Problem Set 3 – Choice

Problem 1.

The following are descriptions of decision making procedures. **Discuss whether the procedures can be described in the framework of the choice model discussed in this lecture and whether they are compatible with the** "**rational man**" **paradigm**.

a. **The DM chooses an alternative in order to maximize another person's suffering**.

Assuming that the relation "the other person suffers more from x than he does from y" is complete and transitive, the DM is maximizing a well-defined preference relation.

b. **The DM asks his two children to rank the alternatives and then chooses the alternative that is the best** "**on average**".

The question is, of course, what does the expression "on average" mean. If the DM ranks all alternatives in X and uses the ranking to attach the number to each alternative a in any set A (independently of A), then the DM's behavior is consistent with rationality. But if the score of an alternative is recalculated for every choice set then his behavior may be inconsistent with the rational man paradigm. For example, assume that one child ranks the alternatives a,d,e,b,c and the other as b,c,a,d,e. Then, the element a is chosen from the set $\{a,b,c,d,e\}$ while b is chosen from $\{a,b,c\}$.

c. **The DM has an ideal point in mind and chooses the alternative that is closest to it**.

Let x be the ideal point and $d(a,b)$ the distance function between $a,b \in X$. The behavior is rationalized by the preferences represented by $u(a) = -d(a,x)$.

d. **The DM looks for the alternative that appears most often in the choice set**.

A choice function C is not well-defined. The DM's behavior is different when faced with the group of elements (a,a,b) than when faced with the group (a,b,b), even though in both cases he chooses from the set $\{a,b\}$.

e. **The DM has an ordering in mind and always chooses the median element**.

C violates condition α. Assume that the order of the grand set $X = \{a,b,c,d,e\}$ is alphabetical. Then, $C(\{a,b,c,d,e\}) = \{c\}$ but $C(\{a,b,c\}) = \{b\}$.

Problem 2.

Let's say that you are to make a choice from a set A. **Consider two procedures**:

(**a**) **You choose from the entire set or** (**b**) **You first partition** A **into the subsets** A_1 **and** A_2 ($A_1, A_2 \neq \emptyset$, $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$), **then make a selection from each of the subsets and finally make a choice from the two selected elements**. **We say that a choice procedure** C **statifies the** ***path independence property*** **if** $C(A) = C(\{C(A_1), C(A_2)\})$.

a. **Show that the rational decision maker satisfies this property**.

Let A_1, A_2 be a partition of A. Let $x = C_{\succsim}(A)$ and $x \in A_i$. Then $x = C_{\succsim}(A_i)$ and x is also the best choice from $\{C_{\succsim}(A_1), C_{\succsim}(A_2)\}$.

b. **Find examples of choice procedures that do not satisfy this property**.

(i) The "second best" procedure. If $x \succ y \succ z$, then $C(\{x,y,z\}) = y$ while $C(\{C(\{x,y\}), C(\{z\})\}) = \{z\}$.

(ii) Let X be partitioned into Y and Z and let $\succ$ be an ordering on X. Let $C(A)$ be the $\succ$-minimal element if all alternatives in A are in Y and the $\succ$-maximal alternative otherwise. If $Y = \{a,b\}$, $Z = \{c,d\}$ and $a \succ b \succ c \succ d$, then $\{C(\{a,b,c,d\}) = a$ but $C(\{C(\{a,b\}), C(\{c,d\})\}) = b$.

c. **Show that if a choice function satisfies path independence**, **then it is consistent with rationality**.

We will show that Condition α is satisfied. Let $A \subset B \subseteq X$ be such that $C(B) \in A$. By path independence, $C(B) = C(\{C(A), C(B \setminus A)\}$. Since $C(B)$ is in A, then it is not in $B \setminus A$. Therefore $C(B)$ is identical to $C(A)$.

d. **Find an example of a multi-valued choice function satisfying path independence which cannot be rationalized**.

In the context of choice correspondences, path independence implies:

$$C(A) = C(C(A_1) \cup C(A_2)).$$

Let $a \succ b \succ c \succ d$ and $C(A)$ be a set which contains the best and worst elements in A. C satisfies path independence (verify) but violates WA since $a \in C(\{a,b,c,d\})$ and $c \in C(\{a,b,c\})$ but $c \notin C(\{a,b,c,d\})$.

Problem 3.

Let X be a finite set. **Check whether the following three choice correspondences satisfy WA**:

1) $C(A) = \{x \in A \mid$ **the number of** $y \in X$ **for which** $V(x) \geq V(y)$ **is at least** $|X|/2\}$ **and if this set is empty**, **then** $C(A) = A$.

