Problem Set 6 – More Economic Agents: a Consumer Choosing Budget Sets, a Dual Consumer and a Producer

Problem 1.

In a world with two commodities, consider a consumer's preferences that are represented by the utility function $u(x_1, x_2) = \min\{x_1, x_2\}$.

a. Calculate the consumer's demand function.

A solution x^* must satisfy $x_1^* = x_2^*$ and $px^* = w$. Thus, $x(p,w) = (w/(p_1+p_2), w/(p_1+p_2))$.

b. Verify that the preferences satisfy convexity.

Let $u(y), u(z) \geq u(x)$, that is $\min\{y_1, y_2\} \geq \min\{x_1, x_2\}$ and $\min\{z_1, z_2\} \geq \min\{x_1, x_2\}$. Then $u(\alpha y + (1-\alpha)z) = \min\{\alpha y_1 + (1-\alpha)z_1, \alpha y_2 + (1-\alpha)z_2\} \geq$

$\alpha \min\{y_1, y_2\} + (1-\alpha)\min\{z_1, z_2\} \geq \min\{x_1, x_2\} = u(x)$.

c. Calculate the indirect utility function $v(p,w)$.

$v(p,w) = u(x(p,w)) \ = w/(p_1+p_2)$.

d. Verify Roy's Identity.

$$x_i(p,w) = -\frac{\partial v(p,w)/\partial p_i}{\partial v(p,w)/\partial w} = -\frac{-w/(p_1+p_2)^2}{1/(p_1+p_2)} = \frac{w}{p_1+p_2}.$$

e. Calculate the expenditure function $e(p,z)$ **and verify the Dual Roy's Identity.**

Obviously, $h_i(p,z) = min\{z_1, z_2\}$. Thus, $e(p,z) = min\{z_1, z_2\}(p_1+p_2)$.

Then $h_i(p,z) = \partial e(p,z)/\partial p_i = min\{z_1, z_2\}$.

Problem 2.

Imagine that you are reading a paper in which the author uses the indirect utility function $v(p_1, p_2, w) = w/p_1 + w/p_2$. **You suspect that the author's conclusions in the paper are the outcome of the"fact" that the function** v **is inconsistent with the model of the rational consumer**. **Take the following steps to make sure that this is not the case**:

a. **Use Roy's Identity to derive the demand function**.

$$x_i(p,w) = -\frac{\partial v(p,w)/\partial p_i}{\partial v(p,w)/\partial w} = -\frac{-w/p_i^2}{(p_1+p_2)/p_1p_2} = \frac{wp_j}{p_i(p_1+p_2)}.$$

b. **Show that if demand is derived from a smooth utility function**, **then the indifference curve at the point** (x_1, x_2) **has the slope** $-\sqrt{x_2}/\sqrt{x_1}$.

By part (a), $x_i(p,w) > 0$ for $i = 1,2$. Note that $x_2(p,w) = (\frac{p_1}{p_2})^2 x_1(p,w)$.

If u is quasi-concave, then

$$\frac{\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2} = \frac{p_1}{p_2} = \sqrt{\frac{x_2}{x_1}}$$

c. **Construct a utility function with the property that the ratio of the partial derivatives at the bundle** (x_1, x_2) **is** $\sqrt{x_2}/\sqrt{x_1}$.

$u(x) = (\sqrt{x_1} + \sqrt{x_2})$.

d. **Calculate the indirect utility function derived from this utility function**. **Do you arrive at the original** $v(p_1, p_2, w)$? **If not**, **can the original indirect utility function still be derived from another utility function satisfying the property in** (**c**)?

The indirect utility function derived from u is

$u(x(p,w)) = (\sqrt{wp_1/p_2(p_1+p_2)} + \sqrt{wp_2/p_1(p_1+p_2)}) = \sqrt{w(p_1+p_2)}/\sqrt{p_1p_2}$.

The function $u^2(x)$ represents the same preference relation and $u^2(x(p,w)) = v(p,w)$.

Problem 3.

A consumer with wealth w **is interested in purchasing only one unit of one of the items included in a** (**finite**) **set** A. **All items are indivisible**. **The consumer does not derive any** "**utility**" **from leftover wealth**. **The consumer evaluates commodity** $x \in A$ **by the number** V_x (**where the value of not purchasing any of the goods is 0**). **The price of commodity** $x \in A$ **is** $p_x > 0$.

a. **Formulate the consumer problem**.

