Problem Set 8 – Risk Aversion

Problem 1.

a. **Show that a sequence of numbers** $(a_1, .., a_K)$ **satisfies that** $\sum a_k x_k \geq 0$ **for all vectors** $(x_1, .., x_K)$ **such that** $x_k > 0$ **for all** k **iff** $a_k \geq 0$ **for all** k.

⇒If $a_{k^*} < 0$ then take $x_{k^*} = 1$ and $x_k = \varepsilon > 0$. For ε small enough $\sum a_k x_k < 0$.

⇐If $a_k \geq 0$ for all k then $\sum a_k x_k \geq 0$ for all vectors $(x_1, .., x_K)$ such that $x_k \geq 0$.

b. **Show that a sequence of numbers** $(a_1, .., a_K)$ **satisfies that** $\sum a_k x_k \geq 0$ **for all vectors** $(x_1, .., x_K)$ **such that** $x_1 > x_2 > \ldots x_K > x_{K+1} = 0$ **iff** $\sum_{k=1}^{l} a_k \geq 0$ **for all** l.

It follows, like in part (a), from the equality:

$\sum a_k x_k = \sum_{k=1}^{K} a_k \sum_{l=k}^{K} (x_l - x_{l+1}) = \sum_{l=1}^{K} (x_l - x_{l+1}) \sum_{k=1}^{l} a_k$.

Problem 2.

We say that p second-order stochastically dominates q and denote this by pD_2q if $p \succsim q$ for all preferences $\succsim$ satisfying the vNM assumptions, monotonicity and risk aversion.

a. **Explain why pD_1q implies pD_2q.**

If pD_1q, then $p \succsim q$ for all preferences satisfying the vNM assumptions and monotonicity. Thus $p \succsim q$ for all preferences satisfying the vNM assumptions, monotonicity *and* risk aversion.

b. **Let p and ϵ be lotteries. Define $p+\epsilon$ to be the lottery that yields the prize t with the probability $\sum_{\alpha+\beta=t} p(\alpha)\epsilon(\beta)$. Interpret $p+\epsilon$. Show that if ϵ is a lottery with expectation 0, then for all p, $pD_2(p+\epsilon)$.**

$p+\epsilon$ is the combination of two independent lotteries p and ϵ. Let the agent satisfy vNM assumptions, monotonicity and risk aversion. Then

$$\begin{aligned} U(p+\epsilon) &= \sum_{\alpha\in Z} p(\alpha) \sum_{\beta\in Z} \epsilon(\beta)u(\alpha+\beta) \\ &\leq \sum_{\alpha\in Z} p(\alpha)u\Big(\sum_{\beta\in Z} \epsilon(\beta)(\alpha+\beta)\Big) \quad \text{by } u \text{ concave} \\ &= \sum_{\alpha\in Z} p(\alpha)u(\alpha) \quad \text{by } Ex(\epsilon) = 0 \\ &= U(p). \end{aligned}$$

c. **Show that pD_2q iff for all $t < K$, $\sum_{k=0}^{t} [G(p,x_{k+1}) - G(q,x_{k+1})][x_{k+1} - x_k] \geq 0$, where $x_0 < \ldots < x_K$ are all the prizes in the support of either p or q and $G(p,x) = \sum_{z\geq x} p(z)$.**

Let $\succsim$ satisfy the vNM axioms, monotonicity and risk aversion. Then $\succsim$ is represented by $U(p) = \sum_{k=0}^{K} u(x_k)p(x_k)$, with u increasing and concave. Define

$$\alpha_k = \begin{cases} \frac{u(x_{k+1})-u(x_k)}{x_{k+1}-x_k} & \text{if } k < K \\ 0 & \text{if } k = K \end{cases}$$

By u increasing and concave, $\alpha_k \geq \alpha_{k+1}$ for all k. Then

$$
\begin{aligned}
U(p) - U(q) &= \sum_{k=0}^{K} (p(x_k) - q(x_k))u(x_k) \\
&= \sum_{k=0}^{K-1} [G(p, x_{k+1}) - G(q, x_{k+1})][u(x_{k+1}) - u(x_k)] && \text{by algebra} \\
&= \sum_{k=0}^{K-1} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k]\alpha_k && \text{by def. of } \alpha_k \\
&= \sum_{k=0}^{K-1} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k] \sum_{t=k}^{K-1} (\alpha_t - \alpha_{t+1}) && \text{telescopic sum and } \alpha_K = 0 \\
&= \sum_{t=0}^{K-1} (\alpha_t - \alpha_{t+1}) \sum_{k=0}^{t} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k] && \text{by algebra} \\
&\geq 0 \qquad \text{by } \alpha_t \geq \alpha_{t+1} \text{ and } \sum_{k=0}^{t} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k] \geq 0.
\end{aligned}
$$

