Normative Equilibrium: The permissible and the forbidden as devices for bringing order to economic environments*

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ABSTRACT: We introduce the notion of a *normative equilibrium* which brings harmony to "general equilibrium"- like environments. Norms stipulate what is permissible and what is forbidden. The main solution concept is a maximally permissive set of alternatives together with a feasible profile of optimal choices. The norms are uniform and play a role analogous to that of price systems in competitive equilibrium. The solution concept is analysed and applied to a variety of economic settings.

KEYWORDS: Normative Equilibrium, Envy-Free, General Equilibrium, Convexity.

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1. Introduction

We ask the reader to forget for a moment the economic models he is familiar with and to imagine we are back to the beginning. Imagine we restart the modelling of social arrangements that can bring harmony to environments with conflicting individual interests due to economy-wide feasibility constraints. We believe that before coming up with ideas like ownership and trade under linear prices, we would think about simpler institutions, such as social norms that restrict what individuals are permitted to do.

In what follows, we investigate *social norms* that specify which alternatives are permitted and which are forbidden. The social norms we have in mind, like competitive prices, apply uniformly to all individuals in the society. Our proposed solution concept of *maximal normative equilibrium* is a set of permissible alternatives combined with a feasible profile of permissible alternatives, such that (i) each agent's alternative is optimal from among those permitted and (ii) there is no larger set of permissible alternatives from which a feasible assignment that satisfies (i) can be made.

We have in mind that in response to disharmony, the social norm adjusts until harmony is achieved. We also assume that if there are unnecessarily forbidden alternatives such that harmony would prevail even if they were allowed, then the set of normative alternatives will expand to permit them. We do not assume that there is an authority that determines these norms, but rather that the same invisible hand that calculates equilibrium prices so effectively is also able to determine a minimal set of forbidden alternatives for which optimal individualistic behavior is compatible. In several examples, we will demonstrate how a tatonnement-like process can work to achieve harmony.

There are three leading scenarios we have in mind:

1. A bundle is to be distributed among a group of people. Social norms determine the set of permissible bundles from which group members can choose. The set of permissible bundles should be such that total demand does not exceed the total bundle.

An example of such a norm is derived from the notion of egalitarian competitive equilibrium in which all agents are initially endowed with an equal share of the total bundle. The equilibrium prices and the equal endowment define a common budget set. This budget set together with the individuals' optimal choices from it constitute a normative equilibrium.

2. The survival of a group depends on the ability of its members to reach one another within a certain amount of time in the case of danger. Therefore, they need to live within a certain distance from one another. The members have preferences over where they live and the equilibrium normative set specifies where they are allowed to. An equilibrium imposes minimal restrictions on the permitted locations so that individuals' choices fulfill the closeness requirement.

Alternatively, one can think of a political party in which each member chooses a platform. In order to maintain party unity, the chosen platforms must be sufficiently close to one another. We look for a broadest set of permitted platforms that maintain party harmony.

We will see that in this environment both fundamental welfare theorems hold. That is, every maximal normative equilibrium outcome is Pareto-efficient and every Pareto-efficient profile of locations is a maximal normative equilibrium outcome.

3. A group owns limited quantities of a number of resources. Each member of the group can choose a quantity of only one resource and feasibility requires that the total demand for each resource not exceed its available quantity. A norm specifies the maximal quantity that each member is allowed to consume from each resource.

We will see that in a maximal normative equilibrium some of the resources might not be exhausted and therefore the result may be Pareto-inefficient. Nonetheless, we will see that in equilibrium the amount of inefficiency is "small".

We will present existence theorems which cover the above examples. However, the social institution which determines what is allowed and what is not cannot always achieve harmony. For example, if all agents share the same preferences and feasibility requires that all agents cannot make the same choice, then no normative equilibrium exists.

In the rest of the paper, we formally define the solution concept, prove some general results about its nature, especially its connection to Pareto optimality, and apply the concept to seven economic examples.

2. Maximal Normative Equilibrium

We start with the definitions of an economy and the basic solution concept:

Definition 1 *An **economy** is a tuple $\langle N, X, \{\succsim^i\}_{i\in N}, F\rangle$ where N is a finite set of agents, X is a set of alternatives, $\succsim^i$ is agent i's preference and F is a subset of X^N (the set of profiles). The set F contains all profiles of choices that are feasible.*

Definition 2 *A **Normative Equilibrium** (NorE) is a pair $\langle Y, (y^i)\rangle$ where $Y \subseteq X$ and (y^i) is a profile of elements in Y satisfying that:*

(i) for all i, y^i is a $\succsim^i$-maximal alternative in Y; and

(ii) $(y^i) \in F$.

*We refer to the two components of a NorE as a normative set (**NorE set**) and a normative outcome (**NorE outcome**). In the case that the set X is a subset of a Euclidean space, we additionally require that the NorE set be closed.*

Thus, normative equilibrium has a structure analogous to that of competitive equilibrium. The price system that is applied uniformly to all economic agents in the competitive case is replaced by a normative set which is applied uniformly to all agents.

We view the normative set as being generated by forces analogous to the "invisible hand" which guides an exchange economy towards competitive equilibrium through small price changes until aggregate demand equals the aggregate endowment. Likewise, the normative set is not chosen by some authority but rather evolves through an invisible-hand-like process in which alternatives are slowly added or removed to the normative sets until harmony is achieved (that is, the profile of chosen alternatives is compatible with the feasibility constraint). Typically we focus on those normative sets with the feature that whatever can be added to the normative set without disturbing the harmony will indeed be added. This brings us to the following central solution concept:

Definition 3 *A NorE $\langle Y, (y^i)\rangle$ is a **maximal NorE** if there is no NorE $\langle Z, (z^i)\rangle$ such that $Z \supset Y$.*

Notice, in the above definition, the larger NorE may have a different profile of optimal choices than the smaller NorE. The maximality of the NorE set among those which induce harmony in the society captures a liberty concept: if a normative equilibrium evolves which is overly strict, in the sense that loosening constraints does not interfere with feasibility, then those constraints are loosened.

In any normative equilibrium, all agents face the same choice set and since all individuals are rational, no agent strictly desires the alternative chosen by another. In other words, the profile of choices is "envy-free" (see Foley (1967)). Formally:

Definition 4 *A profile* (y^i) *is* ***envy-free*** *if for all* $i \neq j$, $y^i \succsim^i y^j$.
A profile (y^i) *is* ***strictly envy-free*** *if for all* $i \neq j$, $y^i \succ^i y^j$.

A maximal NorE outcome can be Pareto-inefficient. To see this, consider the "housing economy" where $N = \{1,2\}$, $X = \{a,b,c,d,e\}$ (the set of houses) and F is the set of all profiles that assign each house at most once. Suppose that the agents' preferences are $a \succ^1 b \succ^1 c \succ^1 d \succ^1 e$ and $a \succ^2 c \succ^2 b \succ^2 e \succ^2 d$. Every Pareto-efficient allocation assigns a to one of the agents. But in any NorE $\langle Y,(y^i)_{i\in N}\rangle$ it must be that $a \notin Y$ (because both agents would demand it). One NorE is $Y = \{d,e\}$, $y^1 = d$, $y^2 = e$, which is Pareto-inefficient and not a maximal NorE. The unique maximal NorE is $Y = \{b,c,d,e\}$ and $y^1 = b$, $y^2 = c$ which Pareto-dominates the other NorE, but is still Pareto-inefficient.

