Problem Set 7 - Expected Utility

Problem 1.

Consider the following preference relations that were described in the text: “**the size of the support**” **and** “**comparing the most likely prize**".

a. **Check carefully whether they satisfy axioms** I **and** C.

Both preference relations violate both axioms:

the size of the support

Let p_t be the lottery: $p_t(z_1) = t$ and $p_t(z_2) = 1 - t$.

not I: $[z_1] \succ p_{1/2}$, but for $1 > \alpha > 0$, $\alpha[z_1] \oplus (1-\alpha)p_{1/2} \sim p_{1/2}$.

not C: For any $1/n > 0$, $p_{1/n} \sim p_{1/2}$, but in the limit $[z_1] \succ p_{1/2}$.

comparing the most likely prize

Assume that z_1 is better than z_2 and that "ties are broken in favor" of z_1:

not I: $[z_1] \succ [z_2]$, but $p_{1/4} \sim [z_2]$.

not C: $p_{1/2-1/n} \sim [z_2]$ for all n but in the limit $p_{1/2} \succ [z_2]$.

b. **These preference relations are not immune to a certain** "**framing problem**". **Explain**.

Both preference relations strictly prefer the lottery $50 with probability 0.4 and $100 with probability 0.6 to the lottery $50 with probability 0.4, one blue note of $100 with probability 0.3 and one green note of $100 with probability 0.3, even though the lotteries seem to be the "same".

Problem 2.

One way to construct preferences over lotteries with monetary prizes is by evaluating each lottery L **on the basis of two numbers**: $Ex(L)$, **the expectation of** L, **and** $var(L)$, L**'s variance**. **Such a construction may or may not be consistent with vNM assumptions**.

a. **Show that** $u(L) = Ex(L) - (1/4)var(L)$ **induces a preference relation that is not consistent with the vNM assumptions**.

$[1] \sim 0.5[0] \oplus 0.5[4]$ since $u([1]) = u(0.5[0] \oplus 0.5[4]) = 1$.

However, for $\alpha = 1/2$:

$\alpha[1] + (1-\alpha)[0.5[0] \oplus 0.5[2]] \succ \alpha[0.5[0] \oplus 0.5[4]] + (1-\alpha)[0.5[0] \oplus 0.5[2]]$

since the utility of the left lottery, $7/8$, is greater than the utility of the right lottery, $13/16$.

b. **Show that** $u(L) = Ex(L) - (Ex(L))^2 - var(L)$ **is consistent with vNM assumptions**.

Using the formula $var(X) = Ex(X^2) - (ExX)^2$ we get $u(L) = Ex(L) - (Ex(L))^2 - var(L) = Ex(L) - (Ex(L))^2 - (\sum_{z \in Z} L(z)z^2 - (Ex(L))^2) = \sum_{z \in Z} L(z)(z - z^2)$

is an expected utility function with vNM value $v(z) = z - z^2$.

Problem 3.

A decision maker has a preference relation $\succsim$ **over the space of lotteries** $L(Z)$ **having a set of prizes** Z. **On Sunday he learns that on Monday he will be told whether he has to choose between** L_1 **and** L_2 (**probability** $1 > \alpha > 0$) **or between** L_3 **and** L_4 (**probability** $1 - \alpha$). **He will make his choice at that time**. **Let us compare between two possible approaches the decision maker can take**.

Approach 1: **He delays his decision to Monday** (“**why bother with the decision now when I can make up my mind tomorrow**.”).

Approach 2: **He makes a contingent decision on Sunday regarding what he will do on Monday**, **that is**, **he decides what to do if he faces the choice between** L_1 **and** L_2 **and what to do if he faces the choice between** L_3 **and** L_4 (“**On Monday morning I will be so busy**. . .”).

a. **Formulate Approach 2 as a choice between lotteries**.

The DM chooses one of the four “compound” lotteries in the set

$$\{\alpha L_i \oplus (1-\alpha)L_j \mid i \in \{1,2\}, j \in \{3,4\}\}.$$

b. **Show that if the preferences of the decision maker satisfy the independence axiom**, **then his choice under Approach 2 will always be the same as under Approach 1**.

Let L_i (L_j) be the preferred lottery in $\{L_1, L_2\}$ ($\{L_3, L_4\}$), and L_{-i} (L_{-j}) be the other lottery. Under approach 1, the DM selects L_i (L_j) if the choice set on Monday is $\{L_1, L_2\}$ ($\{L_3, L_4\}$). Let $\succsim$ be the DM’s preferences over the compound lotteries in (a). By I,

$\alpha L_i \oplus (1-\alpha)L_j \succ \alpha L_i \oplus (1-\alpha)L_{-j} \succ \alpha L_{-i} \oplus (1-\alpha)L_{-j}$ and

$\alpha L_i \oplus (1-\alpha)L_j \succ \alpha L_{-i} \oplus (1-\alpha)L_j$. Thus $\alpha L_i \oplus (1-\alpha)L_j$ is the best of the “compound” lotteries.

Problem 4.

