Problem Set 5– Demand: Consumer Choice

Problem 1.

Verify that when preferences are continuous, the demand function $x(p,w)$ is continuous in prices and in wealth (and not only in p).

Let $\{(p^n, w^n)\}$ converge to (p,w). Since $x(p,w)$ is homogeneous of degree zero, then

$$x(p^n, w^n) = x(\frac{p^n}{w^n}, 1).$$

Since demand is continuous in p, then

$$x(\frac{p^n}{w^n}, 1) \to x(\frac{p}{w}, 1) = x(p,w),$$

where the equality follows from $x(p,w)$ being homogeneous of degree zero.

Problem 2.

Show that if a consumer has a homothetic preference relation, then his demand function is homogeneous of degree one in w.

Let $\lambda > 0$ and $y \in B(p, \lambda w)$. Then $y/\lambda \in B(p, w)$. Since $x(p, w) \succsim y/\lambda$ and preferences are homothetic, then $\lambda x(p, w) \succsim y$, and thus, $\lambda x(p, w)$ is the best element in $B(p, \lambda w)$, that is $x(p, \lambda w) = \lambda x(p, w)$.

Problem 3.

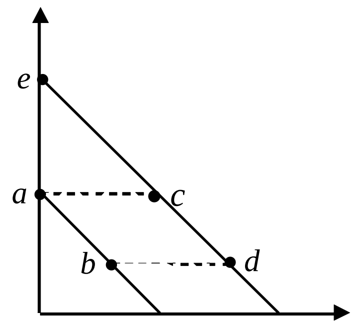
Consider a consumer in a world with $K = 2$, **who has a preference relation that is monotonic**, **continuous**, **strictly convex**, **and quasi-linear in the first commodity**. **How does the demand for the first commodity change with** w?

Claim: For any p either there is no w such that $x_1(p,w) > 0$ or there exists an $w^* \geq 0$ such that if $w \leq w^*$, then the consumer does not consume the first commodity, and if $w \geq w^*$, then the first commodity absorbs all changes in wealth, that is $x_1(p,w) = \frac{w-w^*}{p_1}$.

Proof: Normalize $p_1 = 1$. First, we show if $x_1(p,w) = 0$, then $x_1(p,w') = 0$ for $w' \leq w$.

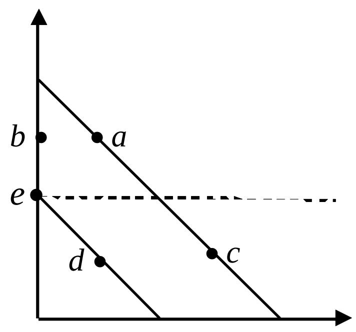
Let $w' \leq w$. Denote $a = (0, w'/p_2)$, $b \in B(p,w')$ such that $pb = w'$, $c = (a_1 + (w - w'), a_2)$, $d = (b_1 + (w - w'), b_2)$ and $e = (0, w/p_2)$.

Note that $pd = w$ and c is between e and d. Since $e \succsim d$ then $c \succsim d$. By the quasi-linearity $a \succsim b$. Therefore $x(p,w') = a = (0, w'/p_2)$.



Now, if there is w such that $x_1(p,w) > 0$ then by continuity of the demand function $w^* = \max\{w \mid x_1(p,w) = 0\}$ exists. We need to show that $x(p,w) = (w - w^*, w^*/p_2)$ for all $w \geq w^*$. If $x_2(p,w) > w^*/p_2$, for some $w > w^*$, let $a = x(p,w)$ and by the quasi linearity in the first commodity $b = (0, a_2) \succsim y$ for all $y \in B(p, p_2 x_2(p,w))$ contradicting the definition of w^* (see graph). If $x_2(p,w) < w^*/p_2$, for some $w > w^*$, then $c = x(p,w)$ and

$d = (x_1(p,w) - (w - w^*), x_2(p,w)) \succsim e = (0, w^*/p_2)$, a contradiction to the definition of w^*.



Problem 4.

Let $\succsim$ be a continuous preference relation (not necessarily strictly convex) and w a number. Consider the set $G = \{(p,x) \in \Re^K_{++} \times \Re^K_{+} \mid x$ is optimal in $B(p,w)\}$. (For some price vectors, there could be more than one $(p,x) \in G$.) Calculate G for the case of $K = 2$ and preferences represented by $x_1 + x_2$.

Show that for any preference relation, G is a closed set.

If preferences are represented by $x_1 + x_2$, then the consumer will buy the cheapest good. Therefore, G is the set

$\{(p,(0,w/p_2))|p_2 < p_1\} \cup \{(p,(w/p_1,0))|p_1 < p_2\} \cup \{(p,(\alpha, w/p_2 - \alpha))|p_2 = p_1,\ \alpha \in [0,w/p_1]\}$.

