Problem Set 4 – Consumer Preferences

Problem 1.

Consider the preference relations on the interval $[0,1]$ **which are continuous**. **What can you say about those preferences which are also strictly convex**?

We will show that a continuous preference relation $\succsim$ on $X = [0,1]$ is strictly convex iff there exists a point x^* such that $b \succ a$ for all $a < b \leq x^*$ or all $x^* \geq b > a$.

(a) Let $\succsim$ be continuous and strictly convex. Since the preferences are continuous and X is compact there exists a unique $x^* \in X$ that maximizes the preferences (see Lecture 5). Let $0 \leq a < b \leq x^*$. By definition $a \precsim x^*$ and $b = \alpha a + (1-\alpha)x^*$ for some $\alpha \in [0,1)$ and thus, by strict convexity $a \prec b$. The case, for two points in $[x^*,1]$ is analogous .

(b) Assuming that the preferences are increasing in $[0,x^*]$ and decreasing in $[x^*,1]$, we will show that they satisfy strict convexity. Let $\alpha \in (0,1)$ and $a,b \in X$ be such that $a \neq b$ and $a \succsim b$. It must be that $\alpha a + (1-\alpha)b$ is either between a and x^* or between b and x^*. If $\alpha a + (1-\alpha)b$ is between b and x^*, then $\alpha a + (1-\alpha)b \succ b$. If it is between a and x^*, then $\alpha a + (1-\alpha)b \succ a \succsim b$.

Problem 2.

Show that if the preferences $\succsim$ satisfy continuity and monotonicity, then the function $u(x)$ defined by $x \sim (u(x), \ldots, u(x))$ is continuous.

Let x be a point in X. By definition $u(x) \geq 0$. We need to show that for any $\epsilon > 0$ there exists δ such that $|u(x) - u(y)| < \epsilon$ for any $y \in Ball(x, \delta)$.

If $u(x) - \epsilon \geq 0$, then by monotonicity, $x \succ (u(x) - \epsilon, \ldots, u(x) - \epsilon)$. By continuity, there exists $\delta_1 > 0$ such that $u(y) > u(x) - \epsilon$ for $y \in Ball(x, \delta_1)$.

If $u(x) - \epsilon < 0$, then for any $\delta_1 > 0$, $u(y) > u(x) - \epsilon$ for $y \in Ball(x, \delta_1)$.

Similarly, by monotonicity, $(u(x) + \epsilon, \ldots, u(x) + \epsilon) \succ x$ and thus $(u(x) + \epsilon, \ldots, u(x) + \epsilon) \succ Ball(x, \delta_2)$ for some $\delta_2 > 0$ by continuity. Therefore, $u(x) + \epsilon > u(y)$ for $y \in Ball(x, \delta_2)$.

Define $\delta = \min\{\delta_1, \delta_2\}$. Then, $|u(x) - u(y)| < \epsilon$ for any $y \in Ball(x, \delta)$.

Problem 3. **In a world with two commodities**, **consider the following condition**: **The preference relation $\succsim$ satisfies Convexity 4 if for all x and $\epsilon > 0$**

$$(x_1,x_2) \sim (x_1-\epsilon,x_2+\delta_1) \sim (x_1-2\epsilon,x_2+\delta_1+\delta_2) \text{ } \textit{implies} \text{ } \delta_2 \geq \delta_1.$$

Interpret Convexity 4 and show that for strong monotonic and continuous preferences, **it is equivalent to the convexity of the preference relation**.

Interpretation: If after an x_1 is reduced by ϵ, the consumer must be compensated with δ units of good 2 in order to remain indifferent to x, then he must be compensated with *at least* 2δ units of good 2 if his consumption of x_1 is decreased by 2ϵ.

