Problem Set 1 - Preferences

Problem 1.

Let $\succsim$ be a preference relation on a set X. Define $I(x)$ to be the set of all $y \in X$ for which $y \sim x$. Show that the set (of sets!) $\{I(x) | x \in X\}$ is a partition of X.

(i) For all x and y, either $I(x) = I(y)$ or $I(x) \cap I(y) = \emptyset$.

Note that the indifference relation is symmetric and transitive.

Let $x, y \in X$, and assume that $I(x) \cap I(y) \neq \emptyset$, which means that there exists $z \in I(x) \cap I(y)$. Let a be any element in one of the sets, let us say $I(x)$. It means that $a \sim x$. But $z \sim x$ and $z \sim y$ since $z \in I(x) \cap I(y)$. By symmetry of the indifference relation, $x \sim z$. By transitivity of $\sim$, it follows that $a \sim y$, and thus $a \in I(y)$.

(ii) For every $x \in X$, there is $y \in X$ such that $x \in I(y)$.

Let $x \in X$. The completeness of $\succsim$ means that $x \sim x$. Consequently, $x \in I(x)$.

Problem 2.

Kreps (**1990**) **introduces another formal definition for preferences**. **His primitive is a binary relation** P **interpreted as** "**strictly preferred**". **He requires** P **to satisfy**:

Asymmetry: **For no** x, y **do we have both** xPy **and** yPx.

Negative Transitivity (**NT**): $\forall x, y, z \in X$, **if** xPy, **then either** xPz **or** zPy (**or both**).

Explain the sense in which Kreps' formalization is equivalent to the traditional definition.

We will closely follow the proof presented in the lecture notes. The following steps are required:

a. Construct an interpretation-preserving function T that maps a binary relation P satisfying asymmetry and negative-transitivity into preference relations.

Consider the following candidate function T, which maps any relation P into a binary relation defined by

$xT(P)y$ if not yPx.

Note that T preserves interpretation: If "y is not strictly preferred to x" according to Kreps' formalization, then "x is at least as good as y".

b. Prove that $T(P)$ is a preference relation.

Completeness of $T(P)$: Kreps' asymmetry property of P says that for any x and y in X, either not xPy or not yPx. Thus either $xT(P)y$ or $yT(P)x$.

Transitivity: Let $x, y, z \in X$ be such that $xT(P)y$ and $yT(P)z$. If not $xT(P)z$, then zPx. By NT, either yPx or zPy. Thus either not $xT(P)y$ or not $yT(P)z$, a contradiction.

c. Prove that T is one-to-one.

Let P_1 and P_2 be two different relations satisfying Kreps' properties. Then there is a pair x,y such that xP_1y and not xP_2y (or the opposite), and thus not $yT(P_1)x$ and $yT(P_2)x$. Thus, $T(P_1) \neq T(P_2)$, implying that T is one to one.

d. Prove that T maps onto all preference relations.

Let $\succsim$ be a preference relation. Define P, by:

xPy if not $y \succsim x$.

P preserves Kreps' properties:

Asymmetry: Since $\succsim$ is complete, then we never have both xPy and yPx.

NT: Let $x, y, z \in X$ be such that xPy, and thus not $y \succsim x$. Therefore it is not true that both $y \succsim z$ and $z \succsim x$. Therefore either zPy or xPz.

Finally, note that $T(P) = \succsim$.

Problem 3.

Let Z be a finite set and let X be the set of all nonempty subsets of Z. Let $\succsim$ be a preference relation on X (not Z). An element $\mathbf{A} \in \mathbf{X}$ is interpreted as a "menu", i.e. "the option to choose an alternative from the set $\mathbf{A}$". Consider the following two properties of preference relations on X:

1. **If $A \succsim B$ and C is a set disjoint to both A and B, then $A \cup C \succsim B \cup C$, and if $A \succ B$ and C is a set disjoint to both A and B, then $A \cup C \succ B \cup C$.**
2. **If $x \in Z$ and $\{x\} \succ \{y\}\ \forall y \in A$, then $A \cup \{x\} \succ A$, and if $x \in Z$ and $\{y\} \succ \{x\}\ \forall y \in A$, then $A \succ A \cup \{x\}$.**

a. Discuss the plausibility of the properties.

Consider an appealing interpretation of the formal model: The elements in Z are the alternatives which might be chosen at the end of a decision process, a set A is a set of candidates to be considered seriously in the second stage. If we have in mind that the economic agent is certain about his preferences in the later stage then (1) is problematic: if the best element in menu A is better than the best in menu B, and menu C includes an even better element, then $A \succ B$ but $A \cup C \sim B \cup C$, violating (1). Also, if any element of menu A is strictly better than $z \in Z$, then $A \succ \{z\}$ but $A \sim A \cup \{z\}$, violating (2). The properties make more sense if the decision maker has in mind a tiny possibility that he will err in his choice or that there is a possibility that an alternative which he chooses will not be feasible at the end.

b. Provide an example of a preference relation that:

(**1**) **Satisfies both properties.**

The relation $\succsim$ defined by $A \succsim B$ if $|A| \geq |B|$ satisfies (1) and (2) in a degenerate way (since for all x and y we have $\{x\} \sim \{y\}$).