Although a maximal NorE outcome may be Pareto-inefficient, the following proposition demonstrates that the maximal normative equilibrium outcomes are precisely the profiles that are Pareto-efficient *among the set of envy-free profiles*. An immediate consequence is that any profile that is both envy-free and Pareto-efficient (among *all* feasible profiles) is a maximal NorE outcome.

Proposition 1 *A profile is a maximal NorE outcome if and only if it is Pareto-efficient among all feasible envy-free profiles.*

Proof. Let $\langle Y,(y^i)\rangle$ be a maximal NorE outcome. The profile (y^i) is envy-free. If it is Pareto-inefficient among the feasible envy-free profiles, then there is a feasible envy-free profile (z^i) that Pareto-dominates (y^i). Clearly, $\langle Y \cup \{z^1,..,z^n\},(z^i)\rangle$ is a NorE and for at least one agent i, $z^i \succ^i y^i$ and therefore $z^i \notin Y$, contradicting the maximality of $\langle Y,(y^i)\rangle$.

Let (y^i) be Pareto-efficient among the feasible envy-free profiles. Define $Y = \cup_i\{y^i\}\cup \{x|$ for all i, $y^i \succsim^i x\}$. Clearly, $\langle Y,(y^i)\rangle$ is a NorE. Suppose that $\langle Z,(z^i)\rangle$ is a NorE with

$Z \supset Y$. Then, $z^i \succsim^i y^i$ for all i and (z^i) is envy-free. Take an $x \in Z - Y$. There is an agent j for whom $x \succ^j y^j$ and consequently, $z^j \succsim^j x \succ^j y^j$. Therefore, (z^i) is an envy-free profile which Pareto-dominates (y^i), contradicting (y^i) being Pareto-efficient among the envy-free profiles. Thus, no such $\langle Z, (z^i)\rangle$ can exist and $\langle Y, (y^i)\rangle$ is a maximal NorE. □

A condition which guarantees that any maximal normative equilibrium outcome is Pareto-efficient is given in the following proposition. The condition requires that if a profile is feasible, then so is any profile for which one agent adopts the alternative chosen by another agent instead of his own. Examples E and F in Section 4 satisfy this condition.

Proposition 2 *Assume that F satisfies the following imitation property: if $a \in F$, then any profile b, which differs from a only in that there is a unique i for which $b^i \neq a^i$ and $b^i = a^j$ for some j, is also in F. Then, a profile is a maximal normative equilibrium outcomes if and only if it is Pareto-efficient.*

Proof. Let $\langle Y, (y^i)\rangle$ be a maximal NorE with a Pareto-inefficient outcome. Then, there is a feasible profile (z^i) which Pareto-dominates (y^i). We will define an envy-free feasible profile (x^i) that also Pareto-dominates (y^i) and then use it to construct a larger NorE.

Assign x^1, a $\succsim^1$-maximal alternative from $\{z^1, ..., z^N\}$, to agent 1. Likewise, assign x^2, a $\succsim^2$-maximal alternative from $\{x^1, z^2, \ldots, z^N\}$, to agent 2, and so on to form the profile (x^i). In this construction: 1) the profile selected at each stage is feasible (because of the imitation property of F) and weakly Pareto-dominates the previous one and 2) at stage j, no agent $i \leq j$ envies any other agent. Therefore, (x^i) is envy-free and Pareto-dominates (y^i) (because it is a Pareto improvement at each stage), violating Proposition 1.

The other direction follows from Proposition 1 because under the imitation condition on F, every Pareto-efficient profile is envy-free and therefore is efficient among the envy-free allocations. □

Example A *Splitting cakes economy*

There are two goods, 1 and 2, with total bundle (α,β). Each agent can choose a quantity of only one of the two goods. Thus, the set of alternatives X consists of all objects of the type $(a,0)$ and $(0,b)$. A profile is feasible if for each good the sum of the agents' assignments of that good does not exceed its total amount. Agents have continuous and strictly monotonic preferences. A social norm determines what quantities are allowed to be chosen from each good and in equilibrium it guarantees that total demands do not exceed the available quantities.

In this example, any maximal NorE set is comprehensive. The following result shows that there is a unique specification of maximal "quotas" in which the demand for each of the goods does not exceed the supply. Note that the result applies equally to any economy of K goods where each agent consumes only one good.

Claim A *In any splitting cakes economy:*

(i) There is a unique maximal NorE set.

(ii) In any maximal NorE, at least one of the goods is fully consumed.

(iii) In any maximal NorE, each good is allocated in a fixed quantity and if a good is not fully allocated, then the unallocated portion is not larger than that fixed quantity.

Proof. **Step 1:** If $\langle Y,(y^i)\rangle$ and $\langle Z,(z^i)\rangle$ are NorE, then there is a NorE with the NorE set $Y\cup Z$.

For any closed subset W of X, let $a_W=\max(a:(a,0)\in W)$ and $b_W=\max(b:(0,b)\in W)$. If $a_Y\geq a_Z$ and $b_Y\geq b_Z$, then $\langle Y\cup Z,(y^i)\rangle$ is a NorE. Similarly, if $a_Z\geq a_Y$ and $b_Z\geq b_Y$, then $\langle Y\cup Z,(z^i)\rangle$ is a NorE. Otherwise, without loss of generality, we have $a_Y>a_Z$ and $b_Z>b_Y$. Then, $a_{Y\cup Z}=a_Y$ and $b_{Y\cup Z}=b_Z$. Take the NorE set to be $Y\cup Z$. Total consumption of the first good is then bounded above by $\#\{i:(a_Y,0)\succsim^i(0,b_Z)\}*a_Y\leq\#\{i:(a_Y,0)\succ^i(0,b_Y)\}*a_Y\leq\alpha$. An analogous argument applies to the second good, and thus the total consumption of both goods is less than the total endowment.

Step 2: Existence of a maximal NorE set.

Let $a^*=\sup\{a_Y:Y\text{ is a NorE set in some NorE}\}$ and similarly define b^*. We now show that $M=\{(x_1,x_2)\in X:x_1\leq a^*,\ x_2\leq b^*\}$ is a NorE set in some NorE. By definition, there are sequences of NorE sets (Y_n) and (Z_n) such that $a_{Y_n} \to a^*$ and $b_{Z_n} \to b^*$. By Step 1, $W_n = Y_n \cup Z_n$ is a sequence of NorE sets and $(a_{W_n}, b_{W_n}) \to (a^*, b^*)$. This sequence has a subsequence in which a fixed set of agents Q choose $(a_{W_n}, 0)$ and the remainder $N-Q$ choose $(0, b_{W_n})$. We now show that $\langle M, (m^i)\rangle$ is a NorE where $m^i = (a^*, 0)$ if $i \in Q$ and $m^i = (0, b^*)$ if $i \in N-Q$. To verify feasibility of (m^i), notice that $a_{Wn} \cdot |Q| \leq \alpha$ and therefore $a^* \cdot |Q| \leq \alpha$ and similarly for the other good. To verify individual optimality, notice that since $(a_{Wn}, 0) \succsim^i (0, b_{W_n})$ for all $i \in Q$, then by continuity $(a^*, 0) \succsim^i (0, b^*)$ for all $i \in Q$. Similarly, $(0, b^*) \succsim^i (a^*, 0)$ for all $i \in N-Q$.

