Lecture G-05: Rationalizability and Iterated Elimination of Dominated Actions

Readings: Osborne and Rubinstein Ch 4

Up to now we discussed solution concepts for strategic games in which each player's choice is required to be optimal given his belief about the other players' behavior, a belief that is required to be correct. That is, we assumed that in equilibrium each player knows the other players' equilibrium behavior. If the players participate repeatedly in the situation that the game models then they can obtain this knowledge from the steady state behavior that they observe (if indeed they observe a steady state behavior). However, if the game is a one-shot event in which all players choose their actions simultaneously then it is not clear how each player can know the other players' equilibrium actions; for this reason we are interested in solution concepts that do not entail this assumption.

We will study solution concepts in which the players' beliefs about each other's actions are not assumed to be correct, but are constrained by considerations of rationality: each player believes that the actions taken by every other player is a best response to some belief, and, further, each player assumes that every other player reasons in this way and hence thinks that every other player believes that every other player's action is a best response to some belief, and so on.

The solution concepts that we study are weaker than Nash equilibrium. In fact, in many games they do not exclude any action from being used. But we find the approach is interesting in that it explores the logical implications of assumptions about the players' knowledge that are weaker than those in the previous chapters.

Fix a strategic game $\langle N, (A_i), (u_i) \rangle$ where A_i is finite. A belief of player i (about the actions of the other players) is a probability measure on A_{-i} $(= \times_{j \in N \setminus \{i\}} A_j)$. Note that this definition allows a player to believe that the other players' actions are correlated: a belief is not necessarily a product of independent probability measures on each of the action sets A_j for $j \in N \setminus \{i\}$. An action $a_i \in A_i$ of player i is a best response to a belief if there is no other action that yields player i a higher payoff given the belief. The phrase "player i thinks that some other player j is rational", takes to mean that player i thinks that whatever action player j chooses is a best response to player j's belief about the actions of the players other than j.

If player i thinks that every other player j is rational then he must be able to rationalize his belief μ_i about the other players' actions as follows: every action of any other player j to which the belief μ_i assigns positive probability must be a best response to a belief of player j. If player i further thinks that every other player j thinks that every player $h \neq j$ (including player i) is rational then he, player i, must also have a view about player j's view about player h's beliefs. If player i's reasoning has unlimited depth, we are led to the following definition.

Definition 1: An action a_1 is **rationalizable** if there exists a collection

$X_1^0 = \{a_i\}$

$,X_2^1,\ldots,X_N^1$

$X_1^2,X_2^2,\ldots,X_N^2$

....

$X_1^t,X_2^t,\ldots,X_N^t$

....

with $X_j^t \subseteq A_j$ for all j and t,

and for each $j \in N$, each $t \geq 0$, and each $a_j \in X_j^t$, a belief $\mu_j^{t+1}(a_j)$ of player j whose support is a subset of X_{-j}^{t+1} such that

1) for every player $j \in N$ and every t every action $a_j \in X_j^t$ is a best response to the belief $\mu_j^{t+1}(a_j)$ of player j

2) for each t and each $j \in N$ the set X_j^t is the set of all $a_j' \in A_j$ such that there is some player $k \in N \setminus \{j\}$, some action $a_k \in X_k^{t-1}$, and some a_{-k} in the support of $\mu_k^t(a_k)$ for which $a_j' = a_j$.

Note that for $|N|= 2$ the definition is superfluous as we can make do with X_i^t for any odd t and X_j^t for any even t.

The set X_j^1 for $j \neq 1$ is the set of actions of player j that are assigned positive probability by the belief μ_1^1 of player i about the actions of the players other than 1 that justifies 1 choosing a_1. For any $j \in N$ the interpretation of X_j^{t+1} is that it is the set of all actions a_j of player j such that there exists at least one action $a_k \in X_k^t$ of some player $k \neq j$ that is justified by the belief $\mu_k^t(a_k)$ that assigns positive probability to a_j.

Example: Let us consider the traveler's dilemma (with each player having to choose an integer between 180 and 300). Player 1's action 298 is optimal against a belief that player 2 chooses 299 which is optimal agents a belief that player 1 chooses 300 but 300 cannot be rationalized in any way. Actually the optimal response to any belief with the highest number in the support being K, is not higher than $K - 1$ since the action $K - 1$ is strictly better for the player than any action of K or more.