A "better" class of examples (including the previous one): Let X be divided to two sets G and B. Define a preference relation by the utility function $u(A) = |A \cap G| - |A \cap B|$. Clearly it satisfies both properties.

(**2**) **Satisfies the first but not the second property.**

Let $z^* \in Z$. Define $\succsim$ over X whereby $A \succ B \Leftrightarrow z^* \in A, z^* \notin B$ and $A \sim B$ otherwise.

(1) Let $A, B, C \in X$ be such that $A \succsim B$ and C is disjoint to both A and B. If $A \succ B$, then $z^* \in A$ and $z^* \notin B, C$, which implies that $A \cup C \succ B \cup C$. If $A \sim B$, then either z^* is a member of both sets $A \cup C$ and $B \cup C$ or of none. In both cases $A \cup C \sim B \cup C$.

(not 2) Let $A = \{z^*\}$ and $y \neq z^*$. Then $A \succ \{y\}$ but $A \sim A \cup \{y\}$ violating the second part of (2).

More generally, attach to each element x a non-negative number $v(x)$ and define a preference relation by a utility function $U(A) = \Sigma_{a \in A} v(a)$. Then the preference relation satisfies (1) but not (2).

(**3**) **Satisfies the second but not the first property**.

Let $\succsim^*$ be a preference relation on Z. Define $\succsim$ by $A \succsim B$ if

(a) the $\succsim^*$ –best element in A is strictly better than the $\succsim^*$ –best element in B, or

(b) the agent is indifferent between the best elements but the $\succsim^*$ –worst element in A is weakly better than the $\succsim^*$ –worst element in B.

(not 1) Let $a \succ^* b \succ^* c \succ^* d$. Then $\{b\} \succ \{c\}$, but $\{b\} \cup \{a,d\} \sim \{c\} \cup \{a,d\}$.

(2) Let $A \in X$ and $z \in Z$. If z is strictly $\succsim^*$ –better (strictly $\succsim^*$ –worse) then all $a \in A$, then $A \cup \{z\} \succ A \quad (A \succ A \cup \{z\})$.

c. **Show that if there are** $x,y,z \in Z$ **such that** $\{x\} \succ \{y\} \succ \{z\}$, **then there is no preference relation satisfying both properties**.

Assume $\succsim$ satisfies (1) and (2), with $\{x\} \succ \{y\} \succ \{z\}$ for some $x,y,z \in Z$. From (2), $\{x\} \succ \{x,y\}$ and $\{y,z\} \succ \{z\}$. Applying (1) to the above, $\{x,z\} \succ \{x,y,z\}$ and $\{x,y,z\} \succ \{x,z\}$, a contradiction.

Problem 4.

Let $\succ$ be an asymmetric binary relation on a finite set $\mathbf{X}$ that does not have cycles. **Show** (**by induction on the size of $\mathbf{X}$**) **that $\succ$ can be extended to a complete ordering**.

Note that if a set A is finite and $\succ$ is an acyclic relation on A (there are no cycles), then there must exist an $x \in A$ such that there is no $y \in A$ such that $y \succ x$.

Since X is finite, then there exists an $x_1 \in X$ such that there is no $y \in X$ such that $y \succ x_1$. Define $x_1 \succ^* y$ for all such $y \in X - \{x_1\}$. Again, there exists an $x_2 \in X - \{x_1\}$ such that there is no $y \in X - \{x_1\}$ such that $y \succ x_2$. Define $x_2 \succ^* y$ for all such y, and so on. By induction we can define $\succ^*$ for all $x \in X$.

By construction, the relation $\succ^*$ is complete, asymmetric, extends $\succ$ and transitive: let $x_i \succ^* x_j$ and $x_j \succ^* x_h$. Then $i < j$ and $j < h$ and therefore, $x_i \succ^* x_h$.

Problem 5.

You have read an article in a "prestigious" journal about a decision maker (DM) whose mental attitude towards elements in a finite set X is represented by a binary relation $\succ$, which is a-symmetric and transitive but <u>not necessarily complete</u>. The incompleteness is the result of an assumption that a DM is sometimes unable to compare between alternatives.

Another, presumingly stronger, assumption made in the article is that the DM uses the following procedure: he has n criteria in mind, each represented by an ordering (a-symmetric, transitive and <u>complete</u>) $\succ_i$ $(i = 1,\ldots,n)$. The DM decides that $x \succ y$ if and only if $x \succ_i y$ for every i.