Let (p^n,x^n) be a sequence of points in G converging to (p,x). Then $p^n x^n \leq w$ for every n, and thus $px = \lim_{n\to\infty} p^n x^n \leq w$, i.e. $x \in B(p,w)$.

If x is not optimal in $B(p,w)$, then there exists a $y \in B(p,w)$ such that $y \succ x$. By continuity, there exists an $\epsilon > 0$ such that $Ball(y,\varepsilon) \succ Ball(x,\varepsilon)$, and thus there exists a bundle $z \in Ball(y,\varepsilon)$ such that $pz < w$ and $z \succ x$. Since $p^n \to p$, then $p^n z \leq w$ for n large. Since $x^n \to x$, then $x^n \in Ball(x,\varepsilon)$ for n large, and thus $z \succ x^n$, a contradiction to x^n be optimal in $B(p^n,w)$.

Problem 5.

Determine whether the following consumer behavior patterns are fully rationalized (**assume K**=**2**):

a. **The consumer's demand function is** $x(p,w) = (2w/(2p_1+p_2), w/(2p_1+p_2))$.

Yes. $x(p,w)$ is rationalized by the monotonic preferences represented by $u(x) = \min\{x_1, 2x_2\}$.

b. **The consumer consumes up to quantity 1 of commodity 1 and spends his excess wealth on commodity 2**.

Yes. $x(p,w)$ is rationalized by the monotonic preferences represented by

$$u(x) = \begin{cases} x_1 & \text{if } x_1 < 1 \\ 1 + x_2 & \text{if } x_1 \geq 1 \end{cases}$$

c. **The consumer chooses the bundle** (x_1,x_2) **which satisfies** $\frac{x_1}{x_2} = \frac{p_1}{p_2}$ **and costs** w. (**Does the utility function** $u(x) = x_1^2 + x_2^2$ **rationalize the consumer's behavior**?)

No. The behavior violates the WA. $x((2,1),5) = (2,1)$ and $x((1,2),5) = (1,2)$. Both bundles are affordable in both budget sets.

The function $u(x) = x_1^2 + x_2^2$ does not rationalize $x(p,w)$ since a consumer maximizing u would allocate all wealth to the cheapest good. The "first order condition" approach is not appropriate because preferences represented by u are not convex.

Problem 6.

In this question, we consider a consumer who behaves differently from the classic consumer we talked about in the lecture. Once again we consider a world with K commodities. The consumer's choice will be from budget sets. The consumer has in mind a preference relation that satisfies continuity, monotonicity, and strict convexity; for simplicity, assume it is represented by a utility function u.

The consumer maximizes utility up to utility level u^0. If the budget set allows him to obtain this level of utility, he chooses the bundle in the budget set with the highest quantity of commodity 1 subject to the constraint that his utility is at least u^0.

a. Formulate the consumer's problem.

$$\begin{array}{lll} \max_{x \in B(p,w)} u(x) & & \text{if } \max_{x \in B(p,w)} u(x) < u^0 \text{, and} \\ \max_{x \in B(p,w)} x_1 & \text{s.t. } u(x) \geq u^0 & \text{if } \max_{x \in B(p,w)} u(x) \geq u^0. \end{array}$$

b. Show that the consumer's procedure yields a unique bundle.

If $\max_{x \in B(p,w)} u(x) < u^0$, then the consumer acts as in the standard framework. $x(p,w)$ exists because preferences are continuous and is unique because preferences are strictly convex.

If $\max_{x \in B(p,w)} u(x) \geq u^0$, define $\tilde{B} = \{x \in B(p,w) \mid u(x) \geq u^0\}$, which is compact, and convex (by strict convexity). Then $\max_{x \in \tilde{B}} x_1$ exists. If both y and z are solutions then by the strict convexity $u((y+z)/2) > u_0$ and thus there is a vector x such that $u(x) > u^0$ and $x_1 > y_1$ contradicting the optimality of y in $\tilde{B}$.

c. Is this demand procedure rationalizable?

Yes. The procedure is rationalized by

$$v(x) = \begin{cases} u(x) & \text{if } u(x) < u^0 \\ u^0 + x_1 & \text{if } u(x) \geq u^0. \end{cases}$$

d. Does the demand function satisfy Walras Law?

Yes. Preferences are monotonic.

e. **Show that in the domain of (p, w) for which there is a feasible bundle yielding utility of at least u^0 the consumer's demand function for commodity 1 is decreasing in p_1 and increasing in w.**

In both cases the budget set is enlarging and the consumer could obtain more x_1 and preserve u^0.

f. **Is the demand function continuous**?

Yes.

Let $z(p, w)$ be the solution of $\max_{x \in B(p,w)} u(x)$. Note that in this case $z(p, w)$ is not necessarily the consumer's demand $x(p, w)$.