By C, a satisfactory element is one which is in the upper half of the elements in the grand set.

C satisfies WA. Define:

$$G = \{x \in X \mid V(x) \geq V(y) \text{ for at least } |X|/2 \text{ alternatives } y \in X\}.$$

Let $x,y \in A \cap B$ such that $x \in C(A)$ and $y \in C(B)$.

If $x \in G$, then $x \in C(B)$.

If $x \notin G$, then there are no elements of A in G and thus $y \notin G$. Since $y \in C(B)$, then $C(B) = B$ and thus $x \in C(B)$ as well.

Alternatively: define $u(x) = 1$ if $x \in G$ and 0 otherwise. Clearly, $C_u = C$.

2) $D(A) = \{x \in A \mid$ **the number of** $y \in A$ **for which** $V(x) \geq V(y)$ **is at least** $|A|/2\}$.

By D, a satisfactory element is one in the upper half of the elements of the choice set.

It is not necessarily consistent with the rational man paradigm:

Let $V(a) > V(b) > V(c) > V(d) > V(e)$. Then, $c \in C(\{a,b,c,d,e\})$ and $a \in C(\{a,b,c\})$, but $c \notin C(\{a,b,c\})$.

3) $E(A) = \{x \in A \mid x \succsim_1 y$ **for every** $y \in A$ **or** $x \succsim_2 y$ **for every** $y \in A\}$, **where** $\succsim_1$ **and** $\succsim_2$ **are two orderings over** X.

By E, a satisfactory element is one which is optimal according to one of the two criteria. It is not necessarily consistent with the rational man paradigm. Let $x \succ_1 y \succ_1 z$ and $y \succ_2 z \succ_2 x$. Then, $z \in C(\{x,z\})$ and $x \in C(\{x,y,z\})$, but $z \notin C(\{x,y,z\})$.

Problem 4.

Consider the following choice procedure: A decision maker has a strict ordering $\succsim$ over the set X and assigns to each $x \in X$ a natural number $class(x)$ to be interpreted as the "class" of x. Given a choice problem A, he chooses the best element in A from those belonging to the most common class in A (i.e., the class that appears in A most often). If there is more than one most common class, he picks the best element from the members of A that belong to a most common class with the highest class number.

a. Is this procedure consistent with the "rational man" paradigm?

No. Let $a \succ b \succ c \succ d \succ e$, $class(a) = class(b) = class(c) = 1$ and $class(d) = class(e) = 2$. $C(\{a,b,c,d,e\}) = a$ but $C(\{a,d,e\}) = d$, thus violating α.

b. Define the relation $\mathbf{xPy}$ if $\mathbf{x}$ is chosen from $\{\mathbf{x},\mathbf{y}\}$. Show that the relation P is a strict ordering (complete, asymmetric and transitive).

By definition, P is complete and asymmetric. We will see that it is also transitive. That is, if xPy and yPz, then xPz.

If xPy and yPz, then [$class(x) > class(y)$ or $class(x) = class(y)$ and $x \succsim y$], and [$class(y) > class(z)$ or $class(y) = class(z)$ and $y \succsim z$]. If either $class(x) > class(y)$ or $class(y) > class(z)$, then $class(x) > class(z)$ and $C(\{x,z\}) = \{x\}$. Otherwise, $class(x) = class(z)$ and $x \succsim z$ and thus $x \in C(\{x,z\})$.

Alternatively: Note that P is identical to the lexicographic preferences with first priority given to class and second priority to the relation $\succsim$.

Problem 5.

Consider the following two choice procedures. Explain each procedure and try to persuade a skeptic that they "make sense". Determine For each of them whether they are consistent with the "rational man" model.

a. The primitives of the procedure are two numerical (one-to-one) functions u and v defined on X and a number v^*. For any given choice problem A, let $a^* \in A$ be the maximizer of u over A and let $b^* \in A$ be the maximizer of v over A. The decision maker chooses a^* if $v(a^*) \geq v^*$ and b^* if $v(a^*) < v^*$.

One interpretation of this procedure is that the DM actually wants to maximize v but pretends to maximize u. If the maximization of u yields a result which is too bad for him, he abandons the pretense and maximizes v. The procedure may fail condition α. For example,

Element	$u(\cdot)$	$v(\cdot)$
x	3	1
y	2	2
z	1	3

and let $v^* = 2$.

Then $C(\{x,y,z\}) = z$ but $C(\{y,z\}) = y$.

b. The primitives of the procedure are two numerical (one-to-one) functions u and v defined on X and a number u^*. For any given choice problem A, the decision maker chooses the element $a^* \in A$ that maximizes u if $u(a^*) \geq u^*$ and the element $b^* \in A$ that maximizes v if $u(a^*) < u^*$.