Step 3: M is the unique maximal NorE set.

Given any NorE $\langle Y, (y^i)\rangle$, by the definition of a^* and b^*, it is the case that $a^* \geq a_Y$ and $b^* \geq b_Y$. Thus, $Y \subseteq M$.

Step 4: In a maximal NorE, at least one of the goods is fully consumed.

Assume otherwise. Let $\langle M, (y^i)\rangle$ be a maximal NorE where k agents are allocated $(a^*, 0)$ while $N-k$ agents are allocated $(0, b^*)$ and no good is fully consumed, that is, $ka^* < \alpha$ and $(N-k)b^* < \beta$. Take $a' = \alpha/k$ and $b' = \beta/(N-k)$ and notice that $a^* < a'$ and $b^* < b'$. Define $Y^\lambda = \{(x_1, x_2) \in X : (x_1, x_2) \leq (\lambda a' + (1-\lambda)a^*, \lambda b^* + (1-\lambda)b')\}$. When $\lambda = 0$, at least $N-k$ agents prefer $(0, b_{Y^0} = b')$ to $(a_{Y^0} = a_Y, 0)$. When $\lambda = 1$, at least k agents prefer $(a_{Y^1} = a', 0)$ to $(0, b_{Y^1} = b^*)$. By continuity, there is some intermediate λ where at least k agents weakly prefer $(\lambda a' + (1-\lambda)a^*, 0)$ to $(0, \lambda b^* + (1-\lambda)b')$ and at least $N-k$ agents prefer $(0, \lambda b^* + (1-\lambda)b')$ to $(\lambda a' + (1-\lambda)a^*, 0)$. Then, Y^λ is a larger NorE set and therefore M could not have been maximal.

Step 5: For any maximal NorE, if a good is not fully consumed, then its unallocated portion is not larger than each allocated portion of that good.

Suppose $\langle M, (y^i)\rangle$ is a maximal NorE where k agents are allocated $(a^*, 0)$ and $\alpha - ka^* > a^*$. If every agent who is allocated $(0, b^*)$ strictly prefers $(0, b^*)$ to $(a^*, 0)$, then a^* can be slightly increased without changing consumption patterns, thus violating the maximality of Y. Otherwise, for at least one i, $y^i = (0, b^*)$ and $(a^*, 0) \sim_i (0, b^*)$. Then, $\langle M, (z^i = (a^*, 0), z^{-i} = y^{-i})\rangle$ is also a NorE where no good is fully consumed. By Step 4, there is a NorE with a larger NorE set, contradicting the maximality of $\langle M, (y^i)\rangle$. □

The unique equilibrium for this economy has an intuitive structure. All agents face a choice between a fixed quantity of each good. The unique equilibrium can emerge through a dynamic process where the limits of permitted consumption are adjusted according to excess supply or demand. The main economic insights are that an equilibrium is almost Pareto-efficient in the sense that at most one "portion" of one of the goods is wasted and this waste is necessary for harmony.

Example B *Quorum economy*

Let X be a finite set of clubs. Each agent chooses one club to be a member of. Feasibility requires that club x is empty or chosen by at least $m_x \leq N$ members. If each agent were to choose his most beloved club, then typically there will be insufficiently occupied non-empty clubs. The role of the maximal normative equilibrium is to facilitate coordination with minimal restrictions on the agents.

There are many normative equilibria (for example, any set $Y = \{x\}$ combined with all agents choosing x). Since there are finitely many NorE sets, a maximal normative equilibrium always exists as well.

Claim B *In some quorum economies, every maximal normative equilibrium is Pareto-inefficient.*

Proof. Consider the quorum economy with $N = 6$, $X = \{a, b, c\}$, and $m_x = 3$ for all x. Suppose that agents 1 and 2 have the preferences $a \succ b \succ c$, agents 3 and 4 have the preferences $b \succ c \succ a$ and agents 5 and 6 have the preferences $c \succ a \succ b$. There is no feasible allocation with three active clubs. Furthermore, there is no normative set with two clubs because one of the clubs is preferred by only two agents, violating feasibility. Therefore, any normative set has only one club, but any constant profile is Pareto-inefficient since there is a different club that is strictly preferred by four agents, and so there is a Pareto improvement whereby exactly three of them move. Thus, every NorE outcome is Pareto-inefficient. $\square$

Two natural forces may lead to equilibrium in this example. First, a club that does not have a quorum is removed from the normative set, which leads to feasibility. Second, from time to time, a forbidden club may become socially acceptable for a while until it is determined whether it attracts a sufficient crowd. Resolving the conflict in this economy requires either coordination in the agents' moves or in their preferences (i.e. single-peakedness). Without such coordination, the above dynamic may lead to cyclical behavior and instability.

Of special interest are the *Euclidean economies* in which the set of alternatives is embedded in a Euclidean space. In such economies, we introduce standard restrictions of differentiability, closedness and convexity on the parameters of the model (the set of alternatives, the preference relations and the feasibility set):

Definition 5 *An economy $\langle N, X, \{\succsim^i\}_{i\in N}, F\rangle$ is a* ***Euclidean economy*** *if:*

(i) The set X is a closed convex subset of some Euclidean space.

(ii) The preferences $\{\succsim^i\}_{i\in N}$ are continuous and convex.

(iii) The feasibility set F is closed and convex.

We say that a Euclidean economy is ***differentiable*** *if the preferences are strictly convex and differentiable (differentiable preferences have differentiable utility representations or more generally satisfy the condition suggested in Rubinstein (2007)).*

The following proposition demonstrates that for a Euclidean economy, when X is compact and F is closed under permutations (anonymity), then there exists a maximal NorE.

Proposition 3 *For Euclidean economies, if F is compact and closed under permutations, then a maximal normative equilibrium exists.*

Proof. Let EFF be the set of envy-free feasible profiles. To see that EFF is not empty, let $(x^i) \in F$ be a feasible profile. By assumption, all permutations of this profile are in F as well. The average of these permutations is a constant profile ($y^i = y^*$) and it is in F because F is convex. Clearly this profile is envy-free and thus it is in EFF.

