Problem Set 9 – Social Choice

Problem 1.

Assume that the set of social alternatives, X, **includes only two alternatives**. **Define a social welfare function to be a function that attaches a preference to any profile of preferences** (**allow indifference for the SWF and the individuals**' **preference relations**). **Consider the following axioms**:

Anonymity: **If** σ **is a permutation of** N **and if** $p = \{\succsim_i\}_{i \in N}$ **and** $p' = \{\succsim'_i\}_{i \in N}$ **are two profiles of preferences on** X **so that** $\succsim'_{\sigma(i)} = \succsim_i$, **then** $\succsim(p) = \succsim(p')$.

Neutrality: **For any preference** $\succsim_i$, **define** $(-\succsim_i)$ **as the preference satisfying** $x(-\succsim_i)y$ **iff** $y \succsim_i x$. **Then**, $\succsim(\{-\succsim_i\}_{i \in N}) = -\succsim(\{\succsim_i\}_{i \in N})$.

Positive Responsiveness: **If the profile** $\{\succsim'_i\}_{i \in N}$ **is identical to** $\{\succsim_i\}_{i \in N}$ **with the exception that for one individual** j **either** ($x \sim_j y$ **and** $x \succ'_j y$) **or** ($y \succ_j x$ **and** $x \sim'_j y$) **and if** $x \succsim y$, **then** $x \succ' y$.

a. **Interpret the axioms**.

A: The social aggregation treats any two individuals symmetrically.

N: The social aggregation treats the two alternatives symmetrically.

PR: The fact that one individual changed his mind in favor of an alternative cannot harm (and in some cases is required to improve) the social status of that alternative.

b. **Show that majority rule satisfies all of them**.

A: trivially satisfied.

N: Let $N(x,p)$ be the number of individuals that strictly prefer x to y in profile p. If $x \succsim (p)y$, then $N(x,p) \geq N(y,p)$, and thus $y \succsim (-p)x$, proving N.

PR: p' is identical to p, except that one individual "increases" his preference for x, and $N(x,p) \geq N(y,p)$ then $N(x,p') > N(y,p')$, and thus $x \succ (p')y$.

c. **Prove May's theorem by which the majority rule is the only SWF satisfying the above axioms**.

Assume $N(x,p) = N(y,p)$. Let σ be a permutation such that $\sigma(i)$'s preference is the reverse of i's preference. Let $p' = \{\succsim'_i\}_{i \in N}$ be a profile so that $\succsim'_{\sigma(i)} = \succsim_i$. By A $x \succsim (p)y$ iff $x \succsim (p')y$. Since $p' = -p$ by N $x \succsim (p')y$ iff $y \succsim (p)x$ and thus $x \sim (p)y$.

Assume p a profile such that $N(x,p) > N(y,p)$. Assume $y \succsim (p)x$. We can move from p to a profile p' by changing the preferences of $N(x,p) - N(y,p)$ individuals who prefer x to y to indifference. By PR we would get $y \succ (p')x$ although $N(x,p') = N(y,p')$, a contradiction.

d. **Are the above three axioms independent**?

Yes.

A and N, but not PR: $x \sim (p)y$ for all p.

A and PR, but not N: Let $X = \{a,b\}$. $a \succ (p)b$ if $N(a,p) > N(b,p) + 1$, otherwise $b \succ (p)a$.

N and PR, but not A: For any profile of preferences we attach the "lexicographic" preferences with some fixed priority of the individuals (For example,

$x \succsim (p)y$ if [$x \succ_1 y$] or [$x \sim_1 y$ and $x \succ_2 y$] and so on...)

Problem 2.

Assume that the set of alternatives, X, **is the interval** $[0,1]$ **and that each individual's preference is single-peaked**, **i.e.**, **for each** i **there is an alternative** a_i^* **such that if** $a_i^* \geq b > c$ **or** $c > b \geq a_i^*$, **then** $b \succ_i c$. **Show that for any odd** n, **if we restrict the domain of preferences to single-peaked preferences**, **then the majority rule induces a** "**well-behaved**" **SWF**.