In this case, the DM, cares about the value of u only if it is at least u^*. Otherwise, he cares about v. The DM behaves as if he is maximizing lexicographic preferences with first priority given to the function $u'(x)$, which receives the value $u(x)$ if $u(x) \geq u^*$ and $u^* - 1$ otherwise, and second priority to $v(x)$.

Problem 6.

The standard economic model assumes that choice is made from a set. Let us construct a model where the choice is assumed to be made from a list (note that the list <a,b> is distinct from <a,a,b> and <b,a>). Let X be a finite grand set. A list is a nonempty, finite vector of elements in X. In this problem, consider a choice function C to be a function that assigns a single element from $\{a_1,\dots,a_k\}$ to each vector $L = \langle a_1,\dots,a_k\rangle$. Let $\langle L_1,\dots,L_m\rangle$ be the concatenation of the m lists $L_1,\dots,L_m$ (note that if the length of L_i is k_i, then the length of the concatenation is $\sum_{i=1,\dots,m} k_i$). We say that L' extends the list L if there is a list M such that $L' = \langle L,M\rangle$. We say that a choice function C satisfies property I if for all $L_1,\dots,L_m$, $C(\langle L_1,\dots,L_m\rangle) = C(\langle C(L_1),\dots,C(L_m)\rangle)$.

a. Interpret Property I. Give two examples of choice functions that satisfy I and two examples that do not.

Property I is analogous to path independence.

Two choice functions that satisfy I:

(i) Choose the first alternative in L.

(ii) Choose the first alternative in L that is "at least as good as" some $\tilde{x} \in X$ and choose the last element in L if there is no such alternative.

Two choice functions that violate I:

(i) Choose the second alternative in L.

(ii) Choose the last alternative such that the alteratives from the start of the sequence up to that alternative are in ascending order.

b. Define formally the following two properties of a choice function:

Order Invariance OI: A change in the order of the elements in the list does not alter the choice.

Let $L = \langle a_1,\dots,a_K\rangle$. A permutation of L is a list $L^\pi = \langle a_{\pi(1)},\dots,a_{\pi(K)}\rangle$, where π is a permutation of $\{1,\dots,K\}$. C satisfies OI if $C(L) = C(L^\pi)$ for every permutation π.

Duplication Invariance DI: Deleting an element that appears elsewhere in the list does not change the choice.

C satisfies DI if $C(L) = C(L')$ whenever (i) $L = \langle\langle L_1\rangle, x, \langle L_2\rangle\rangle$, $L' = \langle\langle L_1\rangle,\langle L_2\rangle\rangle$ and x appears in either L_1 or L_2, or (ii) $L = \langle x,\langle L'\rangle\rangle$ and x appears in L', or (iii) $L = \langle\langle L'\rangle, x\rangle$ and x appears in L'.

c. **Characterize the choice functions that satisfy the following three properties together**: **Order Invariance**, **Duplication Invariance and Property** I.

Claim: Let C be a choice function over the *lists* of X. If C satisfies OI, DI and I, then there exists a rationalizable choice function $\overline{C}$ over the *sets* of X such that $C(L) = \overline{C}(\{L\})$, where $\{L\}$ is the *set of elements* in L.

Proof: Let K,L be two lists such that $\{K\} = \{L\}$. By OI and DI, choice is preserved when the duplicate alternatives in both lists are removed and the resulting lists are reshuffled so that the remaining alternatives appear in the same order. Thus, $C(K) = C(L)$ and thus $\overline{C}$ is well-defined and single-valued.

By Problem 2(d), showing that $\overline{C}$ satisfies path independence is sufficient for rationalizability. For any *set* $S \subseteq X$, define $\langle S \rangle$ to be some *list* of elements in S. Let $A,B \subseteq X$ be disjoint. Then:

$$\begin{aligned}
\overline{C}(A \cup B) &= C(\langle\langle A\rangle,\langle B\rangle\rangle) && \text{by def. of } \overline{C} \\
&= C(\langle C(\langle A\rangle), C(\langle B\rangle)\rangle) && \text{by property } I \\
&= C(\langle \overline{C}(A), \overline{C}(B)\rangle) && \text{by def. of } \overline{C} \\
&= \overline{C}(\{\overline{C}(A), \overline{C}(B)\}) && \text{by def. of } \overline{C}
\end{aligned}$$

Assume now that at the back of the decision maker's mind there is a value function u **defined on the set** X (**such that** $u(x) \neq u(y)$ **for all** $x \neq y$). **For any choice function** C, **define** $v_C(L) = u(C(L))$. **We say that** C **accommodates a longer list if whenever** L' **extends** L, $v_C(L') \geq v_C(L)$ **and there is a pair of lists L' and L**, **such that L' extends L and** $v_C(L') > v_C(L)$.

d. **Give two interesting examples of choice functions that accommodate a longer list**.