Since each preference $\succsim^i$ is continuous and X is a subset of a Euclidean space, there is a continuous utility function u^i representing $\succsim^i$. Also, by continuity of the preferences and F being closed, the set EFF (which is defined by weak inequalities) is closed. Since F is compact, the set EFF is also compact. Thus, there is at least one profile $z \in EFF$ that maximizes $\sum u^i(x^i)$ over EFF and therefore this profile z is Pareto-efficient in EFF. By Proposition 1, z is a maximal NorE outcome. $\square$

Proposition 1 characterizes maximal normative equilibrium outcomes as being precisely the Pareto-efficient allocations from among the envy-free allocations, but does not guarantee overall Pareto efficiency. For discrete environments, we have seen examples of maximal NorE outcomes that are Pareto-inefficient. Furthermore, example A was an economy with Euclidean alternatives in which a maximal NorE outcome can be Pareto-inefficient. When this occurs, there is an agent who is indifferent between his assigned alternative and someone else's. The next proposition shows that for differentiable Euclidean economies, the gap between maximal NorE outcomes and Pareto efficient profiles is within the set of envy-free allocations with indifferences.

Proposition 4 *In a Euclidean economy, a strictly envy-free profile is a maximal normative equilibrium outcome if and only if it is (overall) Pareto-efficient.*

Proof. One direction follows directly from Proposition 1. If a profile is both envy-free and Pareto-efficient, then it is Pareto-efficient among envy-free profiles, and consequently, it is a max NorE profile.

For the other direction, assume $\langle Y, (y^i) \rangle$ is a maximal NorE and $y^i \succ^i y^j$ for all $i \neq j$. If (y^i) is Pareto-inefficient, then there is (z^i) such that $z^i \succsim^i y^i$ for all i and $z^k \succ^k y^k$ for some k. By convexity, any convex combination $(\lambda z^i + (1-\lambda) y^i)$ weakly Pareto-dominates (y^i). Let $\bar{\lambda} < 1$ be the largest λ for which $\lambda z^i + (1-\lambda) y^i \sim_i y^i$ for all i. By the continuity of the agents' preferences, for $\varepsilon > 0$ small enough, the profile $((\bar{\lambda} + \varepsilon) z^i + (1 - \bar{\lambda} - \varepsilon) y^i)$ is envy-free and by definition, it Pareto-dominates (y^i). But, this violates Proposition 1 as (y^i) is then not Pareto-efficient among envy-free profiles.

$\square$

3. Convex Normative Equilibrium

Up to this point, we have not imposed any restrictions on the structure of the normative set. In this section, we study Euclidean economies and require that the normative set be convex, which we suggest is an intuitive condition. There is a natural asymmetry between what is allowed and what is forbidden. Thus, one would certainly conclude that if driving on a highway at 60 mph and at 80 mph are permitted, then driving 70 mph is as well. On the other hand, knowing that driving at 110 mph and at 10 mph are forbidden does not lead one to believe that driving at 60 mph is forbidden. This is consistent with the intuition that forbidden actions are typically extreme in some sense and that the normative set captures some middle ground. In the language of our formal model, this intuition is captured by the requirement that the normative set be convex. We will also demonstrate (in Proposition 7 below) that the convexity requirement is equivalent to the requirement that the normative set be defined by a small system of linear inequalities, in which case the convexity requirement essentially captures the requirement that the normative set be simple to describe.

Definition 6 *For Euclidean economies, a* ***convex normative equilibrium*** *is a NorE* $\langle Y,(y^i)\rangle$ *such that* Y *is closed and convex. A* ***maximal convex normative equilibrium*** *is a convex NorE* $\langle Y,(y^i)\rangle$ *such that there is no other convex NorE* $\langle Z,(z^i)\rangle$ *with* $Z \supset Y$.

One direction of Proposition 1 implies that a Pareto-efficient profile which is a NorE outcome is also a *maximal* NorE outcome. Proposition 4 below is analogous: if a Pareto-efficient profile is a convex NorE outcome, then it is also a maximal convex NorE outcome. The other direction of Proposition 1 states that any maximal NorE outcome is Pareto-efficient among the envy-free profiles. This cannot generally be extended to the case of maximal convex normative equilibria (see Example E). Note that the following proposition does not rely on F being closed or convex.

Proposition 5 *For Euclidean economies, if (y^i) is a convex NorE outcome and is Pareto-efficient (even if only among the convex NorE outcomes), then (y^i) is a maximal convex NorE outcome.*

Proof. We first show that there is a set Y^* such that $\langle Y^*, (y^i)\rangle$ is a convex NorE and is maximal among all convex Y for which $\langle Y, (y^i)\rangle$ is a NorE. To do so, we apply Zorn's Lemma. (A reminder: A chain is a completely ordered subset of P. Given a partially ordered set P , if every chain in P has an upper bound in P, then the set P has at least one maximal element.) Here, P consists of all sets Y for which $\langle Y, (y^i)\rangle$ is a convex NorE and the partial order is $\supseteq$.

In order to show that any chain C of elements in P has an upper bound in P, it suffices to show that $\overline{U}$, the closure of the union of the sets in C, is in P. To be in P means that $\overline{U}$ is a closed convex set for which $\langle \overline{U}, (y^i)\rangle$ is a NorE. By definition, $\overline{U}$ is closed. To show that $\overline{U}$ is convex, it suffices to show that U, the union of the sets in C, is convex. Given any two points x, y in U, there is some $Y \in C$ such that $x, y \in Y$ and therefore all points between x and y are in Y and therefore in U. To see that $\langle \overline{U}, (y^i)\rangle$ is a NorE, by continuity of preferences it suffices to show that for each i the element y^i is $\succsim^i$ top-ranked in U. Suppose that there is an $x \in U$ such that $x \succ^i y^i$ for some i. Then, there is some $Y \in C$ such that $x \in Y$, contradicting that $\langle Y, (y^i)\rangle$ is a NorE.

To prove the maximality of Y^*, suppose that there is another convex NorE $\langle Z, (z^i)\rangle$ such that $Z \supset Y^*$. As $\langle Z, (z^i)\rangle$ is a NorE, it must be that $z^i \succsim^i y^i$ for all i. If $z^i \sim^i y^i$ for all i, then $\langle Z, (y^i)\rangle$ is a convex NorE, contradicting the maximality of Y^*. On the other hand, if $z^i \succsim^i y^i$ for all i with at least one strict inequality, then the profile (z^i) is a convex NorE outcome that Pareto-dominates (y^i), contradicting (y^i) being Pareto-efficient. □

The following proposition demonstrates that when X is compact and F is closed under permutations, then there exists a maximal convex normative equilibrium.

Proposition 6 *For Euclidean economies, if F is compact and closed under permutations, then a maximal convex normative equilibrium exists.*

Proof. Let O be the set of convex NorE outcomes. The set O is not empty, since as in Proposition 3, there is a constant profile $(y^i = y^*)$ in F. Thus, the pair $\langle \{y^*\}, (y^i = y^*)\rangle$ is a convex NorE.

The rest of the proof proceeds as in Proposition 3: The set O is compact, and there is a profile z which is Pareto-efficient in O. By Proposition 5, z is a maximal convex NorE outcome. □

Propositions 3 and 6 establish that for Euclidean economies, when X is compact and F is closed under permutations then both a max NorE and a max convex NorE exist. However, while their proofs are almost identical, they may be quite different, even with respect to outcomes (Claims D and F demonstrate that they are non-nested).