Let "n" denote not purchasing anything, where $p_n = 0$ and $V_n = 0$. Define $B(p,w) = \{x \in A \cup \{n\} \mid p_x \leq w\}$. We get: $v(p,w) = \max_{x \in B(p,w)} V_x$.

b. **Check the properties of the indirect preferences** (**homogeneity of degree zero**, **monotonicity**, **continuity and quasi**-**convexity**).

The proofs of homogeneity, weak monotonicity in prices and wealth, and quasi convexity are valid. The indirect utility function is not continuous: **Take** $K = 1$, $A = \{x\}$, where $V_x = 1$, $(p_x) = (1)$ and $w = 1$. Then $v(1,1) = 1$ but $v(1,1-\epsilon) = 0$ for any $\epsilon > 0$.

c. **Calculate an indirect utility function for the case in which** $A = \{a,b)$ **and** $V_a > V_b > 0$.

$$V(p,w) = \begin{cases} V_a & p_a \leq w \\ V_b & p_b \leq w < p_a \\ 0 & w < p_a, p_b \end{cases}$$

Problem 4.

Show that if the preferences are monotonic, **continuous and strictly convex**, **then the Hicksian demand function** $h(p,z)$ **is continuous**.

Let $\{(p^n,z^n)\}$ converge to (p,z^0). Define $\bar{z}_k = \sup\{z_k^n\}$, $m = \inf\{p_k^n\}$ and $M = \sup\{p_k^n\}$. The consumer does not need more than $M\sum_k \bar{z}_k$ to obtain any z^n. Thus,
$h_l(p^n,z^n) \leq M\sum_k \bar{z}_k/m$ for all l. Thus, WLOG we can assume that $h(p^n,z^n)$ converges to some bundle h^*.

By contradiction, assume that $h^* \neq h(p,z^0)$. By the continuity of the preferences $h^* \succsim z^0$, and thus (assuming that $h(p,z)$ is uniquely defined) $ph^* > ph(p,z^0)$. There exists an $\epsilon > 0$ such that $ph^* > p[h(p,z^0) + (\epsilon,..,\epsilon)]$ and by monotonicity of the preferences $h(p,z^0) + (\epsilon,..,\epsilon) \succ z^0$. Then, for n large enough, $p^n h(p^n,z^n) > p^n[h(p,z^0) + \epsilon e]$ and $h(p,z^0) + \epsilon e \succ z^n$, a contradiction.

Problem 5.

One way to compare budget sets is by using the indirect preferences which involves comparing $\mathbf{x}(p,w)$ **and** $x(p',w)$. **Following are two other approaches to making such a comparison**.

Define:

$$CV(p,p',w) = w - e(p',z) = e(p,z) - e(p',z)$$

where $z = x(p,w)$. **This is the answer to the question**: **What is the change in wealth that would be equivalent**, **from the perspective of** (p,w), **to the change in price vector from** p **to** p'?

Define:

$$EV(p,p',w) = e(p,z') - w = e(p,z') - e(p',z')$$

where $z' = x(p',w)$. **This is the answer to the question**: **What is the change in wealth that would be equivalent**, **from the perspective of** (p',w), **to the change in price vector from** p **to** p'?

Now, **answer the following questions regarding a consumer in a two**-**commodity world with a utility function** u:

a. **For the case of the preferences represented by** $u(x_1,x_2) = x_1 + x_2$, **calculate the two consumer surplus measures**.

$$CV(p,p',w) = w - \frac{w\min\{p_1',p_2'\}}{\min\{p_1,p_2\}} = w\frac{\min\{p_1,p_2\} - \min\{p_1',p_2'\}}{\min\{p_1,p_2\}}$$

$$EV(p,p',w) = \frac{w\min\{p_1,p_2\}}{\min\{p_1',p_2'\}} - w = w\frac{\min\{p_1,p_2\} - \min\{p_1',p_2'\}}{\min\{p_1',p_2'\}}$$

Assume that the price of the second commodity is fixed and that the price vectors differ only in the price of the first commodity.

b. **Assume that the first good is a normal good** (**the demand is increasing with wealth**). **What is the relation of the two measures to the** "**area below the demand function**" (**which is a standard third definition of consumer surplus**)?

Let $b = p_1'' < p_1' = a$ and let A denote the area under the demand curve for commodity 1 between a and b.