Let $x \succsim (p)y$ and $y \succsim (p)z$. By contradiction, assume that $z \succ (p)x$.

(i) $x < y < z$. If a majority find x as good as y then all those strictly prefer y to z and there is a majority who strictly prefer x over z.

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Problem 3.

Each of N individuals chooses a single object from among a set X, interpreted as his recommendation for the social action. We are interested in functions that aggregate the individuals' recommendations (not preferences, just recommendations!) into a social decision (i.e., $F : X^N \to X$). Discuss the following axioms:

Par: If all individuals recommend x^*, then the society chooses x^*.

I: If the same individuals support an alternative $x \in X$ in two profiles of recommendations, then x is chosen in one profile if and only if it chosen in the other.

a. Show that if X includes at least three elements, then the only aggregation method that satisfies P and I is a dictatorship.

L1: For any recommendation profile $(x_1, \ldots, x_N)$, $F(x_1, \ldots, x_N) \in \{x_1, .., x_N\}$.

Otherwise, by I, $F(c,,,.c) = F(x_1, \ldots, x_N)$ for any $c \neq F(x_1, \ldots, x_N)$, contradicting P.

L2: If $F(x_1, \ldots, x_N) = a$ and $\{i|\, x_i = a\} \subset \{i|\, y_i = a\}$ then $F(y_1, \ldots, y_N) = a$.

If not than by L1 $F(y_1, \ldots, y_N) = b \neq a$ where b is one of the alternatives in y.

Let c be a third alternative. Let $(z_1, \ldots, z_N)$ be the same as y with any $i \notin \{i|\, x_i = a\}$ with $y_i = a$ is changed to $z_i = c$. Then, $\{i|\, y_i = b\} = \{i|\, z_i = b\}$and $\{i|\, x_i = a\} = \{i|\, z_i = a\}$ thus by I $F(z)$ should be both b and a. A contradiction.

Let us say that G is *decisive* with respect to $x^* \in X$, if [for all $i \in G$, $x_i = x^*$] then [$F(x_1, \ldots, x_N) = x^*$].

By L2 if $F(x_1, .., x_N) = a$ then $\{i|\, x_i = a\}$ is decisive with respect to a.

L3: If G is decisive with respect to a with $|G| \geq 2$, then for any b there exists $\emptyset \subset G' \subset G$ such that G' is decisive with respect to a or b.

Let c be a third alternative. Since G is decisive with respect to a then $F(x_1, \ldots, x_N) = a$ where $x_i = a$ for $i \in G$ and $x_i = c$ otherwise

Let G_1, G_2 be a partition of G and

$$y_i = \begin{cases} a & \text{if } i \in G_1 \\ b & \text{if } i \in G_2 \\ c & \text{if } i \in N \setminus G \end{cases}$$

By I $F(y_1, \ldots, y_N)$ is not c and by L1 it is either a or b . Thus, by L2 either G_1 or G_2 are decisive with respect to a or b.

L4: There is a singleton i^* who is decisive with respect to some alterative a.

L5: i^* is decisive regarding any alterative b.

Let $x_{i^*} = a$ and $x_i = c$ for all other i. By I, $F(x) = a$.

Let $y_{i^*} = b$ and $y_i = c$ for all other i, By I $F(y) \neq c$ and by L1 $F(y) = b$ and by L2 it is decisive regarding b.

b. **Show the necessity of the three conditions** P, I, **and** $|X|\geq 3$ **for this conclusion**.

Choosing the most popular recommendation (with pre-specified tie breaking rule) satisfies P and I when $|X|= 2$.

When $|X|\geq 3$, the most popular recommendation (with pre-specified tie breaking rule)satisfies P but fails I.

Always choosing action $x^* \in X$, regardless of recommendation, satisfies I but fails P.

Problem 4.

Some proofs of Arrow's theorem use the notion of decisive and almost decisive coalitions.