Convexity 1 $\Rightarrow$ Convexity 4: Let $(x_1,x_2) \sim (x_1-\epsilon,x_2+\delta_1) \sim (x_1-2\epsilon,x_2+\delta_1+\delta_2)$. By convexity 1,

$$\begin{aligned}(x_1-\epsilon,x_2+\frac{\delta_1+\delta_2}{2}) &= \frac{1}{2}(x_1,x_2)+\frac{1}{2}(x_1-2\epsilon,x_2+\delta_1+\delta_2)\\ &\succsim (x_1-\epsilon,x_2+\delta_1).\end{aligned}$$

Then, $(\delta_1+\delta_2)/2 \geq \delta_1$ by monotonicity and thus $\delta_2 \geq \delta_1$.

Convexity 4 $\Rightarrow$ Convexity 1: First, we show that if $x \sim y$, then $(x+y)/2 \succsim y$. If $x \neq y$ then by strong monotonicity we can WLOG assume $x_1 > y_1$ and $y_2 > x_2$. Define $\Delta > 0$ by $\Delta = (y_2-x_2)/2$ and $\epsilon = (x_1-y_1)/2$. By strong monotonicity

$$\begin{aligned}(x_1-\epsilon,x_2+2\Delta) = (\frac{x_1+y_1}{2},y_2) &\succ y \sim x\\ &\succ (\frac{x_1+y_1}{2},x_2) = (x_1-\epsilon,x_2).\end{aligned}$$

By continuity, there exists $\delta > 0$ such that

$$(x_1,x_2) \sim (x_1-\epsilon,x_2+\delta) \sim y = (x_1-2\epsilon,x_2+2\Delta).$$

By Convexity 4, $2\Delta-\delta \geq \delta$ and thus $\Delta \geq \delta$. By monotonicity,

$$\frac{x+y}{2} = (x_1-\epsilon,x_2+\Delta) \succsim (x_1-\epsilon,x_2+\delta) \sim y.$$

Now if $x \succsim y$, then there exists z on the interval which connects 0 and x, such that $z_k \leq x_k$ for all k and $z \sim y$. Then, by monotonicity and the previous result, $(x+y)/2 \succsim (z+y)/2 \succsim y$. The rest follows from the following Lemma:

Lemma: If $\succsim$ are continuous preferences, then $\succsim$ are convex iff [$x \succsim y$ implies $(x+y)/2 \succsim y$] for all $x,y \in X$.

Proof: Assume $x \succsim y$ and $z = \alpha x+(1-\alpha)y$ for $\alpha \in [0,1]$. We will show that $z \succsim y$. Construct a sequence $\{(x^n,y^n)\}$ such that both $x^n,y^n \succsim y$ and z between x^n and y^n. Define $x^0 = x$, $y^0 = y$. Continue inductively. Let $m^n = (x^n+y^n)/2$. Then, m^n is at least as good as either x^n or y^n and the above argument and transitivity imply that it is at least as good as

y. Define:

$x^{n+1} = m^n$ and $y^{n+1} = y^n$ if z lies between y^n and m^n, and
$x^{n+1} = x^n$ and $y^{n+1} = m^n$ otherwise.

Thus, $x^{n+1}, y^{n+1} \succsim y$ and z is between x^{n+1} and y^{n+1}. Since $y^n, x^n \to z$, $z \succsim y$ by continuity.

Problem 4.

Complete the proof (for all K) of the claim that any continuous preference relation satisfying strong monotonicity quasi-linearity in all commodities can be represented by a utility function of the form $\sum_{k=1}^{K} \alpha_k x_k$, where $\alpha_k > 0$ for all k.

Proof by induction on K: We have already proved this for $K = 1$ and 2.

Let $\succsim$ be a preference relation satisfying the problem's assumptions. Consider the preferences restricted to the set of all vectors of the type $(0, x_2, \ldots, x_K)$. The preferences satisfy Continuity, Strong Monotonicity and Quasi-Linearity in goods $2, .., K$. By the induction hypothesis, there is a vector of positive numbers $(\alpha_k)_{k=2,..,K}$ such that $(0, x_2, \ldots, x_K) \sim (0, \sum_{k=2}^{K} \alpha_k x_k, 0, \ldots, 0)$.