Let $u' = v((a,p_2),w)$ and $u'' = v((b,p_2),w)$.

Let $h_1(p,u)$ denote the hicksian demand $h_1(p,z)$ when $u(z) = u$.

If commodity 1 is a normal good, then $h_1(p,u)$ is increasing in u, and thus

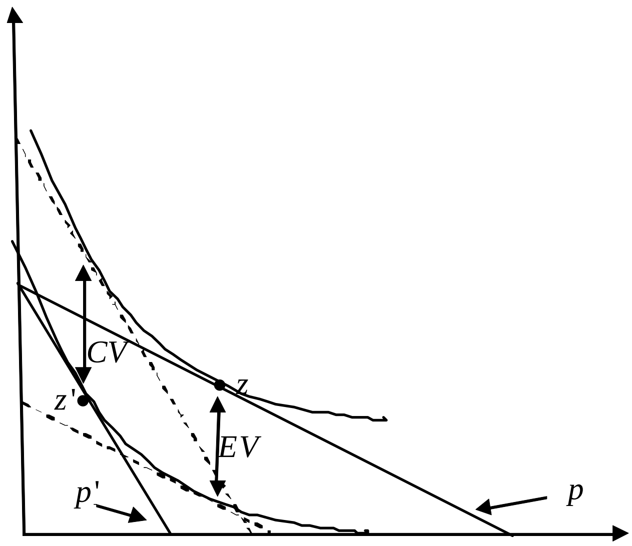
$$h_1((t,p_2),u') \le h_1((t,p_2),v((t,p_2),w)) \le h_1((t,p_2),u'') \quad \text{for } t \in [a,b].$$

Recall that by the Dual Roy's identity, $\frac{\partial e((t,p_2),u)}{\partial p_1} = h_1((t,p_2),u)$ By integrating,

$$
\begin{aligned}
CV(p', p'', w) &= \int_a^b \frac{\partial e((t,p_2),u')}{\partial p_1}dt = \int_a^b h_1((t,p_2),u')dt \\
&\le \int_a^b h_1((t,p_2),v((t,p_2),w))dt = \int_a^b x_1((t,p_2),w))dt = A \\
&\le \int_a^b h_1((t,p_2),u'')dt = \int_a^b \frac{\partial e((t,p_2),u'')}{\partial p_1}dt = EV(p', p'', w).
\end{aligned}
$$

c. **Explain why the two measures are identical if the individual has quasi-linear preferences in the second commodity and in a domain where the two commodities are consumed in positive quantities**.

Recall that by Question 4 of PS5, the consumer's demand adjusts to a change in wealth by adjusting consumption of the quasi-linear good (in this case commodity 2) and the consumption of commodity 1 is constant. Consider the following diagram where CV and EV are drawn. Since the indifference curves are parallel horizontally, the two measures are identical.



Problem 6.

a. **Verify that you know the envelope theorem**, **which states conditions under which the following is correct**:

Consider a maximization problem $\max_x\{u(x,\alpha_1,,..,\alpha_n) \mid g(x,\alpha_1,,..,\alpha_n) = 0\}$. **Let** $V(\alpha_1,,..,\alpha_n)$ **be the value of the maximization**. **Then** $\frac{\partial V}{\partial \alpha_i}(a_1,,..,a_n) = \frac{\partial(u-\lambda g)}{\partial \alpha_i}(x^*(a_1,,..,\alpha_n),a_1,,..,\alpha_n)$ **where** $x^*(a_1,,..,\alpha_n)$ **is the solution to the maximization problem and** λ **is the Lagrange multiplier associated with the solution of the maximization problem**.

b. **Derive the Roy's identity from the envelope theorem** (**hint**: **show that in this context** $\frac{\partial V/\partial \alpha_i}{\partial V/\partial \alpha_j}(a_1,,..,a_n) = \frac{\partial g/\partial \alpha_i}{\partial g/\partial \alpha_j}(x^*(a_1,,..,\alpha_n),a_1,,..,\alpha_n)$).

In the context of the consumer's problem $(a_1,,..,\alpha_n)$ are $(p_1,\ldots,p_k,w)$, $u(x,p_1,\ldots,p_k,w) = u(x)$ and $g(x,p_1,\ldots,p_k,w) = w - px$. Note that the prices and wealth do not effect the function u. The utility depends on the prices and wealth only through their effect on the bundle x. Denote $p = (p_1,,..,p_k)$.

