Solution for Problem Set B-01
Jan 2013

Problem 1:

Show that in this model for any efficient allocation a there is an i such that $a(i) \succsim_i h$ for all h.

Let a be an efficient allocation and assume, in contradiction, that none of the agents receives his preferred house.

Let h_1 be the house preferred the most by agent 1. Let h_2 be the house preferred the most by the agent who currently holds h_1, and so on. By assumption, for no i, $h_i = h_{i+1}$. Since the number of agents is finite, at some point a cycle with at least 2 agents will be created.

An allocation which is identical to a for all houses not in the cycle, and that replaces each h_i with h_{i+1} for all houses that are in the cycle, is a Pareto improvement, which contradicts the efficiency of a.

Problem 2:

Show the existence of a jungle equilibrium in a world with K commodities where an initial bundle w can be divided in any way between the n agents. Assume that each agent has a classical preference relation (satisfying continuity, strict monotonicity and strong convexity) over the set of bundles and is restricted to consume a bundle within a set of bundles X^i which satisfies compactness, free disposal and convexity. Define an equilibrium as an allocation $(a(i))_i$ of w such that there is no i stronger than j such that there exists a bundle $x \leq a(i) + a(j)$ in X^i such that $x \succ_i a(i)$. (That is, i can attack only one weaker agent).

Show that the jungle equilibrium is efficient.

The following procedure will yield a Jungle equilibrium: call the individuals, one by one, according to their order of power in order to pick from whatever is left from the initial bundle under their consumption constraint.

Given the above assumption, this procedure yields a unique allocation which is a jungle equilibrium.

Efficiency: Assume that a is the jungle equilibrium allocation and that b is a Pareto dominating allocation. Let j be the strongest agent for which $b(j) \neq a(j)$. $a(j)$ is j's optimal bundle in $X^j \cap \{x|\, x_k \leq (w - a(1) - \ldots - a(i-1))_k$ for all $k\}$. Thus, it must be that for some k $b(j)_k > (w - a(1) - \ldots - a(i-1))_k$. But, this means that for one of the stronger agents $b(i) \neq a(i)$.

Problem 3:

Show that if agents have preferences with indifferences, **then there might be a jungle equilibrium which is not efficient**.

Show that if agents have preferences with indifferences, **then there might be a competitive equilibrium which is not efficient**.

Let $I = \{1,2\}$ and $H = \{h_1, h_2\}$. Assume that $h_1 \sim_1 h_2$ and $h_1 \succ_2 h_2$.

Jungle:

Let $1S2$. The allocation $a = (h_1, h_2)$ is a jungle equilibrium which is Pareto dominated by (h_2, h_1).

Market:

With the initial allocation $e = (h_1, h_2)$, a possible competitive equilibrium is $p_1 > p_2$ and a final allocation $a = (h_1, h_2)$ which is not efficient.

Problem 4:

Can a competitive equilibrium be always obtained by a chain of pairwise exchanges where each exchange has the property that the exchange improves the situation of the two parties?

Not necessarily. Consider an economy with three agents, 1, 2, 3 and three houses, a, b, c. Let $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$. Let the initial allocation be $w = (b, c, a)$. No pairwise voluntary transaction which improves the situation of the two parties exists. Nevertheless, $p_1 = p_2 = p_3$ with $x = (a, b, c)$ is competitive equilibrium.

Problem 5:

Construct and analyze a model which will be similar to the one analyzed above with $n+1$ agents and n houses. The "new" agent 0 initially owns the n houses and cares only about an additional good, "money". Each agent i $(i = 1,\dots,n)$ initially holds \$$m_i$ and has lexicographic preferences with first priority of the house he will own and second priority about the money left in his pocket after purchasing the house.

Assume $m_1 > m_2 > \ldots > m_n$. Each agent i can hold a pair (m,h) where m is money and h is either one of the houses or 0. We refer to such a pair as a bundle.

A competitive equilibrium is a price vector p and $n+1$ tuple of bundles such that (1) for each agent, the bundle is the best given his budget constraint and (2) the sum of the money holdings is Σm_i and each house appears in only one of the consumption bundles.

There is no equilibrium in which agent 0 holds a house (if there is such an equilibrium then the price of that house is zero and the agent who does not hold a house wants it as well.)

There is a competitive equilibrium where the allocation is determined like in a jungle equilibrium with iSj if $m_i > m_j$. The price of the house that i gets is between m_i and m_{i+1}.

If for some agents $m_i = m_j$ an equilibrium may not exist: if agents i and j have the same (strict) preferences over the houses, for any price vector both agents would like to buy the same house or both wouldn't want to buy any house.

Problem 6:

Show that for every jungle equilibrium $a = \{a(i)\}_{i \in I}$ there is a price vector p such that (a,p) is a competitive equilibrium such that the stronger is the wealthier (measured by the competitive price vector).

Let $p(a(i)) > p(a(j))$ iff iSj. As a is a jungle equilibrium $a(i) \succsim_i a(j)$ for all iSj. Thus, $a(i) \succsim_i h$ for every h such that $p(a(i)) \geq p(h)$. That is, (p,a) is a competitive equilibrium.

Problem 7:

Invent at least two comparative statics results , one for the jungle equilibrium and one for the competitive equilibrium.

Jungle: Let S and $S^{'}$ be strength relations such that only one agent, i^*, is stronger in $S^{'}$ than in S, then $a^{'}(i^*) \succsim_{i^*} a(i^*)$.

Market: If $e(i^*)$ moves up in one of the other agents preferences, then i^* gets a (weakly) better house according to $\succ_{i^*}$.

Problem 8:

Consider the following dynamic process.

Stating point: **start from an arbitrary assignment** (**an agent can be assigned to a house or to the** "**street**")

At stage $\mathbf{t+1}$ **each agent selects the best house** (**from his point of view**) **from the houses which at stage** t **are deserted or held by agents which are not stronger than him**. **At each house which is approached by more than one agent**, **only the stronger stays**. **The rest are sent to the** "**street**".

The process stops at T when the assignment at period T and T - **1 are identical**

Show that the process must stop and will stop at a jungle equilibrium.

Order the set of agents such that iSj for $i < j$.

Claim: Agent t moves to his preferred house, from those not held by stronger agents, not later than stage t.

Proof: In the first stage the strongest agent can move to the house he prefers the most - even if the house is occupied or desired by another agent, the strongest "wins". Moreover, in any subsequent stage he will have no reason to move from it.

Assume that by stage t agents $\{1,2,\ldots,t\}$ have moved to their preferred houses. At stage $t+1$ none of these agent will have a reason to move. Therefore, agent $t+1$ moves to his preferred house from those not held by agents $\{1,2,\ldots,t\}$ - he will surely be the strongest one moving at this stage so he can win whatever house he wishes.

After at most n stages all agents settle and the process stops. This is a jungle equilibrium since each agent holds the best house, according to his preferences, from those held be agents weaker then him.