# Problem Set Two – Utility

**Problem 1.**

**The purpose of this problem is to make sure that you fully understand the basic concepts of utility representation and continuous preferences. Prove or disprove the following:**

**a. Is the statement "if both $U$ and $V$ represent $\succsim$, then there is a strictly monotonic function $f : \Re \to \Re$ such that $V(x) = f(U(x))$" correct?**

False: Let $X = \Re$ and preferences be represented by the utility functions

$$V(x) = x \qquad \text{and} \qquad U(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0. \end{cases}$$

The only increasing function $f : \Re \to \Re$ that satisfies $V(x) = f(U(x))$ is

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 0 & \text{if } 0 < x \leq 1 \\ x - 1 & \text{if } x > 1 \end{cases}$$

which is not *strictly* increasing.

**b. Can a continuous preference relation be represented by a discontinuous utility function?**

True: The preferences ($x \succsim y$ if $x \geq y$) is represented by $U$ in (a) are continuous, though $U$ is discontinuous.

**c. Show that in the case of $X = \Re$, the preference relation that is represented by the discontinuous utility function $u(x) = [x]$ (the largest integer $n$ such that $x \geq n$) is not a continuous relation.**

$1 \succ 1/2$, but $1 - \epsilon \sim 1/2$ for $\epsilon > 0$ small, violating C1.

**d. Show that the two definitions of a continuous preference relation, C1 and C2, are equivalent to**

**Definition C3:** $\forall x \in X$, **the upper and lower contours $\{y \mid y \succsim x\}$ and $\{z \mid x \succsim z\}$ are closed sets in $X$.**

**Definition C4:** $\forall x \in X$, **the sets $\{y \mid y \succ x\}$ and $\{z \mid x \succ z\}$ are open sets in $X$.**

(C3 $\Leftrightarrow$ C4) By completeness, the sets $\{y \mid x \succ y\}$ and $\{y \mid y \succ x\}$ are the complementary to $\{y \mid y \succsim x\}$ and $\{y \mid x \succsim y\}$ correspondingly. Thus the formers are open sets iff the later are closed sets.

(C1 $\Rightarrow$ C4) Let $x \in X$ and $a \in \{y \mid y \succ x\}$. By C1, there exists an $\epsilon > 0$ such that $Ball(a,\varepsilon) \subseteq \{y \mid y \succ x\}$, ($Ball(a,\varepsilon)$ is the set of points in $X$ that are less than $\epsilon$ distance from $a$). Thus $\{y \mid y \succ x\}$ is open. The argument for $\{z \mid x \succ z\}$ open is analogous.

(C4 $\Rightarrow$ C1) Let us use the notation $B \succ x$ to mean that $y \succ x$ for all $y \in B$.

Assume first that there exists a $z \in X$ such that $x \succ z \succ y$. By C4, there exist $\epsilon_1, \epsilon_2 > 0$ such that $Ball(x,\epsilon_1) \succ z$ and $z \succ Ball(y,\epsilon_2)$. Let $\epsilon = \min\{\epsilon_1,\epsilon_2\}$. By transitivity, every point in $Ball(x,\varepsilon)$ is strictly better than every point in $Ball(y,\varepsilon)$.

Next, assume that there does not exist a $z \in X$ such that $x \succ z \succ y$. As above, by C4 there exists an $\epsilon > 0$ such that $Ball(x,\varepsilon) \succ y$ and $x \succ Ball(y,\varepsilon)$. Since there is no $z$ such that $x \succ z \succ y$, then $Ball(x,\varepsilon) \succsim x$ and $y \succsim Ball(y,\varepsilon)$, and thus by transitivity, every point in $Ball(x,\varepsilon)$ is strictly better than every point in $Ball(y,\varepsilon)$.

**Problem 2**.

**Give an example of preferences over a countable set in which the preferences cannot be represented by a utility function that returns only integers as values**.

Let $X = \mathbb{N}$, which is countable. Define preferences to be such that

$$1 \succ 3 \succ 5 \succ \ldots \succ 2 \succ 4 \succ \ldots.$$

By contradiction, assume that there exists a utility function $u : X \rightarrow \mathbb{Z}$ that represents $\succsim$. Then $u(1) = N$ and $u(2) = n$ for some $n, N \in \mathbb{Z}$. But there are an infinite number of odd numbers, implying that $u$ maps to an infinite number of integers between $n$ and $N$, a contradiction.