The envelope theorem states that:

$$\partial V/\partial p_i(p,w) = \partial u/\partial p_i(x^*(p,w),p,w) - \lambda\partial g/\partial p_i(x^*(p,w),p,w)$$

$$\partial V/\partial w(p,w) = \partial u/\partial w(x^*(p,w),p,w) - \lambda\partial g/\partial w(x^*(p,w),p,w)$$

Becasue $\partial u/\partial p_i = \partial u/\partial w = 0$, we obtain

$$\partial V/\partial p_i(p,w) = -\lambda\partial g/\partial p_i(x^*(p,w),p,w) \text{ and}$$

$$\partial V/\partial w(p,w) = -\lambda\partial g/\partial w(x^*(p,w),p,w).$$

By taking ratios and canceling out λ:

$$\frac{\partial V/\partial p_i}{\partial V/\partial w}(p,w) = \frac{\partial g/\partial p_i}{\partial g/\partial w}(x^*(p,w),p,w).$$

Recall that $g(x,p,w) = w - px$, and thus $\partial g/\partial p_i(x^*(p,w),p,w) = -x_i^*$ and $\partial g/\partial w(x^*(p,w),p,w) = 1$. Plugging this we get $\frac{-\partial v/\partial p_i}{\partial v/\partial w}(p,w) = x_i^*$ which is exactly Roy's identity.

c. **What makes it is easy to prove Roy's identity without using the envelope theorem**?

The fact that the utility does not depend directly on the prices (only through the bundle x).

Problem 7.

Assume that technology Z **and the production function** f **describe the same producer who produces commodity** K **using inputs** $1,\ldots,K-1$. **Show that** Z **is a convex set if and only if** f **is a concave function**.

Z is convex

iff $(-v,y),(-v',y') \in Z$, $\lambda \in [0,1]$ implies that $(-\lambda v - (1-\lambda)v', \lambda y + (1-\lambda)y') \in Z$

iff $y \le f(v)$, $y' \le f(v')$, $\lambda \in [0,1]$ implies that $\lambda y + (1-\lambda)y' \le f(\lambda v + (1-\lambda)v')$

iff $\lambda f(v) + (1-\lambda) f(v') \leq f(\lambda v + (1-\lambda) v')$
iff f is concave.

Problem 8.

Consider a producer who uses L inputs to produce $K - L$ outputs. Denote by w the price vector of the L inputs. Let $a_k(w,y)$ be the demand for the k'th input when the price vector is w and the output vector he wishes to produce is y. Show the following:

a. $C(\lambda w,y) = \lambda C(w,y)$.

$C(\lambda w,y) = \min_{\{a\,|\,(-a,y)\in Z\}} \lambda wa = \lambda \min_{\{a\,|\,(-a,y)\in Z\}} wa = \lambda C(w,y)$.

b. **C is nondecreasing in any input price w_k.**

Assume $w_l' \geq w_l$ for all l.

$C(w',y) = w'a(w',y) \geq wa(w',y) \geq wa(w,y) = C(w,y)$.

c. **C is concave in w.**

Let w, w' be input prices, $w'' = \lambda w + (1-\lambda)w'$ for $\lambda \in [0,1]$. Then

$$\begin{aligned} C(w'',y) &= [\lambda w + (1-\lambda)w']a(w'',y) = \lambda wa(w'',y) + (1-\lambda)w'a(w'',y) \\ &\geq \lambda C(w,y) + (1-\lambda)C(w',y). \end{aligned}$$

d. **Shepherd's lemma**: **If C is differentiable, $\frac{\partial C}{\partial w_k}(w^*,y) = a_k(w^*,y)$ (the kth input commodity)**.

Fix y. C is now a function of w. For every w, $C(w,y) \leq wa(w^*,y)$. $C(w^*,y) = w^*a(w^*,y)$. Thus $\{(w,c) \mid c = wa(w^*,y)\}$ is tangent to the graph of the function $C(w,y)$ at $(w^*, C(w^*,y))$. Since C is differentiable $\nabla C(w^*,y) = a(w^*,y)$.

e. **If C is twice continuously differentiable**, **then for any two commodities j and k, $\partial a_j/\partial w_k(w,y) = \partial a_k/\partial w_j(w,y)$.**

By Shapherd's Lemma and Young's Theorem (mixed partial derivatives are equal):

$$\frac{\partial a_j(w,y)}{\partial w_k} = \frac{\partial^2 C(w,y)}{\partial w_k \partial w_j} = \frac{\partial^2 C(w,y)}{\partial w_j \partial w_k} = \frac{\partial a_k(w,y)}{\partial w_j}.$$

Problem 9.