A decision maker is to choose an action from a set A. The set of consequences is Z. For every action $a \in A$, the consequence z^* is realized with probability α and any $z \in Z \setminus \{z^*\}$ is realized with probability $r(a,z) = (1-\alpha)q(a,z)$.

a. Assume that after making his choice he is told that z^* will not occur and is given a chance to change his decision. Show that if the decision maker obeys the Bayesian updating rule and follows vNM axioms, he will not change his decision.

By the vNM Theorem, preferences exhibit expected utility representation. Before learning the information, the DM solves

$$\max_{a \in A}\left[\sum_{z \in Z \setminus \{z^*\}} r(a,z)v(z) + \alpha v(z^*)\right].$$

After learning that z^* will not occur, the DM updates his beliefs so that $r'(a,z) = r(a,z)/(1-\alpha) = q(a,z)$ for $z \in Z \setminus \{z^*\}$ and the DM solves $\max_{a \in A} \sum_{z \in Z \setminus \{z^*\}} r'(a,z)v(z)$, which yields the same solution.

b. Give an example where a decision maker who follows nonexpected utility preference relation or obyes a non-Bayesean updating rule is not time consistent.

Example 1. Assume the DM has a “worst case” preference relation, where z_1 is the best prize, z_2 is the second best and z^* is the worst. Let action a_1 yield z_1 for sure and action a_2 yield z_1 and z_2 with equal probability, conditional on z^* not occurring. Then the DM will initially be indifferent between a_1 and a_2, but will strictly prefer a_1 after the information is revealed.

Example 2. Assume that $Z = \{1, 2, 3, z^* = 0\}$ and that $v(z) = z$. Assume that initially his beliefs are: $q(a_1, 2) = 1$, $q(a_2, 3) = 0.4$ and $q(a_2, 1) = 0.6$. Contingentally the DM chooses a_1. If he updates his beliefs and after he was lucky to avoid z^* he believes that he will be fortunate again, that is $q'(a_2, 3) = 1$, then he will change his mind and choose a_2.

Problem 5. Assume there is a finite number of income levels. An income distribution specifies the proportion of individuals at each level. Thus, an income distribution has the same methematical structure as a lottery. Consider the binary relation "one distribution is more egalitarian than another".

a. Why is the von Neumann-Morgenstern independence axiom inappropriate for characterizing this type of relation?

Assigning all members of the society the income 1 is as egalaterian as assigning all of them the income 2 and under the independence axiom, $0.5[1] \oplus 0.5[2]$ should be as egalaterian as $[1]$, but our intuition is that $0.5[1] \oplus 0.5[2]$ is less egalitarian than assigning equal income to all members of the society.

b. Suggest and formulate a property that is appropriate, in your opinion, as an axiom for this relation. Give two examples of preference relations that satisfy this property.

If p and q are identical distributions, except that the highest (lowest) income level in p is less (more) than in q, then p is more egaletrian than q.

Example 1: $p \succsim q$ if $Var(p) \leq Var(q)$.

Example 2: $p \succsim q$ if $\max_{z \in supp\ p(z)} z - \min_{z \in supp\ p(z)} z \leq \max_{z \in supp\ q(z)} z - \min_{z \in supp\ q(z)} z$.

Problem 6.

A decision maker faces a trade-off between longevity and quality of life. His preference relation ranks lotteries on the set of all certain outcomes of the form (q,t) **defined as "a life of quality** q **and length** t**" (where** q **and** t **are nonnegative numbers). Assume that the preference relation satisfies von Neumann-Morgenstern assumptions and that it also satisfies the following**:

(i) There is indifference between any two certain lotteries $[(q,0)]$ **and** $[(q',0)]$.

(ii) Risk neutrality with respect to life duration: an uncertain lifetime of expected duration T **is equally preferred to a certain lifetime duration** T **when** q **is held fixed**.

(iii) Whatever quality of life, the longer the life the better.

a. Show that the preference relation derived from maximizing the expectation of the function $v(q)t$, **where** $v(q) > 0$ **for all** q, **satisfies the assumptions**.

(i) If $t = 0$, then $v(q)t = 0$ for all q.

(ii) Let p be a lottery over t with expectation T. Then $\sum_t p(t)v(q)t = v(q)\sum_t tp(t) = v(q)T$.

(iii) $v(q)t' > v(q)t$ for all $t' > t$.

b. Show that all preference relations satisfying the above assumptions can be represented by an expected utility function of the form $v(q)t$, **where** v **is a positive function**.

Since $\succsim$ satisfies the v-NM axioms, then $\succsim$ is represented by an expected utility function with values $w(q,t)$.

By the second property, $w(q,t)$ is a affine tranformation of t, that is $w(q,t) = v(q)t + b(q)$.

By propery (i) it must be that $b(q) = b$ as otherwise for some q and q' we would have $w(q,0) \neq w(q',0)$.

By (iii) $v(q) > 0$ for all q.

Problem 7. **Consider a decision maker who systematically calculates that** $2 + 3 = 6$. **Construct a** "**money pump**" **argument against him**. **Discuss the argument**.

Tell the DM: "If you pay me \$5.99, I will give you two checks, one for \$2 and another for \$3." The DM will take the offer since he thinks he profits \$0.01. Then buy from him the checks for \$2.01 and \$3.01 and so on...