By contradiction, assume $\sum_{k=0}^{T} [G(p, x_{k+1}) - G(q, x_{k+1})][x_{k+1} - x_k] < 0$ for some $T < K$. Let $\alpha \in (0, 1)$ and define

$$
u(x) = \begin{cases} x & \text{if } x \leq x_{T+1} \\ x_{T+1} + \alpha(x - x_{T+1}) & \text{if } x > x_{T+1}. \end{cases}
$$

By the above $U(p) < U(q)$ for α small enough.

Problem 3.

Consider a phenomenon called preference reversal. **Let** $L_1 = 8/9[4] \oplus 1/9[0]$ **and** $L_2 = 1/9[40] \oplus 8/9[0]$.

Discuss the phenomenon that many people prefer L_1 **to** L_2 **but when asked to evaluate the certainty equivalence of these lotteries they attach a lower value to** L_1 **than to** L_2.

People often prefer L_1, but they are not willing to pay $4 to play. Nevertheless, some people are willing to pay $4 to play L_2. It seems that people tend to over estimate small probabilities when they evaluate a lottery. See http://www.encyclopedia.com/doc/1O87-preferencereversal.html.

Problem 4.

Consider a consumer's preference relation over K-tuples describing quantities of K uncertain assets. Denote the random return on the kth asset by Z_k. Assume that the random variables $(Z_1,\dots,Z_K)$ are independent and take positive values with probability 1. If the consumer buys the combination of assets $(x_1,\dots,x_K)$ and if the vector of realized returns is $(z_1,\dots,z_K)$, then the consumer's total wealth is $\sum_k x_k z_k$. Assume that the consumer satisfies vNM assumptions, that is, there is a function v (over the sum of his returns) so that he maximizes the expected value of v. Assume that v is increasing and concave. The consumer preferences over the space of the lotteries induce preferences on the space of investments. Show that the induced preferences are monotonic and convex.

Monotonic: Let $x \geq x'$. Whenever the random variable $\sum_k x_k Z_k$ gets a certain value the random variable $\sum_k x'_k Z_k$ an higher value and thus $Ev(\sum_k x_k Z_k) \geq Ev(\sum_k x'_k Z_k)$.

Convex: Let x, x' be two investment combinations, $\lambda \in [0,1]$ and $x'' = \lambda x + (1-\lambda)x'$. By the concavity of v, $v(x'' \cdot z) \geq \lambda v(x \cdot z) + (1-\lambda)v(x' \cdot z)$ for all z, and thus $Ev(\sum_k x''_k Z_k) \geq \lambda Ev(\sum_k x_k Z_k) + (1-\lambda)Ev(\sum_k x'_k Z_k))$. The expectation of v is thus quasi-concave, and preferences are convex.

Problem 5.

Adam lives in the Garden of Eden and eats only apples. Time in the garden is discrete $(t = 1, 2, \ldots)$ **and apples are eaten only in discrete units. Adam possesses preferences over the set of streams of apple consumption. Assume that:**

a) Adam likes to eat up to 2 apples a day and cannot bear to eat 3 apples a day.

b) Adam is impatient. He would be delighted to increase his consumption on day t **from 0 to 1 or from 1 to 2 apples at the expense of an apple he is promised a day later.**

c) In any day in which he does not have an apple, he prefers to get one apple immediately in exchange for two apples tomorrow.

d) Adam expects to live for 120 years.

Show that if (poor) Adam is offered a stream of 2 apples starting in day 4 for the rest of his expected life, he would be willing to exchange that offer for one apple right away.

The following is a sequence of streams, in an increasing ordering:

(0,0,0,2,2,......,2)

(0,0,1,0,2,......,2). and continuing in this way until:

(0,0,1,1,1,......1,0)

(0,0,2,0,2,0...,2,0,0)

(0,1,0,1,0,....1,0,1,0,0,0) and "folding from the end":

(0,1,0,1,0,.1,0..2,0,0,0,0,0)

(0,1,0,1,0,.1,1,0,0,0,0,0,0)...until we reach:

(0,2,0,...0)

(1,0,....)

Problem 6.

In this problem you will encounter Quiggin and Yaari's functional, one of the main alternatives to expected utility theory.