This definition of rationalizability is equivalent to the following.

Definition 2: An action a_i is rationalizable if there is $(Z_j)_{j \in N}$ such that $a_i \in Z_i$ and every action $a_j \in Z_j$ is a best response to a belief $\mu_j(a_j)$ of player j whose support is a subset of Z_{-j}.

Note that if $(Z_j)_{j \in N}$ and $(Z_j')_{j \in N}$ satisfy this definition then so does $(Z_j \cup Z_j')_{j \in N}$, so that the set of profiles of rationalizable actions is the largest set $\times_{j \in N} Z_j$ for which $(Z_j)_{j \in N}$ satisfies the definition.

Lemma: The two definitions are equivalent.

Proof. If $a_i \in A_i$ is rationalizable according to Definition 1 then define $Z_i = \{a_i\} \cup (\cup_{t=1}^{\infty} X_i^t)$ and $Z_j = (\cup_{t=1}^{\infty} X_j^t)$ for each $j \in N \setminus \{i\}$.

If a_i is rationalizable according to Definition 2 then define $\mu_i^1 = \mu_i(a_i)$ and $\mu_j^t(a_j) = \mu_j(a_j)$ for each $j \in N$ and each integer $t \geq 2$. Then the sets X_j^t defined in Definition 1 are subsets of Z_j and satisfy the condition 1.

Any action that a player uses with positive probability in some mixed strategy Nash equilibrium is rationalizable (take Z_j to be the support of player j's mixed strategy). The following result shows that the same is true for actions used with positive probability in some correlated equilibrium.

Lemma: Every action used with positive probability by some player in a correlated equilibrium of a finite strategic game is rationalizable.

Proof: Denote the strategic game by $\langle N, (A_i), (u_i) \rangle$; choose a correlated equilibrium, and for each player $i \in N$ let Z_i be the set of actions that player i uses with positive probability in the equilibrium. Then any $a_i \in Z_i$ is a best response to the distribution over A_{-i} generated by the strategies of the players other than i, conditional on player i choosing a_i. The support of this distribution is a subset of Z_{-i} and hence a_i is rationalizable.

In the Prisoner's Dilemma only the Nash equilibrium action Confess is rationalizable. In the battle of the sees both actions of each player are rationalizable, since in each case both actions are used with positive probability in some mixed strategy Nash equilibrium. Thus rationalizability puts no restriction on the outcomes in these games. For many other games the restrictions that rationalizability imposes are weak. However, in some games rationalizability provides a sharp answer, as the problems 1,2,3 demonstrate.

Note that in both Definitions we take a belief of player i to be a probability distribution on A_{-i}, which allows each player to believe that his opponents' actions are **correlated**. In most of the literature, players are not allowed to entertain such beliefs: it is assumed that each player's belief is a product of independent probability distributions, one for each of the other players. (Such a restriction is obviously inconsequential in a two-player game.) This assumption is consistent with the motivation behind the notion of mixed strategy equilibrium. Our definition of rationalizability requires that at all levels of rationalization the players be rational; the alternative definition of rationalizability requires in addition that at all levels of rationalization the beliefs preserve the assumption of independence. Note that the two definitions have different implications.

Iterated Elimination of Strictly Dominated Actions

Like the notion of rationalizability, the solution concept that we now study looks at a game from the point of view of a single player. Each player takes an action based on calculations that do not "require" knowledge of the actions taken by the other players. To define the solution we start by eliminating actions that a player should definitely not take. We assume that players exclude from consideration actions that are not best responses whatever the other players do. A player who knows that the other players are rational can assume that they too will exclude such actions from consideration. Now consider the game G' obtained from the original game G by eliminating all such actions. Once again, a player who knows that the other players are rational should not choose an action that is not a best response whatever the other players do in G'. Further, a player who knows that the other players know that he is rational can argue that they too will not choose actions that are never best responses in G'. Continuing to argue in this way suggests that the outcome of G must survive an unlimited number of rounds of such elimination. We now formalize this idea and show that it is equivalent to the notion of rationalizability.

Never-Best Responses

Definition: An action of player i is a never best response if it is not a best response to any belief of player i.