1.Verify that the relation $\succ$ generated by this procedure is a-symmetric and transitive. Try to convince a reader of the paper that this is an attractive assumption by giving a "real life" example in which it is "reasonable" to assume that a DM uses such a procedure in order to compare between alternatives.

$\succ$ is a-symmetric: If $x \succ y$ then by definition, $x \succ_i y$ for every i. Since $\succ_i$ are a-syemmtric, $y \not\succ_i x$ for all i, and by definition also $y \not\succ x$.

$\succ$ is transitive: Let $x \succ y$ and $y \succ z$. By definition, $x \succ_i y$ and $y \succ_i z$ for every i. Since $\succ_i$ are transitive, also $x \succ_i z$ for all i, and by definition $x \succ z$.

An example: A parent who considers destinations for a family vacation who ranks the different destinations according to the orderings of his children: he prefers A to B iff all his children prefer A to B.

It can be claimed that the additional assumption regarding the procedure that generates $\succ$ is not a "serious" one since given <u>any</u> asymmetric and transitive relation, $\succ$, one <u>can find</u> a set of complete orderings $\succ_1,\ldots,\succ_n$ such that $x \succ y$ iff $x \succ_i y$ for every i.

2. Demonstrate this claim for the relation on the set $X = \{a,b,c\}$ according to which only $a \succ b$ and the comparison between [b and c] and [a and c] are not determined.

Let $a \succ_1 b \succ_1 c$ and $c \succ_2 a \succ_2 b$. The two relations agree only on $a \succ_i b$.

3. (Main part of the question) Prove this claim for the general case.

Guidance (for c): given an asymmetric and transitive relation $\succ$ on an arbitrary X, define a set of complete orderings $\{\succ_i\}$ and prove that $x \succ y$ iff for every i, $x \succ_i y$.

First, note that if X is a finite set and P is a asymmetric and transitive relation on X then P does not have any cycles and thus P can be extended to a complete ordering of X (see problem 4).

Let Λ be the set of all complete orderings which extends $\succ$. We will see that $a \succ b$ if and only if $a \succ_i b$ for all $\succ_i \in \Lambda$:

(i) If $a \succ b$, then $a \succ_i b$ for all i since any $\succ_i \in \Lambda$ is an extension of $\succ$.

(ii) If not $a \succ b$, then let $\succ^*$ be the relation $\succ$ extended to include also $b \succ^* a$. The relation $\succ^*$ does not have cycles: if there is a cycle $x_1 \succ^* \ldots \succ^* x_n = x_1$ then

(a) if for some i we have $x_i = b \succ^* a = x_{i+1}$ then since $a = x_{i+1} \succ^* x_{i+2} \ldots \succ^* x_n = x_1 \succ^* \ldots \succ^* x_i = b$ by transitivity $a \succ b$ contradicting the assumption.

(b) otherise, by thranstivity $x_1 \succ x_2$ but also $x_2 \succ x_1$ conradicting asymettry.

Thus, $\succ^*$ can be extended to a complete ordering $\succ'$ which will be an extension of $\succ$ as well. Hence, there is an extension $\succ' \in \Lambda$ for which not $a \succ' b$.

Problem 6.

Listen to the illusion called the Shepard Scale. (**You can find it on the internet**. **Currently**, **it is available at http**://**www**.**youtube**.**com**/**watch**?**v**=**boJD\gTLavA**
and http://**en**.**wikipedia**.**org**/**wiki**/**Shepard_tone**.)

Can you think of any economic analogies?

The Shepard Scale consists of three separate scales that play the same tone at different octaves. As the notes ascend, one scale drops its pitch an octave, a change the listener does not notice because the other two scales continue to ascend monotonically, which "covers up" the drop. Several notes later, a second scale drops its pitch an octave, and so on. Thus, the Shepard Scale sounds as if it perpetually ascends, even though the same finite set of notes are repeated. See http://en.wikipedia.org/wiki/Shepard_tone for explanation.

The phenomenon is a reminder of an example due to Fishburn and LaValle (1988):

> Our illustration for decision under uncertainty assumes that states are the faces of a standard die with probability 1/6 for each state. Two acts with results that depend on the up face after one roll are f_1 and f_2:

	1	2	3	4	5	6
f_1	\$1000	\$ 500	\$600	\$700	\$800	\$900
f_2	\$ 900	\$1000	\$500	\$600	\$700	\$800

Many of us prefer the lottery f_2 to the lottery f_1. However, one can easily construct the other 4 lotteries f_3, f_4, f_5, f_6, (increase the \$500 to \$1000 and reduce the other prizes by \$100), such that we would prefer f_3 to f_2,f_4 to f_3, ... and f_1 to f_6.

See Peter C. Fishburn and Irving H. LaValle (1988). "Context-Dependent Choice with Nonlinear and Nontransitive Preferences", Econometrica, Vol. 56, 1221-1239. Stable URL: http://www.jstor.org/stable/1911365