We now turn to the structure of maximal convex NorE sets for Euclidean economies. Recall that any convex set is the intersection of the infinite family of half-spaces that contain it. Proposition 7 states that for differentiable Euclidean economies, any maximal convex NorE set is not just convex but is also a polygon (an intersection of a finite set of half-spaces), with at most one half-space per agent. Thus, the proposition provides a formal basis for the assertion that the convexity of the normative set is a requirement of simplicity.

Proposition 7 *Let $\langle Y,(y^i)\rangle$ be a maximal convex normative equilibrium in a differentiable Euclidean economy and let $J=\{i|\, y^i$ is not $\succsim^i$ -global maximum in $X\}$. Then, there is a profile of closed half-spaces $(H^i)_{i\in J}$, such that $Y=\cap_{i\in J}H^i$.*

Proof. By the differentiability and strict convexity of the agents' preference relations, for every $i\in J$ there is a unique largest closed half-space H^i containing y^i such that y^i is strictly preferred to all other elements in H^i.

Suppose that for some $i\in J$, there is an element $w^i\in Y\backslash H^i$. By the differentiability and strict convexity of i's preferences, and for small $\varepsilon>0$, $\varepsilon w^i+(1-\varepsilon)y^i\succ_i y^i$ and by convexity of Y, $\varepsilon w_i+(1-\varepsilon)y^i\in Y$. Therefore, y^i is not top $\succ_i$-ranked in Y, a contradiction. Thus, $Y\subseteq\cap_{i\in J}H^i$.

Since $\langle Y,(y^i)\rangle$ is a maximal convex NorE it is sufficient to show that $\langle\cap_{i\in J}H^i,(y^i)\rangle$ is a convex NorE. This follows from:

(i) the set $\cap_{i\in J}H^i$ is closed and convex;

(ii) for each agent k, $y^k\in Y\subseteq\cap_{i\in J}H^i$;

(iii) for each $j\notin J$, y^j is a global maximum and therefore preferred by agent j to all other alternatives in $\cap_{i\in J}H^i$;

(iv) for each $j \in J$, y^j is $\succsim^j$-preferred to all other alternatives in H^j and therefore y^j is $\succsim^j$-preferred to all alternatives in $\cap_{i \in J} H^i$. □

4. Examples

We now consider a variety of economic examples illustrating the maximal convex normative equilibrium concept and its relationship to Pareto efficiency.

Example C *Division economy*

A **division economy** $\langle N, X, \{\succsim^i\}_{i \in N}, F \rangle$ is a differentiable Euclidean economy such that:

(i) The set $X = \mathbb{R}^m_+$ is the set of bundles.

(ii) Agents' preferences $\{\succsim^i\}_{i \in N}$ are monotonic (in addition to being continuous, strictly convex and differentiable).

(iii) There is some bundle $e \in \mathbb{R}^m_{++}$ such that $(x^i) \in F$ if and only if $\sum x^i \leq e$.

Note that a division economy is almost identical to an exchange economy with the sole difference being that it lacks an initial distribution of the total bundle e among the individuals.

The following claim demonstrates the special status of the egalitarian equilibrium allocations: they are maximal convex NorE outcomes and are uniquely so among the Pareto-efficient interior profiles. (Note that there may exist an interior maximal convex NorE outcome which is Pareto-inefficient.)

Claim C *For a division economy* $\langle N, X, \{\succsim^i\}_{i \in N}, F \rangle$:

(i) Each egalitarian competitive equilibrium allocation, where at least one agent chooses an interior bundle, together with its associated budget set is a maximal convex NorE.

(ii) If a profile is a maximal convex NorE outcome, Pareto-efficient and consists of interior bundles, then it is an egalitarian equilibrium profile.

(iii) There can exist a Pareto-efficient profile where some agent receives 0 of some good, which is a maximal convex NorE outcome but not an egalitarian equilibrium outcome.

(iv) There can exist a Pareto-inefficient interior profile which is a maximal convex NorE outcome.

Proof. (i) Let $\langle p^*, (y^i)\rangle$ be a competitive equilibrium in the exchange economy where each agent is initially endowed with e/n. Notice that for every agent i, it is the case that $p^* \cdot y^i = p^* \cdot e/n$ and for some agent j, $y^j \gg 0$. The pair $\langle B(p^*, e/n), (y^i)\rangle$ is a convex NorE where $B(p^*, e/n)$ is the budget set given by the equilibrium price vector p^* and the initial endowment e/n. To see that this is indeed a maximal convex NorE, suppose that $\langle Z, (z^i)\rangle$ is a larger convex NorE and let $z \in Z$ be an alternative such that $p^* \cdot z > p^* \cdot e/n$. By the monotonicity, differentiability and strict convexity of $\succsim^j$, and the fact that $y^j \gg 0$ $\varepsilon z + (1-\varepsilon)y^j \succ^j y^j$ for some $\varepsilon > 0$. Furthermore, $\varepsilon z + (1-\varepsilon)y^j \in Z$ by the convexity of Z. Since $\langle Z, (z^i)\rangle$ is a NorE, it is the case that $z^j \succsim^j \varepsilon z + (1-\varepsilon)y^j \succ^j y^j$. Therefore, $z^j \notin Y$ and so $p^* \cdot z^j > p \cdot e/n$. For every other agent i, either $z^i = y^i$ or $p^* \cdot z^i > p^* \cdot y^i$. But, then $\sum_i p^* \cdot z^i > p^* \cdot e$, and so (z^i) is not feasible, a contradiction.

(ii) Let $\langle Y, (y^i)\rangle$ be a maximal convex NorE and (y^i) be interior. For each agent i, the chosen alternative y^i is not $\succsim^i$-globally maximal and thus by Proposition 7, $Y = \bigcap_{i \in N} H^i$ where each H^i is a half-space. Since for each agent $y^i \in \mathbb{R}^m_{++}$ and the allocation is Pareto-efficient the half-spaces must be parallel (otherwise, any two agents on non-parallel half spaces could make a Pareto-improving local exchange). By monotonicity, the half-spaces must be identical and equal to $Y = \{x | \lambda x \leq w\}$ for some positive vector λ and a number w. For each i, the bundle y^i is optimal in Y and by monotonicity, $\lambda y^i = w$. Therefore, $\sum_{i \in N} \lambda y^i = nw$, which implies that $\lambda y^i = w = \lambda(\sum_{i \in N} y^i/n) = \lambda(e/n)$. Thus, (y^i) is a competitive equilibrium allocation for the market with prices λ where each agent is initially endowed with e/n.