Consider a firm producing one commodity using L inputs, **which maximizes production subject to the constraint of achieving a level of profit ρ** (**and does not produce at all if he cannot**). **Show that under reasonable assumptions**:

a. **The firm's problem has a unique solution for every price vector**.

Denote the production function by $y = f(a)$. For a given a vector of inputs $a = (a_1, \ldots a_L)$, and price vector $p = (p_{a_1}, \ldots, p_{a_L}, p_y)$, the profit function is $\pi(a,p) = p_y y - p_a a$.

Let $D(p) = \{a \in \Re_+^L \mid p_y f(a) - p_a a \geq \rho\}$. The firm solves $\max_{a \in D(p)} f(a)$. Let $a(p)$ be the firm's input demand and $y(p) = f(a(p))$ be the firm's optimal output (if $D(p) = \phi$ then $a(p) = y(p) = 0$).

If the production technology is strictly convex ($f(a)$ strictly concave, decreasing returns to scale), bounded from above, and all prices are strictly positive then we have a unique solution to our problem.

Assume for contradiction that there are two solutions, a and a'. It must be that $f(a) = f(a')$ and $\pi(a,p) = \pi(a',p) = \rho$, otherwise we would be able to increase production. Now look at a convex combination of these two points. For $\lambda \in (0,1)$, due to strict concavity of $f(a)$, $f(\lambda a + (1-\lambda)a') > f(a) = f(a')$, and therefore $\pi(\lambda a + (1-\lambda)a', p) > \rho$ (since $p_a a = p_a a' = p_a(\lambda a + (1-\lambda)a')$ and $p_y f(a) = p_y f(a') < p_y f((\lambda a + (1-\lambda)a'))$). But this means $\lambda a + (1-\lambda)a'$ is better than a and a'.

b. **The firm's supply function satisfies monotonicity and continuity in prices**.

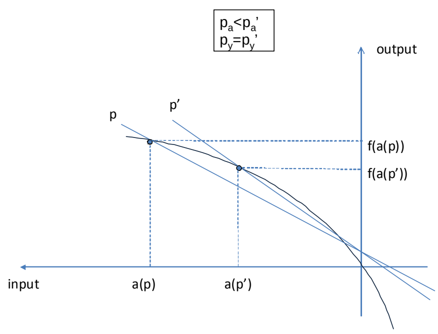
Monotonicity:

Consider increasing the output price: let p and p' be price vectors s.t. $p'_y > p_y$ and $p'_a = p_a$. For all a if $p_y f(a) - p_a a \geq \rho$ then $p'_y f(a) - p'_a a > \rho$ and for $\epsilon > 0$ small enough also $p'_y f(a + (\epsilon, \ldots, \epsilon)) - p'_a(a + (\epsilon, \ldots, \epsilon)) > \rho$.

Therefore, $f(a(p', \rho)) \geq f(a(p, \rho) + (\varepsilon, \ldots, \varepsilon)) > f(a(p, \rho))$ (assuming that f is increasing).

Now consider increasing the input price: let p and p' be price vectors s.t. $p'_y = p_y$ and $p'_a > p_a$. Similarly, For all a if $p'_y f(a) - p'_a a \geq \rho$ then $p_y f(a) - p_a a > \rho$ and for $\epsilon > 0$ small enough also $p_y f(a + (\epsilon, \ldots, \epsilon)) - p_a(a + (\epsilon, \ldots, \epsilon)) > \rho$.

Therefore, $f(a(p, \rho)) \geq f(a(p', \rho) + (\varepsilon, \ldots, \varepsilon)) > f(a(p', \rho))$ (assuming that f is increasing).



