Solution for Problem set 2

1.

(a) Let u_i be player i's payoff function in the game G, let w_i be his payoff function in G', and let (x^*, y^*) be a Nash equilibrium of G'. Then, using part (b) of the Proposition in L-2, we have $w_1(x^*,y^*) = \min_y \max_x w_1(x, y) \geq \min_y \max_x u_1(x,y)$, which is the value of G.

(b) This follows from part (b) of the Proposition in L-2 and the fact that for any function f we have $\max_{x\in X} f(x) \geq \max_{x\in Y} f(x)$ if $Y \subseteq X$.

(c) In the unique equilibrium of the game

	L	R
T	3,3	1,1
B	1,0	0,1

player 1 receives a payoff of 3, while in the unique equilibrium of

	L	R
T	3,3	1,1
B	4,0	2,1

she receives a payoff of 2. If she is prohibited from using her second action in this second game then she obtains an equilibrium payoff of 3, however.

2.

Part 1:

First note, that in a symmetric zero sum game, for every $a_1, b_1 \in A_1$, (a_1, a_1) and (b_1, b_1) yield the same payoff to each player. To see this note that from the symmetry, if $(a_1, a_1) \succeq_1 (b_1, b_1)$ then $(a_1, a_1) \succeq_2 (b_1, b_1)$, from the strict competitiveness of the game, it cannot be that one of the players is strictly better off and the other is not worse off. Hence, both players are indifferent between (a_1, a_1) and (b_1, b_1) for every $a_1, b_1 \in A_1$. This means that along the main diagonal of the matrix, all squares contain the same pair of values. Now, referring to the equilibrium in a symmetric zero sum game we can say:

(a) If a profile of actions (a,b) is a Nash equilibrium, so are the profiles (b,a), (a,a) and (b,b)..

Proof:

From symmetry, it is straightforward that if (a,b) is a Nash equilibrium, so is (b,a).

If (a,b) is a Nash equilibrium then $(a,b)\succeq_1(b,b)$. If $(a,b)\succ_1(b,b)$ by the strict competitiveness we get that $(b,b)\succ_2(a,b)$ and since all payoffs of each player are constant along the main diagonal we have that $(a,a)\succ_2(a,b)$ a contradiction to (a,b) being Nash equilibrium. Hence, $(a,b)\sim_1(b,b)$.Using the symmetry and the strict competitiveness again (and transitivity) we have that $(a,b)\sim_i(b,b)\sim_i(a,a)\sim_i(b,a)$ for $i=1,2$. Moreover, since (a,b) is a Nash equilibrium we have $\forall_{c\in A}.(a,b)\succeq_1(c,b)$ and using $(a,b)\sim_i(b,b)$ we get that $\forall_{c\in A}.(b,b)\succeq_1(c,b)$. Therefore, player 1 cannot profit from deviating from (b,b).. Symmetry implies that $\forall_{c\in A}.(b,b)\succeq_2(b,c)$ meaning that player 2 cannot profit from deviating from (b,b).. Hence, (b,b). is a Nash equilibrium.

Using a similar argument one can show that (a,a) is a Nash equilibrium.

(b) If a Nash equilibrium exists than there is an action $a^* \in A$ such that (a^*,a^*) is a Nash equilibrium of the game (note that this claim goes beyond the claim proved in problem set 1 question 7 because we do not demand here that the game satisfy the conditions of the existence theorem). This claim is derived directly from (a).

Note that if $|A|=2$ a Nash equilibrium of the symmetric zero sum game must exist (show!!). However, there is no guarantee for the existence of a Nash equilibrium in the general case.

Part 2:

In a zero sum game, player 1 is in a better position if the next two conditions hold:

- The game can be described as a result of taking a symmetric zero sum game and increasing the payoffs of player one.
- The game is not symmetric.

The second condition is needed to rule out increments of the payoffs of player 1 that keep the symmetry of the game.

3.

The game includes a set of players $N = \{1,2\}$, a set of actions (the same set for both players) $A_1 = A_2 = \{x \mid x\, is\, a\, 20\, digits\, number\}$, and a pair of payoff functions

$$u_1(a,b) = -u_2(a,b) = \begin{cases} 1 & if \quad a \neq b \\ -1 & if \quad a = b \end{cases}.$$

If we consider the sequential nature of the game, repeating player 1's number is a dominant strategy for player 2 and insures that she is to win the game. In this case the value of the game is (-1,1).

Clearly, in real life one will prefer to be player 1 than 2, but this is due to reasonable memory constraints which are missing from the model (note, that if you are to play the game using a computer you will probably prefer being player 2 as predicted by the model).