Given the SWF we say that:

i) **a coalition** G **is decisive with respect to** x,y **if** [**for all** $i \in G$, $x \succ_i y$] **implies** [$x \succ y$], **and**

ii) **a coalition** G **is almost decisive with respect to** x,y **if** [**for all** $i \in G$, $x \succ_i y$ **and for all** $j \notin G$, $y \succ_j x$] **implies** [$x \succ y$].

Note that if G **is decisive with respect to** x,y, **then it is also almost decisive with respect to** x,y, **since** "**almost decisiveness**" **refers only to the subset of profiles in which all members of** G **prefer** x **to** y **and all members of** $N - G$ **prefer** y **to** x.

We say that a coalition G **is decisive if it is decisive with respect to all** x,y.

Let F **be an SWF satisfying** Par **and** IIA.

a. **Prove the** "**Field Expansion Lemma**": **If** G **is almost decisive with respect to** x,y, **then** G **is decisive with respect to** x,z **and with respect to** y,z.

Consider a profile $(\succ_1, \dots, \succ_n)$ such that $x \succ_i z$ for all $i \in G$.

Consider another profile $(\succ_1^*, \dots, \succ_n^*)$ which ranks x,y and z as follows:

if $i \in G$	*if* $i \in N \setminus G$ and $z \succ_i x$	*if* $i \in N \setminus G$ and $x \succ_i z$
x	y	y
y	z	x
z	x	z

Since G is almost decisive with respect to x,y, then $x \succ^* y$. By *Par*, $y \succ^* z$. By transitivity $x \succ^* z$. By IIA also $x \succ z$.

Now, consider a profile $(\succ_1, \dots, \succ_n)$ such that $z \succ_i y$ for all $i \in G$.

Consider another profile $(\succ_1^*, \dots, \succ_n^*)$ which ranks x,y and z as follows:

if $i \in G$	*if* $i \in N \setminus G$ and $z \succ_i y$	*if* $i \in N \setminus G$ and $y \succ_i z$
z	z	y
x	y	z
y	x	x

Since G is almost decisive with respect to x,y, then $x \succ^* y$. By *Par*, $z \succ^* x$. By transitivity $z \succ^* y$. By IIA also $z \succ y$.

b. **Conclude that if G is almost decisive with respect to x,y, then G is decisive**.

Let w,z be any two alternatives.

If G is almost decisive with respect to x,y, then by part a it is decisive with respect to x,z. Thus, G is also almost decisive with respect to x,z. By part a, G is decisive with respect to w,z.

c. **Prove the "Group Contraction Lemma": If G is decisive.and $|G|\geq 2$, then there exists $G' \subset G$ such that G' is decisive**.

Let $G = G_1 \cup G_2$, where G_1 and G_2 are nonempty and $G_1 \cap G_2 = \emptyset$. By the Field Expansion Lemma it is enough to show that G_1 or G_2 is almost decisive with respect to some alternatives.

Take three alternatives x,y, and z and a profile of preference relations $(\succ_i)_{i\in N}$ satisfying:

if $i \in G_1$	*if* $i \in G_2$	*if* $i \in N \setminus \{G_1 \cup G_2\}$
z	x	y
x	y	z
y	z	x

The coalition G is decisive, thus $x \succ y$.

If G_1 is not almost decisive with respect to z,y, then there is a profile $(\succ_i')_{i\in N}$ such that $z \succ_i' y$ for all $i \in G_1$ and $y \succ_i' z$ for all $i \notin G_1$, such that $F(\succ_1', \ldots, \succ_n')$ determines $y \succ' z$. Therefore, by IIA, $y \succ z$.

Similarly, if G_2 is not almost decisive with respect to x,z, then $z \succ x$. Thus, by transitivity $y \succ x$, but since G is decisive, $x \succ y$, a contradiction. Thus, G_1 or G_2 is almost decisive with respect to some alternatives.

d. **Show that there is an individual i^* such that $\{i^*\}$ is decisive**.

By *Par*, the set N is decisive. By the Group Contraction Lemma, every decisive set that includes more than one member has a proper subset that is decisive. Thus, there is a set $\{i^*\}$ that is decisive, which means that $F(\succ_1, \ldots, \succ_n) \equiv \succ_{i^*}$.