Continuity:

Assume that f is continuous and strictly concave. We will show that $a(p)$ and $y(p)$ are continuous:

Let $\{p^n\}$ converge to p. Define $\overline{p_y} = \sup\{p_y^n\}$ and $\underline{p_a} = \inf\{a_L^n\}$, and thus $D(p^n) \subseteq D(\underline{p_a},\ldots,\underline{p_a},\overline{p_y})$ for all n. Then

$$a(p^n) \in \{a \in \Re_+^L \mid \overline{p_y}f(a(\underline{p_a},\ldots,\underline{p_a},\overline{p_y})) - (\underline{p_a},\ldots,\underline{p_a}) \cdot a \geq \rho\}$$

since $\overline{p_y}f(a(\underline{p_a},\ldots,\underline{p_a},\overline{p_y})) \geq p_y^n f(a^n)$ and $(\underline{p_a},\ldots,\underline{p_a}) \leq p_a^n$ for all n. If $\{a(p^n)\}$ does not converge to $a(p)$ then there is a subsequence that converges to some $a^* \neq a(p)$. Since f is continuous, $p_y f(a^*) - p_a a^* \geq \rho$. Thus, by strict concavity, $f(a^*) < f(a(p))$.

Now (i) $f((a^* + a(p))/2) > f(a^*)$ and (ii) $p_y f(a^*) \geq p_a a^* + \rho$ and $p_y f(a(p)) \geq p_a a(p) + \rho$ and by the strict concavity of f,

$p_y f((a^* + a(p))/2) > (p_y f(a^*) + p_y f(a(p)))/2 \geq p_a((a^* + a(p)/2) + \rho$.

Therefore we found a' such that $f(a') > f(a^*)$ and $p_y f(a') - p_a a' > \rho$. Therefore, for n large enough, $p_y^n f(a') - p_a^n a' \geq \rho$ and $f(a') > f(a^n)$, a contradiction.

c. The firm's supply function is monotonic in ρ.

Intuitively, as ρ increases, keeping the prices the same, we shift the price line with same slope up. Due to convexity of the production technology this will lead to less production.

Formally, assume $\rho' > \rho$. For all a if $p_y f(a) - p_a a \geq \rho'$ then $p_y f(a) - p_a a > \rho$ and for $\epsilon > 0$ small enough also $p_y f(a + (\epsilon,\ldots,\epsilon)) - p_a(a + (\epsilon,\ldots,\epsilon)) > \rho$. Therefore, $f(a(p,\rho)) \geq f(a(p,\rho') + (\varepsilon,\ldots,\varepsilon)) > f(a(p,\rho'))$.

Problem 10.

It is usually assumed that the cost function C is convex in the output vector. Much of the research on production has been aimed at investigating conditions under which convexity is induced from more primitive assumptions about the production process. Convexity often fails when the product is related to the gathering of information or data processing.

Consider, for example, a firm conducting a telephone survey immediately following a TV program. Its goal is to collect information about as many viewers as possible within 4 units of time. The wage paid to each worker is w (even when he is idle). In one unit of time, a worker can talk to one respondent or be involved in the transfer of information to or from exactly one colleague. At the end of the 4 units of time, the collected information must be in the hands of one colleague (who will announce the results).

Define the firm's product, calculate the cost function and examine its convexity.

The firm's product is units of information.

Denote the agents by 1,...,n. Let $i \to j$ stands for i transfers information to j and let $\to j$ stands for j collects information from a viewer. Denote a procedure by a sequence of square brackets, each stands for one period and contains the transfers of information during that period. Agent 1 will be the agent who announces the result.

$n = 1$. Since there is only one agent, he can use the four units of time only for collecting information from four viewers. Thus, the maximum number of responses is 4. An optimal procedure:

$[\to 1][\to 1][\to 1][\to 1]$

2. The last unit of time has to be used to transfer information from some agent, 2, to agent 1. Therefore, both agents can collect information only for the first three units of time. Thus, the maximum number of responses is 6.

An optimal procedure:

$[\to 1, \to 2][\to 1, \to 2][\to 1, \to 2][2 \to 1]$

3. Again, the last unit of time has to be used to transfer information from one agent, let us say 2, to agent 1. However, one of these two agents has to get the information from agent 3 one period earlier. Thus, there are two agents who are free to collect information for two periods and one agent who is free for three periods. The total number of collected responses is thus, bounded by 7.

An optimal procedure:

$[\to 1, \to 2, \to 3][\to 1, \to 2, \to 3][\to 1, 3 \to 2][2 \to 1]$

4. If the firm employs 4 or more agents it can collect at most 8 responses.

Again, the last unit of time has to be used to transfer information from one agent, let us say 2, to 1. It is sufficient to show that each of them cannot hold more than 4 units of information after three periods. To see it note that after two periods each agent can hold not more than 2 units and thus, after three periods he will have 3 units if he makes a call himself, or 4 units if he gets the information collected earlier by another agent.