(i) Choose the u-maximal element in L.

(ii) Choose the second u-best alternative in L.

e. **Give two interesting examples of choice functions which satisfy property** I **but do not accommodate a longer list**.

(i) Choose the first alternative in L that yields at least utility $\tilde{u}$ and choose the last alternative in L if $u(x) < \tilde{u}$ for all $x \in L$.

(ii) Choose the first element in L.

Problem 7.

Let X be a finite set. We say that a choice function c is lexicographically rational if there exists a profile of preference relations $\{\succ_a\}_{a\in X}$ (not necessarily distinct) and an ordering O over X such that for every set $A \subset X$, $c(A)$ is the $\succ_a$ -maximal element in A, where a is the O -maximal element in A.

A decision maker who follows this procedure is attracted by the most notable element in the set (as described by O). If a is that element, he applies the ordering $\succ_a$ and chooses the $\succ_a$ -best element in the set.

We say that C satisfies the reference point property if for every set A, there exists $a \in A$ such that if $a \in A'' \subset A' \subset A$ and $C(A') \in A''$, then $C(A'') = C(A')$.

a. Show that a choice function C is lexicographically rational if and only if it satisfies the reference point property.

($\Rightarrow$) Assume C is lex. rational. For every set A we'll show that the $O-maximal$ element $a \in A$ satisfies the requirement of the reference point property. Note that for any $A' \subset A$ containing a, a is still the $O-maximal$ element. Thus, for all subsets of A containing a, $C(A')$ is determined by $\succ_a$. Thus condition α is satisfied for all subsets of A containing a and the reference point property holds.

($\Leftarrow$) Assume C satisfies the reference point property. We build the representation recursively. Consider the set X. By the reference point property there exists an element a_1 such that for all subsets of X that contain a_1, condition α holds. Define $Y = \{x \in X|C(\{x,a_1\} = x\}$. For any $x,y \in Y$, $x \neq y$, define $x \succ_{a_1} y$ if $x = C(\{a_1,x,y\})$ (including the case $y = a_1$), and for any $x \notin Y$ define $a_1 \succ_{a_1} x$. Extend $\succ_{a_1}$ such that the elements in $X \backslash Y$ are ordered arbitrarily. This preference relation, $\succ_{a_1}$, is well defined by condition α, and it rationalizes the choices of C whenever a_1 is available. Now consider the set $X \setminus \{a_1\}$, and repeat the procedure. We'll find an a_2 and $\succ_{a_2}$ such that $\succ_{a_2}$ represents all choices of C whenever a_2 is available in the subsets of $X \setminus \{a_1\}$. For completeness, assume that for any $i \neq 1$, $a_i \succ_{a_2} a_1$. Also construct the $O-preference$ such that $a_1 \succ_O a_2$. By induction we can complete the $O-preference$, and $\succ_a$ for every a in X.

b. Try to come up with a procedure satisfying the reference point axiom which is not stated explicitly in the language of the lexicographical rational choice function.

Choosing the second best alternative:

For every set A, the best alternative a satisfies the requirement of the reference point property: let $a \in A'' \subset A' \subset A$ and let $C(A')$ be the second best alternative in A'. If

$C(A') \in A''$ then $C(A')$ is also the second best alternative in A'' and thus $C(A'') = C(A')$.

In order to decsribe this procedure as lexicographically rational, assume the DM chooses the second best alternative according to some ordering $\succ$ and define:

1. the ordering O to be the original ordering $\succ$
2. $\succ_a$ to be an ordering indentical to $\succ$ except that a is moved to be the worst alternative.

Given a set A the DM is attracted by the O–maximal alternative a, which is the best alternative, but applies an ordering $\succ_a$ in which a is last. Therefore, the best alternative in A according to $\succ_a$ is in fact the second best alternative according to $\succ$.

Problem 8.

Consider a decision maker who has in mind a set of rationales and a preference relation and chooses the best alternative that he can rationalize.