By quasi-linearity in good 1, $(x_1, x_2, \ldots, x_K) \succsim (y_1, y_2, \ldots, y_K)$ iff
$(x_1, \sum_{k=2}^{K} \alpha_k x_k, 0, \ldots, 0) \succsim (y_2, \sum_{k=2}^{K} \alpha_k y_k, 0, \ldots, 0)$.

The relation over all vectors of the type $(x_1, x_2, 0, \ldots, 0)$ satisfies the three properties in the first two dimensions. Thus, there exists $\beta_1, \beta_2 > 0$ such that $(x_1, x_2, 0, \ldots, 0) \sim (\beta_1 x_1 + \beta_2 x_{2,} 0, , 0, \ldots, 0)$ and thus $x \sim (\beta_1 x_1 + \sum_{k=2}^{K} \beta_2 \alpha_k x_k, 0 \ldots, 0)$ and by strong monotonicity in the first good, the preferences have a linear utility representation.

Problem 5.

Show that for any consumer's preference relation $\succsim$ satisfying continuity, monotonicity, strong monotonicity with respect to commodity 1 and quasi-linearity with respect to commodity 1, there exists a number $v(x)$ such that $x \sim (v(x), 0, \ldots, 0)$ for every vector x.

Since $\succsim$ satisfies continuity and monotonicity every bundle is indifferent to a bundle on the main diagonal. Thus, it is sufficient to show the claim for bundles on the main diagonal.

Let $e = (1, \ldots, 1)$ and define

$$T = \{\alpha \in \Re_+ \mid \alpha e \succ (x_1, 0, \ldots, 0) \text{ for all } x_1 \in \Re_+\}.$$

We will see that $T = \emptyset$. Assume that $T \neq \emptyset$. Let $\gamma = \inf T$. There are two cases:

Case 1: $\gamma \in T$. Then $\gamma > 0$ and by strict monotonicity of commodity 1, $(1+\gamma, \gamma, \ldots, \gamma) \succ \gamma e$. By continuity, there exists $\epsilon > 0$ such that

$$(1+\gamma, \gamma - \epsilon, \ldots, \gamma - \epsilon) \succ \gamma e \succ (x_1, 0, \ldots, 0)$$

for all x_1.

Since $\gamma - \epsilon < \inf T$, there exists an x_1^* such that $(x_1^*, 0, \ldots, 0) \succsim (\gamma - \epsilon, \gamma - \epsilon, \ldots, \gamma - \epsilon)$ and by quasi-linearity in commodity 1,

$$(x_1^* + 1 + \epsilon, 0, \ldots, 0) \succsim (1+\gamma, \gamma - \epsilon, \ldots, \gamma - \epsilon), \text{ a contradiction.}$$

Case 2: $\gamma \notin T$. Then $(\beta, 0, \ldots, 0) \succsim \gamma e$ for some β. By strong monotonicity of commodity 1, $(\beta + 1, 0, \ldots, 0) \succ \gamma e$. By continuity, there is an $\epsilon > 0$ such that $(\beta + 1, 0, \ldots, 0) \succ (\gamma + \epsilon)e$, which contradicts $\gamma = \inf T$.

Thus, $T = \emptyset$ and for any bundle on the main diagonal, αe, there exists a bundle $(x_1, 0, \ldots, 0)$ such that $(x_1, 0, \ldots, 0) \succsim \alpha e \succsim (0, \ldots, 0)$. By continuity there exists a number $v(\alpha e)$ such that $(v(\alpha e), 0, \ldots, 0) \sim \alpha e$.

Problem 6.

We say that a preference relation satisfies separability if it can be represented by an additive utility function, that is, a function of the type $u(x) = \sum_k v_k(x_k)$.