Problem 5

Who is an economist? Departments of economics are often sharply divided over this question. Investigate the approach according to which the determination of who is an economist is treated as an aggregation of the views held by department members on this question.

Let $N = \{1, \ldots, n\}$ **be a group of individuals** ($n \geq 3$). **Each** $i \in N$ **"submits" a set** E_i, **a proper non empty subset of** N, **which is interpreted as the set of "real economists" in his view. An aggregation method** F **is a function that assigns a proper non empty subset of** N **to each profile** $(E_i)_{i=1,\ldots,n}$ **of proper subsets of** N. $F(E_1, \ldots, E_n)$ **is interpreted as the set of all members of** N **who are considered by the group to be economists. (Note that we require that all opinions be proper subsets of** N.**)**

Consider the following axioms on F:

Consensus: If $j \in E_i$ **for all** $i \in N$, **then** $j \in F(E_1, \ldots, E_n)$ **and if** $j \notin E_i$ **for all** $i \in N$, **then** $j \notin F(E_1, \ldots, E_n)$.

Independence: If $(E_1, \ldots, E_n)$ **and** $(G_1, \ldots, G_n)$ **are two profiles of views so that for all** $i \in N$, $[j \in E_i$ **iff** $j \in G_i]$, **then** $[j \in F(E_1, \ldots, E_n)$ **iff** $j \in F(G_1, \ldots, G_n)]$.

a. Interpret the two axioms.

C: If all individuals include (omit) an economist from their list, then that option is included (omitted) from the aggregation.

I: If all individuals in profiles $\{E_i\}$ and $\{G_i\}$ have the same opinion regarding j, then j is either included or excluded in both aggregations.

b. Find one aggregation method that satisfies C **but not** I **and one that satisfies** I **but not** C.

Select the set with the smallest number of elements suggested by one of the members (for tie breaking rule, from among the sets with minimal number of individuals choose the one suggested by the member with the smallest index) This methods satisfies C but not I.

Always selecting $F(E_1, \ldots, E_n) = \{1\}$ satisfies I but not C.

c. Provide a proof similar to that of Arrow's Impossibility Theorem of the claim that the only aggregation methods that satisfy the above two axioms are those for which there is a member i^* **such that** $F(E_1, \ldots, E_n) = E_{i^*}$.

L1: Assume that G is almost decisive regarding j (that is j is an economist whenever the group of people who consider j an economist is precisely G) then it is almost decisive regarding any h (and thus will be almost decisive).

Consider the profile where the supporters of j are the members of G, the supporters of i (a third member) are $N-G$ and everybody supports all other members. By C those must be in E and since E is a proper subset and $j \in E$, it must be that i is not.

Now consider the profile where the set of supporters of h is G, the supporters of i are $N-G$ and nobody considers any other member to be an economist. By C all those members besides i and h are not in E. By I i is not in E. Since E is a proper subset it must be that h is in E. Thus, G is almost decisive regarding h.

L2: If G is almost decisive and contains more than one element then there is a proper subset of G which is almost decisive.

Partition G to G_1 and G_2. Consider the profile where 1 is considered an economist by exactly the members of G_1, 2 is considered an economist by exactly the members of G_2 and all other individuals are considered economists by the members of $N-G$ only. As we have seen in L1, since G is almost decisive then $N-G$ is not and thus all individuals besides 1 and 2 are not determined by the aggregator to be economists. It follows that either 1 or 2 must be an economist and thus (using I) either G_1 or G_2 is almost decisive in regard 1 and 2 and by L1 at least one of them is almost decisive.

L3: There is an i^* who is almost decisive.

L4: If i is supported by G which contains i^* then i is an economist.

Consider the profile where i is supported by G, j by $N-\{i^*\}$ and all the rest by no one. By C all members $N-\{i^*,j\}$ are not economists. From the argument above neither is j so i must be an economist with respect to this profile and by I with respect to any profile.