(iii) Consider the division economy with three agents, $e = (5,5)$ and preferences represented by the following utility functions (a slight modification of the preferences will strengthen the weak convexity of the preference relations to strict convexity.):

$$u^1(x_1, x_2) = x_1 + 5x_2$$
$$u^2(x_1, x_2) = x_1 + x_2$$
$$u^3(x_1, x_2) = 5x_1 + x_2$$

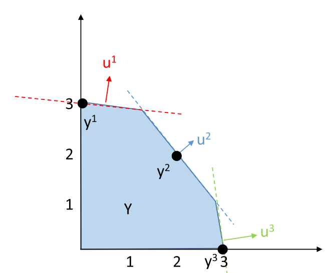

Figure 1. A convex NorE with a non-egalitarian Pareto-efficient assignment (Example C)

Let $y^1 = (0,3)$, $y^2 = (2,2)$ and $y^3 = (3,0)$ and let Y be the set of all bundles below the three indifference curves of individuals $i = 1,2,3$ through y^i (depicted in Figure 1). The pair $\langle Y,(y^i)\rangle$ is a convex normative equilibrium. The allocation (y^i) is Pareto-efficient since any allocation (z^i) which Pareto-dominates (y^i) satisfies that $z_1^i+z_2^i \geq y_1^i+y_2^i$ for all i with at least one inequality. Thus, $\Sigma_i(z_1^i+z_2^i) > \Sigma(y_1^i+y_2^i) = 10$ which is not feasible. The pair $\langle Y,(y^i)\rangle$ is a maximal convex NorE because in any larger convex NorE, the agents' consumption profile (z^i) will be infeasible for the same reason.

(iv) Consider the division economy (depicted in Figure 2) with two agents, two goods, total bundle $e=(1+\sqrt{3},1+\sqrt{3})$ and utility functions $U^1(x_1,x_2)=x_1^{3/4}x_2^{1/4}$ and $U^2(x_1,x_2)=x_1^{1/4}x_2^{3/4}$. The set Y together with $x^1=\left(\sqrt{3},1\right)$ and $x^2=\left(1,\sqrt{3}\right)$ is a convex NorE.

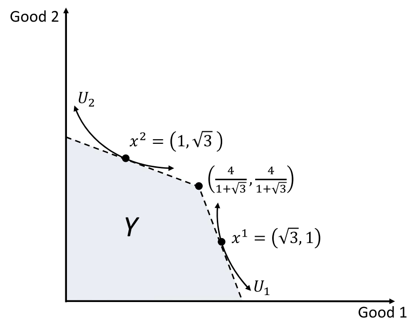

Figure 2. A Pareto Inefficient max convex NorE.

The outcome is inefficient as the bundles are interior and the two agents' indifference curves through their bundles have different slopes. In particular: there is an $\varepsilon > 0$ small enough so that the two agents would happy with agent 1 gets ε more of good 1 and agent 2 gets ε more of good 2.

To see that $\langle Y,(x^i)\rangle$ is a maximal convex NorE, suppose not. In any larger convex NorE, the agents must trade so that agent 1 receives more of good 1 and less of good 2 and agent 2 receives less of good 1 and more of good 2. As exact feasibility need not be maintained, the sum of their trades need only be weakly less than 0. We denote a Pareto dominating allocation as (y^i).

For (y^i) to be the outcome of a larger convex NorE , it must be that agent 1 does not prefer to deviate to any bundle on the line segment between y^1 and $\left(\frac{4}{1+\sqrt{3}},\frac{4}{1+\sqrt{3}}\right)$. Therefore, it must be that the negative slope through agent 1's indifference curve at y^1 (namely, $\frac{3y_2^1}{y_1^1}$) is at least as large as the negative slope of the line segment (namely $\frac{\frac{4}{1+\sqrt{3}}-y_2^1}{y_1^1-\frac{4}{1+\sqrt{3}}}$), that is

$$1+\sqrt{3} \geq \frac{3}{y_1^1}+\frac{1}{y_2^1}.$$

Notice that under the condition $y_1^1 + y_2^1 \leq \sqrt{3}+1$, the RHS is minimized at $(\sqrt{3},1)$, in which case equality occurs. Therefore, for agent 1 to choose a different point and satisfy the above equation, it must be that $y_1^1 + y_2^1 > \sqrt{3}+1$ (and similarly $y_2^1 + y_2^2 > \sqrt{3}+1$), which means that overall consumption must increase, violating feasibility. □

This result provides a new perspective on the egalitarian competitive equilibrium allocations. As long as they involve only interior bundles, a maximal convex normative equilibrium outcome that is also Pareto-efficient must be an egalitarian equilibrium outcome. Note that part (ii) of Claim C is strongly related to a result of Zhou (1992) who showed that an interior feasible allocation is strictly envy-free in all duplicates of an economy if and only if it is an egalitarian competitive equilibrium allocation.

Exchange Economy: The analysis of the division economy can also be applied to the exchange economy with initial distribution (e^i). In this case, $X = R^K$ where a member of X is interpreted as a transfer. The feasibility set F includes profiles (t^i) such that $\sum t^i = 0$ and for each agent i, $t^i + e^i \geq 0$. The preferences of an individual over X are derived from the preferences on the consumption bundles and the initial endowment. Claim C then becomes the conclusion that (i) any competitive equilibrium of the exchange economy is also a maximal convex NorE outcome and (ii) any maximal convex NorE which yields internal bundles for all individuals and is Pareto-efficient is an outcome of a competitive equilibrium of the exchange economy.

Concave Preferences: The existence of a Pareto-efficient envy-free profile may fail in the case of concave preferences (Varian, 1974) and therefore an egalitarian competitive equilibrium may not exist. Nevertheless, Proposition 6 shows that a maximal convex NorE does exist in such a setting, even if neither convexity nor monotonicity hold (continuity is still necessary). By construction, its normative set includes e/N. However, Proposition 7 does not apply in this case, and therefore it cannot be guaranteed that the normative set is a budget set nor even a polygon.

This brings us to the concept of a *budget NorE*, which is a NorE whose normative set is given by a budget set $Y = \{x \in R_+^m : p \cdot x \leq I\}$ for some price vector $p \geq 0$ and $I \geq 0$. A *maximal budget NorE* is a budget NorE for which there is no budget NorE with a strictly larger normative set. In what follows we show that a maximal budget NorE exists and that its budget set is unique. The proof is based on Example A (Splitting Cakes Economy).

The budget NorE notion is an extension of the standard competitive egalitarian equilibrium notion: when the latter exists, it is a maximal budget NorE. While in many economic settings, convexity guarantees "good behavior" of the economic model, here the opposite is true. When preferences are convex, there may be a multiplicity of maximal budget NorE's and when preferences are concave, there is a unique maximal budget NorE.

Claim C (Concavity) For a division economy $\langle N, X, \{\succsim^i\}_{i\in N}, F\rangle$ with strictly concave preferences, there is a maximal budget NorE and its budget set is unique. In this equilibrium, each agent consumes only one good, at least one good is fully consumed and any good that is not fully consumed has leftovers weakly smaller than any agent's non-zero consumption of that good.