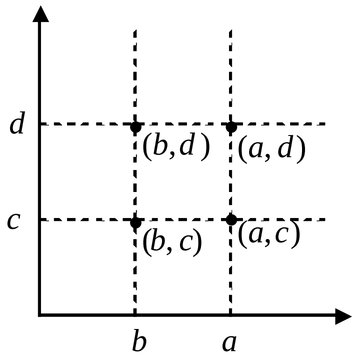
a. **Show that such preferences satisfy condition S**: **for any subset of commodities** J, **and for any bundles** a, b, c, d, **we have**

$$(a_J, c_{-J}) \succsim (b_J, c_{-J}) \Leftrightarrow (a_J, d_{-J}) \succsim (b_J, d_{-J})$$

where (x_J, y_{-J}) **is the vector that takes the components of** x **for any** $k \in J$ **and takes the components of** y **for any** $k \notin J$.

$$\begin{aligned}(a_J, c_{-J}) \succsim (b_J, c_{-J}) &\Leftrightarrow \sum_{k \in J} v_k(a_k) + \sum_{i \notin J} v_i(c_i) \geq \sum_{k \in J} v_k(b_k) + \sum_{i \notin J} v_i(c_i) \\ &\Leftrightarrow \sum_{k \in J} v_k(a_k) + \sum_{i \notin J} v_i(d_i) \geq \sum_{k \in J} v_k(b_k) + \sum_{i \notin J} v_i(d_i) \\ &\Leftrightarrow (a_J, d_{-J}) \succsim (b_J, d_{-J}).\end{aligned}$$

Graphically, if two bundles lie on the same horizontal line and $(a, c) \succsim (b, c)$, then a change of c to d will preserve the preference relation, that is $(a, d) \succsim (b, d)$.



b. **Show that for** $K = 2$ **such preferences satisfy the** $Hexagon-condition$: **If** $(a, b) \succsim (c, d)$ **and** $(c, e) \succsim (f, b)$ **then** $(a, e) \succsim (f, d)$.

$v_1(a) + v_2(b) \geq v_1(c) + v_2(d)$ and

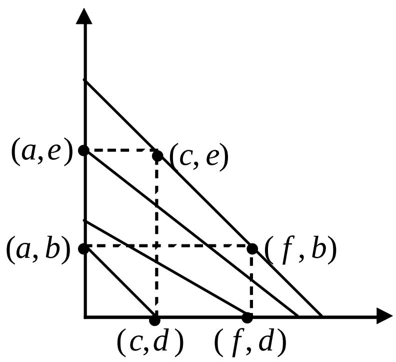
$v_1(c) + v_2(e) \geq v_1(f) + v_2(b)$ implies

$v_1(a) + v_2(e) \geq v_1(f) + v_2(d)$.

c. **Give an example of a continuous preference relation which satisfies condition**

S and does not satisfy separability.

Consider any preference relation with linear indifference curves as depicted:



Such preferences violate the Hexagon Condition.

Problem 7.

a. **Show that the preferences represented by the utility function** $\min\{x_1, \ldots, x_K\}$ **are not differentiable**.

Let $x^* = (a, \ldots, a)$ and let $v(x^*)$ be a candidate set of subjective values. Without loss of generality, let $v_1(x^*) > 0$. Then, $(1, 0, 0, \ldots, 0) \cdot v(x^*) > 0$ but $(a + \epsilon, a, a, \ldots, a) \sim x^*$ for all ϵ, and thus $(+1, 0, 0, \ldots, 0) \notin D(x^*)$, a contradiction.

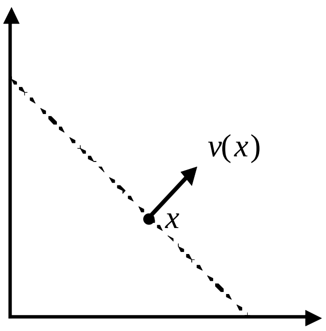
b. **Check the differentiability of the lexicographic preferences in** $\Re^2$.