An optimal procedure:
$[\to 1, \to 2, \to 3, \to 4][\to 1, \to 2, \to 3, \to 4][3 \to 1, 4 \to 2][2 \to 1]$

Let y be the "output", the number of responses the center collected and $C(y,w)$ be the minimum cost of producing y. Then,

y	0	$1,2,3,4$	$5,6$	7	8
$C(y,w)$	0	w	$2w$	$3w$	$4w$

Obviously, C is not convex: $C(2,w) = w > w/2 = .5C(0,w) + .5C(4,w)$.

Problem 11.

An event that could have occurred with probability 0.5 either did or did not occur. A firm must provide a report in the form of "the event occurred" or "the event did not occur". The quality of the report (the firm's product), denoted by q, is the probability that the report is correct. Each of k experts (input) prepares an independent recommendation which is correct with probability $1 > p > 0.5$. The firm bases its report on the k recommendations in order to maximize q.

a. Calculate the production function $q = f(k)$ for (at least) $k = 1,2,3,\ldots$

Experts k	$f(k)$
0	0.5
1	p
2	$p^2 + p(1-p) = p$
3	$p^3 + 3p^2(1-p) > p$

b. We say that a "discrete" production function is concave if the sequence of marginal product is nonincreasing. Is the firm's production function concave?

No. Marginal product is positive from 0 to 1, zero from 1 to 2 and then positive from 2 to 3.

Assume that the firm will get a prize of M if its report is actually correct. Assume that the wage of each worker is w.

c. Explain why it is true that if f is concave, the firm chooses k^* so that the k^*th worker is the last one for whom marginal revenue exceeds the cost of a single worker.

The firm's profits if it employs k workers are: $Mf(k) - kw$. If f is concave, then for any worker $k < k^*$, the firm's marginal profit $M[f(k+1) - f(k)] - w$ is positive, whereas for any worker $k > k^*$, the firm's marginal profit is negative.

d. Is this conclusion true in our case?

No. Since the marginal revenue of the second expert is 0, while it is possible that it is optimal for the firm to hire 3 experts.

Problem 12.

An economic agent is both a producer and a consumer. **He has** a_0 **units of good** 1. **He can use some of** a_0 **to produce commodity** 2. **His production function** f **satisfies monotonicity**, **continuity**, **strict concavity**. **His preferences satisfy monotonicity**, **continuity and convexity**. **Given he uses** a **units of commodity** 1 **in production he is able to consume the bundle** $(a_0 - a, f(a))$ **for** $a \leq a_0$. **The agent has in his** "**mind**" **three** "**centers**":

*__The pricing center declares a price vector__ (p_1, p_2).

*__The production center takes the price vector as given and operates according to one of the following two rules__

Rule 1: **maximizing profits**, $p_2 f(a) - p_1 a$.

Rule 2: **maximizing production subject to the constraint of not making any losses**,, **i**.**e**. $p_2 f(a) - p_1 a \geq 0$.

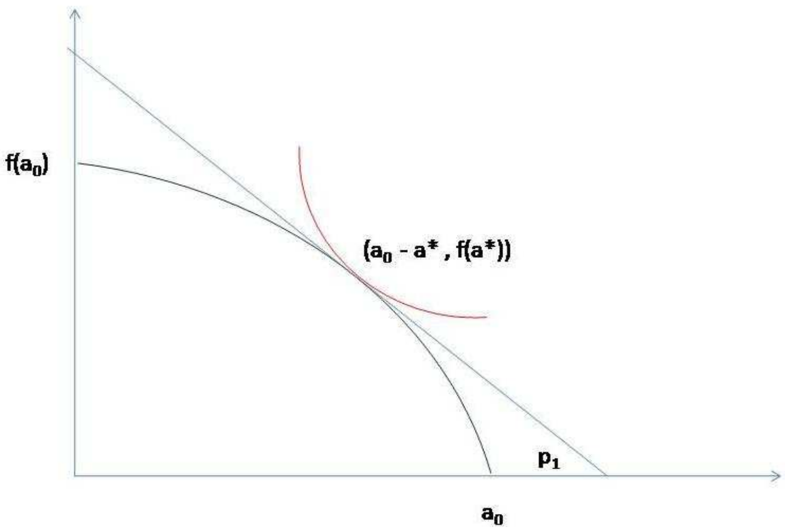
The output of the production center is a consumption bundle.