Proof. From any budget set, an agent with concave preferences always chooses to consume a quantity of only one good. Thus, agents' behavior is identical to that in the splitting cakes economy (Example A), where agents were physically restricted to choosing a quantity of only one good. Then, each budget NorE corresponds to a splitting cakes NorE, and since there is a unique maximal NorE in Example A, there is also a unique maximal budget NorE. Its normative set is the convex hull of the unique cake-splitting maximal NorE given by the agents' preferences, restricted to the boundary. □

Example D *Give-and-take economy.*

Let $X = [-1, 1]$, where a positive x represents a withdrawal of x from a social fund and a negative x represents a contribution of $-x$ to the fund. Feasibility requires that the social fund is balanced, that is $(x^i) \in F$ iff $\sum_i x^i = 0$. All agents have strictly convex and continuous preferences (i.e. single-peaked) with their ideal denoted by $peak^i$.

The following claim characterizes the unique maximal convex normative equilibrium and shows that it is Pareto-efficient. The second part of the claim demonstrates that if we do not require the normative set to be convex, then Pareto-efficiency does not necessarily result.

Claim D *Consider a give-and-take economy with* $\sum peak^i > 0$.

(i) There is a unique maximal convex normative equilibrium $\langle Y, (y^i)\rangle$ *where* Y *takes the form* $[-1, m]$ *and* (y^i) *is Pareto-efficient.*

(ii) There is a give-and-take economy with a maximal NorE outcome which is Pareto-inefficient.

Proof. (i) Given any set $[-1, m]$ with $m \geq 0$, every agent who wants to give will select his peak, and every agent who wants to take is either at his peak or cannot reach his peak and instead makes do with taking m instead. There is a unique m such that $[-1, m]$ is a NorE set. To see this, denote by $D(m)$ the net amount given and taken by all agents when the NorE set is $[-1, m]$. The function D is weakly increasing and continuous. Moreover, $D(0) \leq 0$, $D(1) = \sum peak^i > 0$, and D is strictly increasing whenever $m < \max\{peak^i\}$. Thus, there is a unique $m^* \geq 0$ for which $D(m^*) = 0$. The set $[-1, m^*]$ combined with the agents' optimal choices is a convex NorE. To see that it is also a maximal convex NorE, notice that any larger closed convex set must be of the form $[-1, m]$ where $m > m^*$. However, for such an m, $D(m) > 0$.

We now show that any other convex NorE set $[x, y]$ is smaller than $[-1, m^*]$. In order for the social fund to be balanced, it must be that $x \leq 0 \leq y$. In equilibrium, agents who wish to give will do so at either their peak or at x if $peak^i < x$. Therefore, the total giving in $[x, y]$ is not more than the total giving in $[-1, m^*]$. Since the social fund is balanced, the total taking in $[x, y]$ is also less than or equal to the total taking in $[-1, m^*]$, and therefore $y \leq m$. Thus, $[x, y] \subseteq [-1, m^*]$.

The equilibrium outcome (y^i) is Pareto-efficient since if (z^i) Pareto-dominates (y^i) it must be that $y^i \leq z^i$ for all i with strict inequality for at least one agent, violating the feasibility constraint.

(ii) Consider the two-agent give-and-take economy with preferences represented by the utilities depicted in Figure 3. (By part (i), $[-1, 0]$ is the unique maximal convex NorE set.) The economy has a maximal NorE that is inefficient: $Y = \{-1, 1\}$ and $y^1 = -1$, $y^2 = 1$. To see that it is maximal, suppose that there is a NorE $\langle Z, (z^i)\rangle$ with $Z \supset Y$.

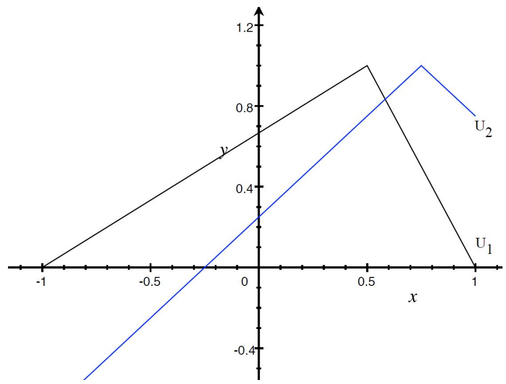

Figure 3. Convex preferences for which there is a Pareto-inefficient maximal NorE (Example D).

Feasibility requires that $z^1 = -z^2$. It must be that $|z^1| \neq 1$ since 1 and -1 are agent 1's two least preferred alternatives in X, and since Z is larger it contains a better alternative. It is impossible that $0 < |z^1| < 1$, because both agents prefer $|z^1|$ to $-|z^1|$. Finally, $z^1 \neq 0$ since if $z^1 = 0$ then $z^2 = 0$, but agent 2 prefers $y^2 = 1$ to $z^2 = 0$. □

This economy was studied by Sprumont (1991) and Richter and Rubinstein (2015). As it turns out, the maximal convex NorE outcome derived above precisely coincides with Sprumont (1991)'s Uniform Rule, which he derives through an axiomatic characterization rather than as an equilibrium outcome.

The give-and-take economy is an economic situation, in which standard market forces do not play a role. An agent can just give or take and there is no room for trade. Claim D demonstrates the effectiveness of norms as a non-market tool for achieving harmony in the absence of markets.

How may a society in disequilibrium reach harmony? Given a convex normative set, it is natural that when there is too much taking then the lower bound on giving is relaxed and if this is not possible, then the upper bound on taking is tightened and vice versa for too much giving. Together with continuous adjustment of the agents' choices, this provides a dynamic process that ends in the maximal convex NorE from any initial starting set.

Example E *The Keeping Close Economy*

Consider a Euclidean economy in which X is a closed convex set of geographic locations or political positions. Motivated by example 2 in the introduction, feasibility requires that each pair of locations be within a distance of 1 from each other.

The following process, in which agents sequentially choose their positions, leads to an envy-free Pareto-efficient outcome: agent 1 selects his top-ranked point $x^1 = peak^1$ from the set $X^1 = X$ and each subsequent agent i selects his most preferred point x^i from the set $X^i = \{x : d(x, x^j) \leq 1, \forall j < i\}$. Furthermore, $\langle X^n, (x^i)\rangle$ is a convex NorE, and since (x^i) is Pareto-efficient, it is also a maximal convex NorE outcome (Proposition 5).

It follows from Proposition 2 (since F satisfies the imitation condition) that the set of maximal NorE outcomes is equal to the set of Pareto-efficient profiles. It will now be shown that the set of maximal convex NorE outcomes for this economy always includes all Pareto-efficient profiles but may also include Pareto-inefficient ones. We provide an example of this economy in which the set of maximal convex NorE outcomes is strictly larger than the maximal NorE outcomes. This expansion occurs because while there are fewer convex normative sets than unrestricted normative sets, there may be more maximal convex normative sets than maximal normative sets.

Claim E *For a keeping close economy:*

(i) Any Pareto-efficient allocation $(y^i) \in F$ *is a maximal convex NorE outcome.*

(ii) A maximal normative convex equilibrium may be Pareto-inefficient.