Clearly any action that is a never-best response is not rationalizable. If an action a_i of player i is a never-best response then for every belief of player i there is some action, which may depend on the belief, that is better for player i than a_i.

We now show that if a_i is a never-best response in a finite game then there is a mixed strategy that, whatever belief player i holds, is better for player i than a_i. This alternative property is defined precisely as follows.

Definition: The action $a_i \in A_i$ of player i in the strategic game $\langle N, (A_i), (u_i) \rangle$ is strictly dominated if there is a mixes strategy α_i of player i suct hat $U_i(a_{-i}, \alpha_i) > u_i(a_{-i}, a_i)$ for all $a_{-i} \in A_{-i}$.

In fact, we show that in a game in which the set of actions of each player is finite an action is a never-best response if and only if it is strictly dominated. Thus in such games the notion of strict domination has a decision-theoretic basis that does not involve mixed strategies. It follows that even if one rejects the idea that mixed strategies can be objects of choice, one can still argue that a player will not use an action that is strictly dominated.

Lemma: An action of a player in a finite strategic game is a never-best response if and only if it is strictly dominated.

Iterated Elimination of Strictly Dominated Actions

We now define formally the procedure that we described at the beginning of the section.

Definition: The set $\times_j X_j \subseteq A$ (compare with the book!) of outcomes of a finite game $\langle N, (A_i), (u_i) \rangle$ survives iterated elimination of strictly dominated actions if there is a collections $((X_j^t)_{j \in N})_{t=0}^T$ of sets such that:

$X_j^0 = A_j$ and $X_j^T = X_j$.

$X_j^{t+1} \subseteq X_j^t$ for each $t = 0, \ldots, T-1$.

For each $t = 0, \ldots, T-1$ every action of player j in $X_j^t \setminus X_j^{t+1}$ is strictly dominated in the game $\langle N, (X_i^t), (u_i^t) \rangle$, where u_i^t for each $i \in N$ is the function u_i restricted to $\times_{j \in N} X_j^t$.

No action in X_j^T is strictly dominated in the game $\langle N, (X_i^T), (u_i^T) \rangle$.

Example

	L	R
T	$3,0$	$0,1$
M	$0,0$	$3,1$
B	$1,1$	$1,0$

B is dominated by the mixed strategy in which T and M are each used with probability $\frac{1}{2}$. After B is eliminated from the game, L is dominated by R; after L is eliminated T is dominated by M. Thus (M, R) is the only outcome that survives iterated elimination of strictly dominated actions.

We now show that in a finite game a set of outcomes that survives iterated elimination of dominated actions exists and is the set of profiles of rationalizable actions.

Proposition: If $X = \times_{j \in N} X_j$ survives iterated elimination of strictly dominated actions in a finite strategic game $\langle N, (A_i), (u_i)\rangle$ then X_j is the set of player j's rationalizable actions for each $j \in N$.

Proof: Suppose that $a_i \in A_i$ is rationalizable and let $(Z_j)_{j \in N}$ be the profile of sets in the definition that supports a_i. For any value of t we have $Z_j \subseteq X_j^t$ since each action in Z_j is a best response to some belief over Z_{-j} and hence is not strictly dominated in the game $\langle N, (X_i^t), (u_i^t)\rangle$. Hence $a_i \in X_i$.

We now show that for every $j \in N$ every member of X_j is rationalizable. By definition, no action in X_j is strictly dominated in the game in which the set of actions of each player i is X_i, so by the Lemma every action in X_j is a best response among the members of X_j to μ_j a belief on X_{-j}. We need to show that every action in X_j is a best response among all the members of the set A_j to μ_j.

Assume that $a_j \in X_j$ but there is a value of t such that a_j is a best response among the members of X_j^t to a belief μ_j on X_{-j}, but is not a best response among the members of X_j^{t-1}. Then there is an action $b_j \in X_j^{t-1} \setminus X_j^t$ that is a best response among the members of X_j^{t-1} to μ_j, contradicting the fact that b_j is eliminated at the tth stage of the procedure.

Note that the procedure does not require that all strictly dominated strategies be eliminated at any stage. Thus it follows from the result that the order and speed of elimination have no effect on the set of outcomes that survive.

The Lemma and the equivalence of the notions of iterated elimination of strictly dominated actions and rationalizability fail if we modify the definition of rationalizability to require the players to believe that their opponents' actions are independent.