Lexicographic preferences are not differentiable. Let $x \in \Re^2$ and assume that $v(x)$ is a vector of subjective values. Since $x + \epsilon(0, 1) \succ x$ for all $\epsilon > 0$, then $(0, 1)$ is an improving direction and $v_2(x) > 0$. Then, for small $\delta > 0$, $(-\delta, 1) \cdot v(x) > 0$. However, $(-\delta, 1)$ is not an improving direction, a contradiction.

c. **Assume that** $\succsim$ **is monotonic**, **convex and differentiable such that for every** x, **we have** (*) $D(x) = \{d \mid (x + d) \succ x\}$. **What can you say about** $\succsim$?

We will show that the indifference curves are linear.

By differentiability and (*) there exists $v(x)$ such that $d \cdot v(x) > 0$ iff $x + d \succ x$. Graphically, any point above the dotted line is strictly better than x:



We will show that for any $z \in X$ on the dotted line (that is $zv(x) = xv(x)$), we have $x \sim z$.

First let us see that (**) any $z \in X$ on the dotted line satisfies $x \succsim z$. If $z \succ x$, then by (*) $(z - x) \in D(x)$ and by differentiability $(z - x) \cdot v(x) > 0$ but as $zv(x) = xv(x)$, a contradiction. Note that it must be that $v(z) = v(x)$, since otherwise there would be a point on $\{y \mid yv(z) = zv(z)\}$such that $yv(x) > xv(x)$ but by (*) $y \succ x$ though $x \succsim z \succsim y$. Thus $(x - z) \cdot v(z) = 0$ and by (**) (applied to z) $z \succsim x$.

d. **Assume that $\succsim$ is a monotonic**, **convex and differentiable preference relation**. **Let** $E(x) = \{d \in \Re^K \mid$ **there exists** $\epsilon^* > 0$ **such that** $x + \epsilon d \prec x$ **for all** $\epsilon \leq \epsilon^*\}$. **Show that** $\{-d \mid d \in D(x)\} \subseteq E(x)$ **but not necessarily** $\{-d \mid d \in D(x)\} = E(x)$.

We first show that $\{-d \mid d \in D(x)\} \subseteq E(x)$. By contradiction, let $d \in D(x)$ be such that $-d \notin E(x)$. WLOG $x + d \succ x$ and $x - d \succsim x$. By definition of $D(x)$, $d \cdot v(x) > 0$ and $e \cdot v(x) > 0$ for some e with $e_k \leq d_k$ with at least one strict inequality. For $\epsilon > 0$ small enough $x + \epsilon e \succ x$. By convexity any convex combination of $x + \epsilon e$ and $x - d$ is at least as good as x but the segment contains points which by monotonicity are at least as bad as x.

Let $\succsim$ be represented by $u(x) = x_1 x_2$. Since u is quasi-concave, has continuous partial derivatives and satisfies $u_i(x) > 0$. Thus, the relation $\succsim$ is convex, monotonic and differentiable. Let $d = (1,-1)$ and note that $-d \in E(2,2)$ but $d \notin D(2,2)$.

e. **Consider the consumer**'**s preferences in a world with two commodities defined by**:

$$\mathbf{u}(\mathbf{x}_1, \mathbf{x}_2) = \begin{cases} x_1 + x_2 & \textit{if} \;\; x_1 + x_2 \leq 1 \\ 1 + 2x_1 + x_2 & \textit{if} \;\; x_1 + x_2 > 1 \end{cases}.$$

Show that these preferences are not continuous but nevertheless are differentiable according to our definition.

If $x_1 + x_2 \leq 1$, then differentiability holds for $v(x) = (1,1)$ and if $x_1 > 1$, then differentiability holds for $v(x) = (2,1)$. The preferences are not continuous, since $(0,2) \succ (1,0)$, but $(0,2) \prec (1+\epsilon,0)$ for $\epsilon > 0$.