The consumption center takes $(a_0 - a, f(a))$ **as endowment**, **and finds the optimal consumption allocation that it can afford according to the prices declared by the pricing center**.

The prices declared by the pricing center are chosen to create harmony between the other two centers in the sense that the consumption center finds the outcome of the production center's activity, $(a_0 - a, f(a))$, **optimal given the announced prices**.

a. **Show that under Rule 1**, **the economic agent consumes the bundle** $(a_0 - a^*, f(a^*))$ **which maximizes his preferences**.

The solution corresponds to the point on the production possibility set where preferences are maximized.



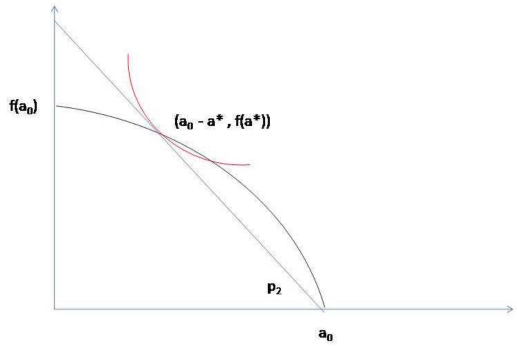
Since the production possibility set is strictly convex, and preferences are convex we know that there is a unique maximum.

Now choose a price vector such that the price line is tangent to this set and the indifference line exactly at the maximum.

By construction, profit is maximized given prices, and preferences are maximized at the intersection point for given prices and endowment point $(a_0 - a^*, f(a^*))$.

b. **What is the economic agent's consumption with Rule 2**?

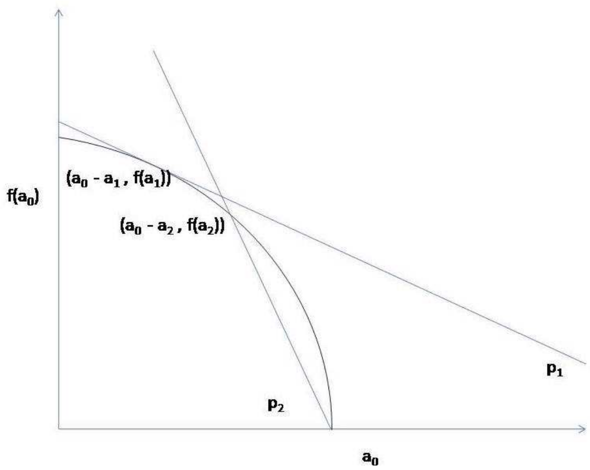
The economic agent chooses $(a_0 - a^*, f(a^*))$ with maximal a^* subject to the constraint that preferences are maximized at this point when we take the line connecting this point to $(a_0, 0)$ as the price line. By construction, production is maximized here subject to the constraint that there are no losses with the given prices. Also the point is chosen to guarantee that the consumer preferences are maximized at the budget set with the same prices.



c. **State and prove a general conclusion about the comparison between the behavior of two individuals**, **one whose production center operates with Rule 1 and one whose production center activates Rule 2**.

Claim: Individual using Rule 2 will always produce more, i.e. for a_1, p^1 and a_2, p^2 denoting the solutions under Rule 1 and Rule 2, $f(a_1) \leq f(a_2)$.

Assume for contradiction that $a_1 > a_2$. This means that the solution with Rule 2 is strictly to the right of the solution with Rule 1. Since f is strictly concave and monotonic, if solution with Rule 2 is to the right of solution with Rule 1, we must have $\frac{p_1^1}{p_2^1} < \frac{p_1^2}{p_2^2}$.



Note that $(a_0 - a_1, f(a_1))$ affordable (and strictly interior) in the budget set defined by p^2. Also $(a_0 - a_2, f(a_2))$ is affordable (and strictly interior) in the budget set defined by p^1.

$\Rightarrow (a_0 - a_2, f(a_2)) \succ (a_0 - a_1, f(a_1))$

$\Rightarrow (a_0 - a_1, f(a_1)) \succ (a_0 - a_2, f(a_2))$

which is a contradiction.