**Problem 3**.

**Let $\succsim$ be continuous preferences on a set $X \subseteq \Re^n$ which contains the interval connecting the points $x$ and $z$. Show that if $y \in X$ and $x \succsim y \succsim z$, then there is a point $m$ on the interval connecting $x$ and $z$ such that $y \sim m$.**

Construct inductively the sequence $\{(x^n, z^n, m^n)\}$ as follows: Start with define $x^0 = x$, $z^0 = z$ and the midpoint $m^0 = 1/2x^0 + 1/2z^0$.

If $m^n \sim y$ then we found the point we look for. Otherwise, $m^n \succ y$ or $y \succ m^n$.

If $m^n \succ y$ let $x^{n+1} = m^n$ and $z^{n+1} = z^n$.

If $y \succ m^n$ let $x^{n+1} = x^n$ and $z^{n+1} = m^n$.

In any case define $m^{n+1} = 1/2x^{n+1} + 1/2z^{n+1}$.

If none of the points $m^n \sim y$ then $x^n \succ y \succ z^n$ for all $n$. Both sequences $(x^n)$ and $(z^n)$ converge to some $m^*$ on the interval between $x$ and $z$.

Since $\succsim$ is continuous, then $m^* \succsim y$ and $y \succsim m^*$, and thus $m^* \sim y$.

Another possible proof: the interval between $x$ and $y$ is a connected set. The two sets $\{a|a \succ y\}$ and $\{a|y \succ a\}$ are disjoint by definition and open by the continuity of $\succsim$. Two disjoint open sets cannot cover a connected set and therefore there is at least one point on this interval such that $y \sim m$.

**Problem 4**.

**Consider the sequence of preference relations** $(\succsim^n)_{n=1,2,\ldots}$**, defined on** $\Re^2_+$ **where** $\succsim^n$ **is represented by the utility function** $u_n(x_1,x_2) = x_1^n + x_2^n$**. We will say that the sequence** $\succsim^n$ **converges to the preferences** $\succsim^*$ **if for every** $x$ **and** $y$ **such that** $x \succ^* y$**, there is an** $N$ **such that for every** $n > N$ **we have** $x \succ^n y$**. Show that the sequence of preference relations** $\succsim^n$ **converges to the preferences** $\succsim^*$ **which are represented by the function** $max\{x_1,x_2\}$**.**

Let $x \succ^* y$. Since $\max\{x_1,x_2\} > \max\{y_1,y_2\}$, then there exists an $\epsilon > 0$ such that $\max\{x_1,x_2\} > (1+\epsilon)\max\{y_1,y_2\}$. Consequently, for $n$ large enough,
$[\max\{x_1,x_2\}]^n > 2[\max\{y_1,y_2\}]^n$. But $x_1^n + x_2^n \geq [\max\{x_1,x_2\}]^n$ and $2\max\{y_1,y_2\}^n \geq y_1^n + y_2^n$, and thus $x \succ^n y$ for $n$ large enough.

**Problem 5**.

**Let $X$ be a finite set and let $(\succsim, \succ\succ)$ be a pair where $\succsim$ is a preference relation and $\succ\succ$ is a transitive sub-relation of $\succ$ (by sub-relation, we mean $x \succ\succ y$ implies $x \succ y$). We can think about the pair as representing the responses to the questionnaire $A$ where $A(x,y)$ is the question:**

**How do you compare $x$ and $y$? Tick one of the following five options:**

☐ **I very much prefer $x$ over $y$** ($x \succ\succ y$)

☐ **I prefer $x$ over $y$** ($x \succ y$)

☐ **I am indifferent** ($I$)

☐ **I prefer $y$ over $x$** ($y \succ x$)

☐ **I very much prefer $y$ over $x$** ($y \succ\succ x$)

**Assume that the pair satisfies extended transitivity: If $x \succ\succ y$ and $y \succsim z$, or if $x \succsim y$ and $y \succ\succ z$ then $x \succ\succ z$. We say that a pair $(\succsim, \succ\succ)$ is represented by a function $u$ if**

$u(x) = u(y)$ **iff** $x \sim y$,

$u(x) - u(y) > 0$ **iff** $x \succ y$, **and**

$u(x) - u(y) > 1$ **iff** $x \succ\succ y$.

