Consider $B(\alpha,\beta)$. Let $\alpha^*$ be the maximum angle $x$ between $\alpha$ and $\beta$ such that $B(\alpha,\beta)$ and $B(x,\beta)$ are indifferent according to the indirect preferences. Then, the point on $B(\alpha,\beta)$ at angle $\alpha^*$ is the chosen alternative from $B(\alpha,\beta)$.