Recall that expected utility can be written as $U(p) = \sum_{k=1}^{K} p(z_k)u(z_k)$ **where** $z_0 < z_1 < \ldots < z_K$ **are the prizes in the support of** p. **Let** $W(p) = \sum_{k=1}^{K} f(G_p(z_k))[z_k - z_{k-1}]$, **where** $f : [0,1] \to [0,1]$ **is a continuous increasing function and** $G_p(z_k) = \sum_{j \geq k} p(z_j)$. ($p(z)$ **is the probability that the lottery** p **yields** z **and** G_p **is the "anti-distribution" of** p.)

a. The literature often refers to W **as the dual expected utility operator. In what sense is** W **dual to** U?

Recall that $Ex(p) = \sum_{k=1}^{K} p(z_k)z_k = \sum_{k=1}^{K} G_p(z_k)[z_k - z_{k-1}]$

While the expected utility functional transforms the prize numbers whereas Quiggin-Yaari functional transofrms the anti-distribution numbers.

b. Show that W **induces a preference relation on** $L(Z)$ **that may not satisfy the independence axiom.**

Let $K = 2$, $f(x) = x^2$, $z_0 = 0$, $z_1 = 1$ and $z_2 = 4$. Define lotteries $p = .75[z_0] \oplus .25[z_2]$ and $p' = .5[z_0] \oplus .5[z_1]$. Then

$$U(p) = 4f(\frac{1}{4}) = \frac{1}{4} = f(\frac{1}{2}) = U(p')$$

but

$U(\frac{1}{2}p \oplus \frac{1}{2}[z_1]) = f(\frac{5}{8}) + 3f(\frac{1}{8}) = \frac{28}{64} < \frac{9}{16} = f(\frac{3}{4}) = U(\frac{1}{2}p' \oplus \frac{1}{2}[z_1])$.

c. What are the difficulties with a functional form of the type $\sum_z f(p(z))u(z)$? **(See Handa (1977))**

(1) If the DM is indifferent between prizes z_1 and z_2, then $[z_1]$ and $0.5[z_1] \oplus 0.5[z_2]$ need not be indifferent. If $f(1/2) \neq 1/2$, then $U([z_1]) \neq U(0.5[z_1] \oplus 0.5[z_2])$.

(2) The DM might be "worse" off if probability weight is shifted to a more preferred alternative:

if $f(1/2) > 1/2$, $0.5[1-\varepsilon] \oplus 0.5[1] \succ [1]$ for $\varepsilon > 0$ small enough while and–

if $f(1/2) < 1/2$, $0.5[1+\varepsilon] \oplus 0.5[1] \prec [1]$ for $\varepsilon > 0$ small enough.

Problem 7. The two envelopes paradox.

Assume that a number 2^n is chosen with probability $2^n/3^{n+1}$ and the amounts of money 2^n, 2^{n+1} are put into two envelopes. One envelope is chosen randomly and given to you and the other is given to your friend. Whatever the amount of money in your envelope, the expected amount in your friend's envelope is larger (verify it). Thus, it is worthwhile for you to switch envelopes with him even without opening the envelope! What do you think about this paradoxical conclusion?

Note that this is indeed a probability distribution: $\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \frac{1}{1-\frac{2}{3}} = 1.$

Assume that in your envelope the sum of money is 2^n. For any $n \geq 1$ this can be either the smaller amount or the larger one. If it is the smaller, then the other envelope contains 2^{n+1} and changing envelopes means a gain of 2^n. The probability for this event is $2^n/3^{n+1}$. If your amount is the larger one, then the other envelope contains 2^{n-1} and changing means a loss of 2^{n-1}. The probability for this event is $2^{n-1}/3^n$. Hence, the expected gain when changing envelopes is $\frac{2^n \cdot 2^n/3^{n+1} - 2^{n-1} \cdot 2^{n-1}/3^n}{2^n/3^{n+1} + 2^{n-1}/3^n} = \frac{2^n}{10}$ which is positive for any n. For $n = 0$, your envelope is surely the smaller one, hence changing envelopes is profitable for any $n \geq 0$. Thus, we can conclude that you should change envelopes without even opening yours.

Note that the random expected amount of money in any envelope, your own and the other one, is $\frac{1}{3} + \sum_{n=1}^{\infty} 2^n \cdot 5 \cdot \frac{2^{n-1}}{3^{n+1}} = \infty$. It is possible to show that if the problem is constructed such that the expected value of each envelope is finite, this paradox does not arise.

Random variables with infinite expectation create many paradoxes. For example, after every draw of such a random variable the decision maker who is risk neutral would prefer to replace the outcome in hand with another draw...