**Show that every extended preference $(\succsim, \succ\succ)$ is represented by a function $u$.**

Denote $A \succ B$ if $a \succ b$ for all $a \in A$ and $b \in B$. Let $X_1, X_2, ..., X_K$ be the $\succsim$ indifference sets such that $X_K \succ X_{K-1} \succ \ldots \succ X_1$. Define first $u(X_1) = 0$.

Let us define $u(X_k)$ for $k > 1$.

(1) if $X_k \succ\succ X_{k-1}$, then $u(X_k) = u(X_{k-1}) + 2$

(2) if $X_k$ is not $\succ\succ$ even of $X_1$, then $u(X_k) \in (u(X_{k-1}), 1)$

(3) otherwise, there exists a maximal $m(k)$ such that $X_k \succ\succ X_{m(k)}$. Define $u(X_k)$ such that $u(X_k) > u(X_{k-1})$ and $1 + u(X_{m(k)+1}) > u(X_k) > u(X_{m(k)}) + 1$.

Clearly, $x \sim y$ iff $u(x) = u(y)$

Also, if $x \succ y$ then $u(x) > u(y)$, since we picked $u(X_k)$ as an increasing sequence.

Finally, if $x \succ\succ y$ , $x \in X_k$ and $y \in X_m$ then $m(k) \geq m$ and $u(x) > u(X_{m(k)}) + 1 \geq u(y) + 1$.

**Problem 6**.

**The following is a typical example of a utility representation theorem**: **Let** $X = \Re^2_+$.
**Assume that a preference relation** $\succsim$ **satisfies the following three properties**:

**ADD**: $(a_1,a_2) \succsim (b_1,b_2)$ **implies that** $(a_1+t,a_2+s) \succsim (b_1+t,b_2+s)$ $\forall s,t$.

**SMON**: **If** $a_1 \geq b_1$ **and** $a_2 \geq b_2$, **then** $(a_1,a_2) \succsim (b_1,b_2)$. **In addition**, **if either** $a_1 > b_1$ **or** $a_2 > b_2$ **then** $(a_1,a_2) \succ (b_1,b_2)$.

**CON**: **Continuity**.

**a**. **Show that if** $\succsim$ **has a linear representation** (**that is**, $\succsim$ **are represented by a utility function** $u(x_1,x_2) = \alpha x_1 + \beta x_2$ **with** $\alpha,\beta > 0$), **then** $\succsim$ **satisfies ADD**, **SMON**, **CON**.

ADD: Let $s,t \in \Re$ and $x,y \in X$ be such that $x \succsim y$. Note that $(x_1,x_2) \succsim (y_1,y_2) \Leftrightarrow \alpha x_1 + \beta x_2 \geq \alpha y_1 + \beta y_2 \Leftrightarrow \alpha(x_1+t) + \beta(x_2+s) \geq \alpha(y_1+t) + \beta(y_2+s) \Leftrightarrow$ $u(x_1+t,x_2+s) \geq u(y_1+t,y_2+s) \Leftrightarrow (x_1+t,x_2+s) \succsim (y_1+t,y_2+s)$.

SMON: Let $x,y \in X$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$ with at least one strict inequality. Since $\alpha,\beta > 0$, then $\alpha x_1 + \beta x_2 > \alpha y_1 + \beta y_2$, which implies that $(x_1,x_2) \succ (y_1,y_2)$.

CON: $u(x_1,x_2)$ is continuous, and thus $\succsim$ is continuous.

**b**. **Show that for any pair of the three properties there is a preference relation that does not satisfy the third property**.

Satisfies only ADD, SMON: Lexicographic preferences satisfy *ADD* and *SMON*, but are not continuous (see the lecture notes).

Satisfies only ADD, CON: The preferences represented by $u(x_1,x_2) = x_1 - x_2$ satisfy *ADD* and *CON*, but not *SMON* since $(1,1) \succ (1,2)$.

Satisfies only MON, CON: Preferences represented by $u(x_1,x_2) = x_1^2 + x_2^2$ satisfy *SMON* and *CON*, but not *ADD* since $(3,0) \succ (2,1)$ and $(3,3) \prec (2,4)$.