Formally, **we say that a choice function** C **is rationalized if there is an asymmetric complete relation** $\succ$ (**not necessarily transitive**!) **and a set of partial orderings** $\{\succ_k\}_{k=1...K}$ (**called rationales**) **such that** $C(A)$ **is the** $\succ$ **-maximal alternative from among those alternatives found to be maximal in** A **by at least one rationale** (**given a binary relation** $\succ$ **we say that** x **is** $\succ$ **-maximal in** A **if** $x \succ y$ **for all** $y \in A$). **Assume that the relations are such that the procedure always leads to a solution**.

We say that a choice function C **satisfies The Weak Weak Axiom of Revealed Preference** (**WWARP**) **if for all** $\{x,y\} \subset B_1 \subset B_2$ ($x \neq y$) **and** $C\{x,y\} = C(B_2) = x$, **then** $C(B_1) \neq y$.

a. **Show that a choice function satisfies WWARP if and only if it is rationalized**. **For the proof**, **construct rationales**, **one for each choice problem**, **that are asymmetric binary relations and allow that** $\succ$ **will not necessarily be transitive**.)

($\Leftarrow$) Let us see first that the axiom is satisfied by any rationalized choice function: If x is chosen from B_2 then has a rationale in B_2 (I.e. there is a rationale $\succ_k$ such that x is the $\succ_k$ -maximal in B_2). Thus, it has a rationale also in B_1. If y were chosen from B_1, then it has a rationale in B_1 as well. Since y is chosen from B_1 it must be that $y \succ x$. For $\{x,y\}$ both x and y have rationales and thus y would have been chosen from $\{x,y\}$, a contradiction.

($\Rightarrow$) Let C be a choice function satisfying WWARP. For every set B, define $x \succ_B y$ iff $x = C(B)$ and $y \in B$. Obviously, this rationale is a very partial ordering. As to the top preferences $\succ$, they are elicited by the choice from the two-element sets: $x \succ y$ if $C(\{x,y\}) = x$.

To see that those definitions "work", assume $C(B_1) = y$ but there is rationale $\succ_{B_2}$ and a x which is $\succ_{B_2}$ -maximal in B_1 such that $x \succ y$. It must be that $B_1 \subset B_2$ and $C(B_2) = x$. By definition of $\succ$ also $C(\{x,y\}) = x$. A contradiction to $WWARP$.

b. **What do you think about the axiomatization**?

There might be other ways in which people's choices satisfy WWARP. Axiomatizing such a choice with partial orderings for each subset might be representing the choice procedure in a much more complex way than actual.

Consider the "**warm-glow**" **procedure**: **The decision maker has two complete**

orderings in mind: one moral $\succsim_M$ and one selfish $\succsim_S$. He chooses the most moral alternative m as long as he doesn't "lose" too much by not choosing the most selfish alternative. Formally, for every alternative s there is some alternative $l(s)$ such that if the most selfish alternative is s then he is willing to choose m as long as $m \succsim_S l(s)$. If $l(s) \succ_S m$, he chooses s.

The function l satisfies $s \succsim_S l(s)$ and $s \succsim_S s'$ iff $l(s) \succsim_S l(s')$.

c. Show that WWARP is satisfied by this procedure.

Assume in contradiction that WWARP is violated, i.e. there exists $\{x,y\} \subset B_1 \subset B_2$ ($x \neq y$) and $C(\{x,y\}) = C(B_2) = x$, and $C(B_1) = y$.

Assume x is the moral maximal in B_2. Clearly x is also the moral maximal in B_1 and this implies that y is the selfish maximal in B_1. Since x is chosen in B_2, it must be that $x \succeq_s l(s_{B_2})$, where s_{B_2} is the selfish maximal in B_2. But $B_1 \subset B_2$ implies $s_{B_2} \succeq_s s_{B_1} = y$. By monotonicity of $l(s)$, $x \succeq_s l(y)$ which contradicts that $C(B_1) = y$.

Assume that x is the selfish maximal in B_2. Clearly x is also the selfish maximal in B_1, hence y is the moral maximal in B_1. This implies that in $\{x,y\}$, x is the selfish maximal and y is the moral maximal. Since x is chosen in $\{x,y\}$, it must be that $y \not\succeq_s l(x)$. But since x is also selfish maximal in B_1 this contradicts that $C(B_1) = y$.

d. Show directly that the warm-glow procedure is rationalized (in the sense of the definition in this problem).

There are two rationales, the selfish and the moral orderings. The final relation $\succ$ is the moral ordering, that is $x \succ y$ if $x \succ_m y$. However, if $l(y) \succ_s x$, then it is reversed, that is $y \succ x$.

To see that this works, given any set we choose the moral and selfish maximal, m and s. Then we apply the final ordering. Note that if $m \succeq_s l(s)$ then the ordering says $m \succ s$. Otherwise, $s \succ m$ as desired.