Course: **Microeconomics, New York University**
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Exam: **Mid-term, October 2004**

A Suggestive Solution

Question 1

a) No. Consider the case $X = \{x_1, x_2, x_3\}$, $x_1 \succ x_2 \succ x_3 \succ x_1$. Take the forward procedure. Let $A = \{x_2, x_3\}, B = \{x_1, x_2, x_3\}$. Then by forward procedure $B \to \{x_1, x_3\} \to x_3$, $A \to \{x_2, x_3\} \to x_2$. For backward procedure take $A = \{x_1, x_3\}, B = \{x_1, x_2, x_3\}$. Then by backward procedure $B \to \{x_1, x_2\} \to x_1$, $A \to x_3$.

b) The example above for the set B.

c) Transitivity.

If transitivity does not hold then there is a cycle on three elements $\{a, b, c\}$ with $n(a) < n(b) < n(c)$. It must be that either $a \succ b \succ c \succ a$ or $a \succ c \succ b \succ a$.

In the first case, given the set $\{a, b, c\}$, a is chosen by the forward procedure and c by the backwards procedure.

In the second case, given the set $\{a, b, c\}$, c is chosen by the forward procedure and a by the backwards procedure.

If transitivity holds then for any set A both procedures yield the most preferred element of the set A which is well defined in this case.

Question 2

a) The consumer's problem.

$\max_{(x_1,x_2)} u(x_1, x_2)$

s.t. $p(x_1) \cdot x_1 + x_2 \le w$

b) Let (x_1^*, x_2^*) be a solution of the problem. By monotonicity $x_2^* = w - p(x_1^*)x_1^*$.
Let $z_1 > x_1^*$ and $z_2 = w - p(x_1^*)x_1$ be a candidate solution of the problem

$\max_{(x_1,x_2)} u(x_1, x_2)$

s.t. $p(x_1^*) \cdot x_1 + x_2 \le w$

The bundle (x_1^*, x_2^*) is feasable in the two problems. It is sufficent to show that (z_1, z_2) was also feasible before the modification and therefore cannot yield higher utility then (x_1^*, x_2^*).

Since $0 < x_1^* < z_1$ we can represent $x_1^* = \lambda z_1 + (1-\lambda)0 = \lambda z_1$ for some $\lambda \in (0,1)$. By the concavity of the function $C(x_1) = p(x_1)x_1$ we have
$C(x_1^*) = p(x_1^*)x_1^* \ge \lambda C(z_1) + (1-\lambda)C(0) = \lambda p(z_1)z_1$ which implies that
$p(x_1^*)z_1 = p(x_1^*)x_1^*/\lambda \ge p(z_1)z_1$ and thus $p(z_1)z_1 + z_2 = p(z_1)z_1 + w - p(x_1^*)z_1 \le w$.

c1) Let (x_1^*, x_2^*) be a solution of the problem

$\min_{(x_1,x_2)} p'(x_1)x_1 + x_2$

s.t. $u(x_1, x_2) \ge u$

The minimal expense for the problem with the price schedule p is not more than
$p(x_1^*)x_1^* + x_2^* \le p'(x_1^*)x_1^* + x_2^* = e(p', u)$ thus $e(p, u) \le e(p', u)$

c2) Let (x_1^*, x_2^*) be the solution to the problem with the price schedule $\lambda p + (1-\lambda)p'$. We

have
$e(\lambda p + (1-\lambda)p', u) = (\lambda p + (1-\lambda)p')(x_1^*)x_1^* + x_2^* = \lambda p(x_1^*)x_1^* + \lambda x_2^* + (1-\lambda)p'(x_2^*)x_2^* + (1-\lambda$
$\lambda e(p,u) + (1-\lambda)e(p',u)$.

Question 3

a) Properties A2-5 are trivially satisfied. For A1 note that if v is a convex function with $v(0) = 0$ we have $v(a+b) + v(0) \geq v(a) + v(b)$. (Note that $a = \frac{b-a}{b}0 + \frac{a}{b}b$, $b = \frac{b-a}{b}(a+b) + \frac{a}{b}a$. Therefore $v(a) \leq \frac{b-a}{b}v(0) + \frac{a}{b}v(b)$, $v(b) \leq \frac{b-a}{b}v(a+b) + \frac{a}{b}v(a)$. Thus, $v(b) + v(a) \leq \frac{a}{b}(v(a) + v(b)) + \frac{b-a}{b}(v(a+b) + v(0))$, and therefore $v(a) + v(b) \leq v(a+b) + v(0))$.

By induction $v(\sum_{k=1}^{K} x_k) \geq \sum_{k=1}^{K} v(x_k) - Kv(0) = \sum_{k=1}^{K} v(x_k)$.

b)

1. all but A1. Let $U(x_1,..,x_K)$ = the number of elements in the sequence x which are strictly positive.

2. all but A2. For v convex, strictly increasing, $v(0) = 0$ set $U(x_1,..,x_K) = \sum_{k=1,..,K} v(x_k) + v(x_1)$. (the first element in the sequence gets double weight)

3. all but A3. Let v be a decreasing convex function with $v(0) = 0$ and let $U(x_1,..,x_K) = \sum v(x_k)$. (another example, the average of the positive amounts).

4. all but A4. The preferences induced from the lexicographic preferences over $(max_k\{x_k\},$ number of k with $x_k > 0)$.

5. all but A5. The preferences induced from the lexicographic preferences over $(\sum_{k=1}^{K} x_k, -K)$.