**c**. **Show that if** $\succsim$ **satisfies the three properties**, **then it has a linear representation**.

Assume first that $x$ and $y$ are two different points such that $x \sim y$. Then:

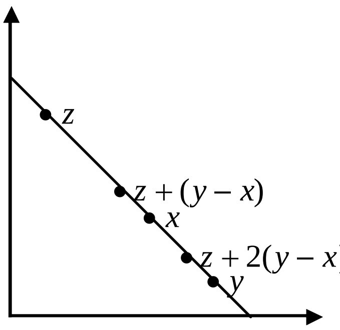
(i) $(x+y)/2 \sim y$. Otherwise, $(x+y)/2 \succ y$ would imply that

$x = \frac{x+y}{2} + \frac{x-y}{2} \succ y + \frac{x-y}{2} = \frac{x+y}{2} \succ y$ by ADD, a contradiction.

(ii) $z = (1-\alpha)x + \alpha y \sim x$ for $\alpha \in [0,1]$. Define $\{(x^n,y^n)\}$ inductively as follows: let $x^0 = x$, $y^0 = y$. Let $m^0 = (x^0+y^0)/2$.

Assume $z$ belongs to $[x^n,y^n]$ and its length is $1/2^n$ the length of $[x,y]$. The point $z$ belongs to at least one of the intervals $[x^n,m^n]$ and $[m^n,y^n]$. Define $[x^{n+1},y^{n+1}]$ to be one of those intervals which contains $z$. Now, all $x^n \sim x$ for all $n$. The sequence $x^n \to z$, therefore by continuity $z \sim x$ .

(iii) Let $z$ be on the line which connects $x$ and $y$, $z \sim x$. Without loss of generality, assume that $z$ is closer to $x$. There is $n$ such that $w = z + n(y - x)$ is between $x$ and $y$. By ADD if $a - x = b - y$ (that is a-b=x-y) then $a \sim b$. Thus by transitivity $z \sim w \sim x$.




By SMON there is an $\varepsilon > 0$ such that $a = (x_1 + \varepsilon, x_2) \succ x \succ (x_1, x_2 - \varepsilon) = b$. By question 3, there exists $y$ (different than $x$) on the interval which connects $a$ and $b$ such that $x \sim y$. Thus, every point is on a difference line which is a line. The indifference lines must be parallel since otherwise we will get a contradiction to *ADD*.

**d**. **Characterize the preference relations which satisfy ADD**, **SMON and an additional property MUL**:

$$(a_1, a_2) \succsim (b_1, b_2) \text{ implies that } (\lambda a_1, \lambda a_2) \succsim (\lambda b_1, \lambda b_2) \text{ for any } \lambda \geq 0.$$

Define $s = sup\{x | (0,1) \succ (x,0)\}$ (by SMON the set is not empty).

Case (1): $s = \infty$ or $s = 0$: the preferences must be lexicographic with priority for the second or first components, respectively.

Assume $s = \infty$.

If $a_2 > b_2$ then $(a_1, a_2) \succ (b_1, b_2)$ iff $(a_1, a_2 - b_2) \succ (b_1, 0)$ (by ADD) iff $(a_1/(a_2 - b_2), 1) \succ (b_1/(a_2 - b_2), 0)$ (by MUL), which is always true (by $s = \infty$).

If $a_2 = b_2$ then $(a_1, a_2) \succ (b_1, b_2)$ iff $a_1 > b_1$ (by SMON).

Thus, we have a lexicographic relation with priority for the second component.

If $s = 0$ then it follows that $s = sup\{y | (1,0) \succ (0,y)\} = \infty$ and the preferences must be lexicographic with priority for the first component.

Case (2): $\infty > s > 0$

Let $(a_1, a_2)$ and $(b_1, b_2)$ be two vectors with $a_1 \leq b_1$. $(a_1, a_2)$ relates to $(b_1, b_2)$ as $(0, a_2 - b_2)$ relates to $(b_1 - a_1, 0)$ (by ADD) and thus as $((b_1 - a_1)/(a_2 - b_2), 0)$ relates to $(0, 1)$. This relation is determined by the comparison of $(b_1 - a_1)/(a_2 - b_2)$ to $s$, which is equivalent to the comparison of $a_1 + sa_2$ and $b_1 + sb_2$.