Example: Note that in the traveler's dilemma the only action that can be deleted at the first stage is 300 (each other k is a best response against a belief that the other player plays $k+1$). It is strongly dominated by a mixed strategy $\alpha/\beta[299] + \alpha^2/\beta[298] + \ldots\ldots + \alpha^{119}/\beta[180]$ where $\beta = \alpha + \alpha^2 + \ldots\ldots + \alpha^{119}$ for α small enough (check!)

Iterated Elimination of Weakly Dominated Actions

We say that a player's action is weakly dominated if the player has another action at least as good no matter what the other players do and better for at least some vector of actions of the other players.

Definition: The action $a_i \in A_i$ of player i in the strategic game $\langle N, (A_i), (u_i) \rangle$ is weakly dominated if there is a mixed strategy dominated if there is a mixes strategy α_i of player i such that $U_i(a_{-i}, \alpha_i) \geq u_i(a_{-i}, a_i)$ for all $a_{-i} \in A_{-i}$ and $U_i(a_{-i}, \alpha_i) > u_i(a_{-i}, a_i)$ for at least one $a_{-i} \in A_{-i}$.

By the Lemma an action that is weakly dominated but not strictly dominated is a best response to some belief. This fact makes the argument against using a weakly dominated action weaker than that against using a strictly dominated action. Yet since there is no advantage to using a weakly dominated action, it seems very natural to eliminate such actions in the process of simplifying a complicated game.

The notion of weak domination leads to a procedure analogous to iterated elimination of strictly dominated actions. However, this procedure is less compelling since the set of actions that survive iterated elimination of weakly dominated actions may depend on the order in which actions are eliminated, as the following two-player game shows:

	L	R
T	$1,1$	$0,0$
M	$1,1$	$2,1$
B	$0,0$	$2,1$

The sequence in which we first eliminate T (weakly dominated by M) and then L (weakly dominated by R) leads to an outcome in which player 2 chooses R and the payoff profile is $(2,1)$. On the other hand, the sequence in which we first eliminate B (weakly dominated by M) and then R (weakly dominated by L) leads to an outcome in which player 2 chooses L and the payoff profile is $(1,1)$.

1: Find the set of rationalizable actions of each player in the two-player game

$$\begin{array}{cccc} 0,7 & 2,5 & 7,0 & 0,1 \\ 5,2 & 3,3 & 5,2 & 0,1 \\ 7,0 & 2,5 & 0,7 & 0,1 \\ 0,0 & 0,-2 & 0,0 & 10,-1 \end{array}.$$

2: (Cournot duopoly) Consider the strategic game $\langle\{1,2\},(A_i),(u_i)\rangle$ in which $A_i = [0,1]$ and $u_i(a_1,a_2) = a_i(1-a_1-a_2)$ for $i = 1,2$. Show that each player's only rationalizable action is his unique Nash equilibrium action.

3: Consider the game where each of n players have to announce a number in $\{0,\ldots,100\}$ and $u_i(a_1,\ldots,a_n) = 1$ if $(2/3)[\sum_{j=1}^{n} a_j/n] = a_i$. ($[x]$ is the largest integer $n \leq x$) and $u_i(a_1,\ldots,a_n) = 0$ otherwise. Show that the only rationazable action is 0.

4: Consider a variant of the Hoteling game in which there are two players, sellers and positions $1,2,..,7$. Consumers are uniformly located at the seven positions. The consumers at a certain position chooses the seller who is closer to its location (and are split equally between the two sellers if they are equally distanced).

Show that the only outcome that survives iterated elimination of weakly dominated actions is that in which both players choose the position 4.

5: A strategic game is dominance solvable if all players are indifferent between all outcomes that survive the iterative procedure in which all the weakly dominated actions of each player are eliminated at each stage.

Consider a game where each of two players announces a nonnegative integer equal to at most 100. If $a_1 + a_2 \leq 100$, where a_i is the number announced by player i, then each player i receives payoff of a_i. If $a_1 + a_2 > 100$ and $a_i < a_j$ then player i receives a_i and player j receives $100 - a_i$; if $a_1 + a_2 > 100$ and $a_i = a_j$ then each player receives 50. Show that the game is dominance solvable and find the set of surviving outcomes.