First, we show that demand is continuous in prices. Let $\{p^n\}$ converge to p. If $u(z(p, w)) \ < u^0$ then for n large enough $max_{x \in B(p^n,w)} u(x) < u^0$ and the demand is $z(p^n, w)$ converges to $z(p, w)$ which is the demand in (p, w).

Assume $u(z(p, w)) \ \geq u^0$. Let $m = \inf_{i,n} p_i^n > 0$, the infimum of the commodity prices. Then $x(p^n, w) \in [0, w/m]^K$ for all n, and thus WLOG we can assume that $x(p^n, w)$ converges to a bundle y. By contradiction, assume that $y \neq x(p, w)$.

If $u(y) < u^0$, then by continuity, there exists an $\epsilon > 0$ such that $Ball(x(p, w), \varepsilon) \succ Ball(y, \varepsilon)$. Then, there exists a $a \in Ball(x(p, w), \varepsilon)$ such that $pa < w$ and $a \succ y$. For n large, $p^n a \leq w$ and $a \succ x(p^n, w)$, a contradiction.

If $u(y) \geq u^0$, then $x_1(p, w) > y_1$. Let $a = \frac{1}{2} x(p, w) + \frac{1}{2} y$. Then $a_1 > y_1$, $a \in B(p, w)$ and $u(a) > u^0$ by strict convexity. By continuity, there exists an $\epsilon > 0$ small such that $a_1 - \epsilon > y_1 + \epsilon$, $p \cdot (a - \epsilon e_1) < w$ and $u(a - \epsilon e_1) \geq u^0$. Thus for n large, $B(p_n, w)$ contains $a - \epsilon e_1$ which yields utility larger than u^0 and quantity larger than $x_1(p^n, w)$, a contradiction.

Now, let $\{(p^n, w^n)\}$ converge to (p, w). Since $x(p, w)$ is homogeneous of degree zero, then

$$x(p^n, w^n) = x(\frac{p^n}{w^n}, 1).$$

Since demand is continuous in p, then

$$x(\frac{p^n}{w^n}, 1) \to x(\frac{p}{w}, 1) = x(p, w),$$

where the equality follows from $x(p, w)$ being homogeneous of degree zero.

Problem 7.

It's a common practice in economics to view aggregate demand as being derived from the behavior of a "representative consumer". Give two examples of "well-behaved" consumer preference relations that can induce average behavior that is not consistent with maximization by a "representative consumer". (That is, construct two "consumers", 1 and 2, who choose the bundles x^1 and x^2 out of the budget set A and the bundles y^1 and y^2 out of the budget set B so that the choice of the bundle $\frac{x^1+x^2}{2}$ from A and of the bundle $\frac{y^1+y^2}{2}$ from B is inconsistent with the model of the rational consumer).

Let $(p^A, w^A) = ((1,2),8)$, $(p^B, w^B) = ((2,1),8)$ and

$$u_1(x) = \begin{cases} x_1 & \text{if } x_1 < 4 \\ 4 + x_2 & \text{if } x_1 \geq 4 \end{cases} \qquad u_2(x) = \begin{cases} x_2 & \text{if } x_2 < 4 \\ 4 + x_1 & \text{if } x_2 \geq 4 \end{cases}$$

The demands of the two agents in A will be $(4,2)$ and $(0,4)$ and thus $\overline{x^A}(p,w) = (2,3)$. Similarly, $\overline{x^B}(p,w) = (3,2)$. Both average bundles are interior in A and in B. Thus, we the average demand violates the weak axiom.

Problem 8.

A commodity k is Giffen if the demand for the k'th good is increasing in p_k. A commodity k is inferior if the demand for the commodity decreases with wealth. Show that if there is a vector (p, w) such that the demand for the k'th commodity is rising after its price has increased, then there is a vector (p', w') such that the demand of the k'th commodity is falling after the income has increased (Giffen implies inferior).

Let e_k be the vector with the k'th component being 1 and all other components being 0. We have $x_k(p + \epsilon e_k, w) > x_k(p, w)$. Let $w' \geq w$ be the "compensating" wealth level, that is $[p + \epsilon e_k] \cdot x(p, w) = w'$. Thus, $x(p + \epsilon e_k, w') \succsim x(p, w)$. By definition,

$px(p + \epsilon e_k, w') + \epsilon x_k(p + \epsilon e_k, w') = [p + \epsilon e_k] \cdot x(p + \epsilon e_k, w') \leq w' = w + \epsilon x_k(p, w)$.

If $x_k(p + \epsilon e_k, w') > x_k(p, w)$ then $px(p + \epsilon e_k, w') < w$ contradicting the optimality of $x(p, w)$ in $B(p, w)$.