Therefore, if $(0,1) \sim (s,0)$ then $x_1 + sx_2$ represents the preferences. If $(0,1) \succ (s,0)$ or $(0,1) \prec (s,0)$ then the preferences are lexicographic with the first priority to $x_1 + sx_2$ and the second to $x_2$ or $x_1$ accordingly.

**Problem 7**.

**Utility is a numerical representation of preferences. One can think about the numerical representation of other abstract concepts. Here, you will try to come up with a possible numerical representation of the concept "approximately the same" (see Luce (1956) and Rubinstein (1988)). For simplicity, let $X = [0,1]$. Consider the following six properties of $S$:....**

**c. Let $S$ be a binary relation that satisfies the above six properties and let $\epsilon > 0$. Show that there is a strictly increasing and continuous function $H : X \to \Re$ such that $aSb \Leftrightarrow |H(a) - H(b)| \leq \epsilon$.**

Note the definitions of $m(x)$ and $M(x)$ in the question.

Define $\{x_n\}$ by $x_0 = 0$, $x_1 = M(0)$, $x_2 = M(x_1) = M(M(0))$ and so on. By S6, $\{x_n\}$ is increasing and bounded above by 1, and thus $\{x_n\}$ converges to $x^* \leq 1$. By S5, there exists an $N$ such that $x_{N-1}Sx^*$, and thus $x^* \leq M(x_{N-1}) = x_N$. Since $x^*$ is the upper bound of $\{x_n\}$, then $x^* = 1$ by S6. Define $N$ to be the smallest integer such that $x_N = 1$, and thus $0 = x_0 <\ldots< x_N = 1$.

**Lemma 1**: If $a \in [x_n, x_{n+1}]$, where $1 \leq n \leq N-1$, then $m(a) \in [x_{n-1}, x_n]$.

Proof: Since $x_nSx_{n+1}$, then $x_nSa$ by S4, and thus $m(a) \leq x_n$. Moreover, $x_{n-1} \leq m(a)$, as otherwise $m(a) < x_{n-1}$ and $M(x_{n-1}) = x_n \leq M(m(a))$, violating the assumption that $M$ increasing.

**Lemma 2**: $m(a)$ is strictly increasing and continuous on $(x_1, 1]$.

Proof: $m(a) > 0$ if $a > x_1$, as otherwise $aS0$, and thus $M(0) \geq a > x_1$, a contradiction. By S6, the lemma is proved.

Define

$$H(a) = \begin{cases} \frac{\epsilon}{x_1}a & \text{if } a \leq x_1 \\ H(m(a)) + \epsilon & \text{if } a > x_1. \end{cases}$$

$H$ is clearly continuous and strictly increasing on $[0, x_1]$, with $H(x_1) = \epsilon$.

If $a \in (x_1, x_2]$, then $H(a) = \epsilon[m(a)/x_1 + 1]$ since $m(a) \in [0, x_1]$ by Lemma 1. Thus $H$ is strictly increasing and continuous on $(x_1, x_2]$ by Lemma 2. Since $m(x_1) = 0$, then $H(x) \to \epsilon$ as $x \to x_1$ from the right, and thus $H$ in continuous and strictly increasing on $[0, x_2]$, with $H(x_2) = 2\epsilon$.

More generally, if $a \in (x_n, x_{n+1}]$, where $n \leq N-1$, then $m(a) \in [x_{n-1}, x_n]$, $m(m(a)) \in [x_{n-2}, x_{n-1}]$ and so on by Lemma 1. Therefore $H(a) = \epsilon[m(\ldots m(a)\ldots)/x_1 + n]$, which is strictly increasing and continuous by Lemma 2, where $m(\ldots m(a)\ldots)$ applies $m$ inductively $n$ times. Since $H(x) \to n\epsilon$ as $x \to x_n$ from the right, then $H$ is strictly increasing and continuous on $[0, x_{n+1}]$.

Let $a, b \in [0,1]$ where $a < b$. If $b \leq x_1$, then $H$ represents $S$ by S4. Otherwise, $aSb$ iff $H(m(b)) \leq H(a) < H(b)$ iff $|H(b) - H(a)| \leq \epsilon$, where the first iff follows from $aSb$ iff $m(b) \leq a < b$ and $H$ strictly increasing, and the second iff follows from $H(b) = H(m(b)) + \epsilon$.