Proof. (i) Given a Pareto-efficient allocation (y^i), define Y to be the convex hull of the set $\{y^1, ..., y^n\}$. Since (y^i) is feasible, any two locations in Y are within distance 1 of each other (since $d(\sum_i \lambda^i y^i, \sum_j \gamma^j y^j) \leq \max_{i,j}\ d(y^i, y^j)$). Thus, any profile within Y is feasible. By the Pareto efficiency of (y^i) , it must be that y^i is i's best alternative within Y. Therefore, $\langle Y, (y^i)\rangle$ is a convex NorE. By Proposition 5, (y^i) is also a maximal convex NorE outcome.

(ii) An example of a Pareto-inefficient maximal convex NorE outcome occurs in a two-agent economy with alternatives $X = \mathbb{R}^2$ and agents who prefer to be located as close as possible to their ideal points, $peak^1 = (-\frac{1}{2}, -\frac{1}{2})$ and $peak^2 = (\frac{1}{2}, \frac{1}{2})$ (see Figure 4). A maximal convex NorE is the profile $\langle Y, (y^i)\rangle$ where Y is the x-axis, $y^1 = (-\frac{1}{2}, 0)$

and $y^2 = (\frac{1}{2}, 0)$. The pair $\langle W, (w^i) \rangle$ where $w^1 = (-0.3, -0.3)$, $w^2 = (0.3, 0.3)$, and $W = conv(w^1, w^2)$ is a convex NorE with a Pareto-superior outcome. However, there is no larger convex NorE $\langle Z, (z^i) \rangle$. This is because Z must be convex and closed and therefore Z must be a horizontal strip that includes the x-axis. Therefore, $z^1 = (-\frac{1}{2}, a)$ and $z^2 = (\frac{1}{2}, b)$ where $b \geq 0 \geq a$ with at least one strict inequality, and any such z^1, z^2 are strictly more than distance 1 apart.

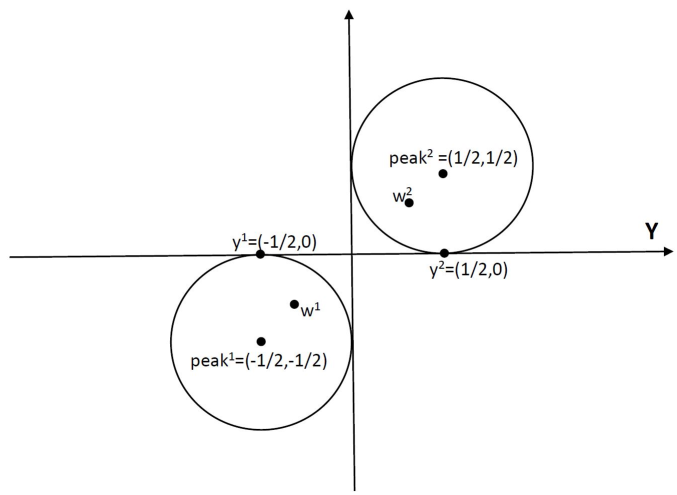

Figure 4. A Pareto-inefficient maximal convex NorE (Example E). □

Example F *Consensus Economy*

A consensus economy is one in which X is a convex set of positions. The set F consists of all the constant profiles. This F satisfies the imitation condition of Proposition 2 and thus a profile is a maximal NorE outcome if and only if it is Pareto-efficient.

The following claim describes the relationship between Pareto efficiency and maximal convex NorE outcomes. It shows that when X is a subset of the real line and agents have convex single-peaked preferences, then a profile is a maximal convex NorE outcome if and only if it is Pareto-efficient, namely a profile of the form $(y^i = \alpha)$ where α is between the left-most and right-most peaks. However, in multidimensional Euclidean economies, maximal convex NorE outcomes may also be Pareto-inefficient.

Claim F *(i) For any consensus economy with $X = [-1, 1]$, a profile is a maximal convex NorE outcome if and only if it is Pareto-efficient.*

(ii) There is a multidimensional consensus economy which has a Pareto-inefficient maximal convex NorE outcome.

Proof. (i) A profile (y^i) is Pareto-efficient if and only if there exists $l \leq y^* \leq r$ such that $y^i = y^*$ for all i, where l is the minimum of the agents' peaks and r is the maximum. Any pair $\langle \{y^*\}, (y^i = y^*) \rangle$ where $l < y^* < r$ is a maximal convex NorE. Furthermore, $\langle [r, 1], (y^i = r) \rangle$ and $\langle [-1, l], (y^i = l) \rangle$ are also maximal convex NorE. In the other direction, if $\langle Y, (y^i) \rangle$ is a maximal convex NorE, then there is a y^* such that $y^i = y^*$ for all i. It cannot be that $y^* > R$, because then $R \notin Y$, and therefore the pair $\langle [R, 1], (z^i = R) \rangle$ would be a larger convex NorE, contradicting Y's maximality. Similarly, y^* cannot be below L.

(ii)

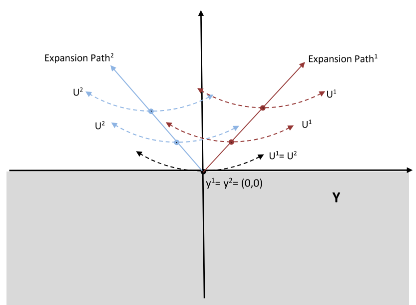

Figure 5. Multidimensional Consensus Economy: Pareto-inefficient maximal Convex NorE

Consider the two-agent consensus economy with $X = R^2$ and agents' preferences represented by $U^1(x_1, x_2) = 2x_2 - |x_2 - x_1|$ and $U^2(x_1, x_2) = 2x_2 - |x_2 + x_1|$. Let $Y = \{(x_1, x_2) : x_2 \leq 0\}$ (see Figure 5). From Y, both agents most prefer $y^1 = y^2 = (0, 0)$. The pair $\langle Y, (y^i) \rangle$ is a convex NorE. If there were a larger convex NorE set Z, it would have to be of the form $\{(x_1, x_2) : x_2 \leq z\}$ with $z > 0$. However, from Z, agent 1 prefers (z, z) and agent 2 prefers $(-z, z)$, and this profile is not in F. However, $(0, 0)$ is Pareto-inefficient since both agents prefer $(0, 1)$ to $(0, 0)$. □

The consensus economy requires that all agents make the same choice. For the one-dimensional case, two forces would lead to an equilibrium: one narrows the normative set when there is disagreement, pushing to consensus. Once a consensus is achieved, the other force extends the normative set and pushes the consensus towards efficiency. Social taboos are a natural non-market mechanism for achieving consensus.

Example G *Near-Average Economy*

Consider a Euclidean economy in which $X = \mathbb{R}$ is a set of positions. Agents have single-peaked preferences. Harmony requires that no agent is an outlier, in the sense that his position is no more than distance 1 from the average. Thus, denoting $Avg(x^i) = \sum x^i/n$, we have $F = \{(x^i) \mid d(x^j, Avg(x^i)) \leq 1 \text{ for all } j\}$. The set F is convex, but does not satisfy the imitation condition of Proposition 2; given a feasible profile, one agent moving to the position of another may cause a third agent to become an outlier.

While our existence result does not apply (since X is not compact), the following claim establishes that maximal convex normative equilibria exist and that their outcomes are a (typically strict) subset of the set of Pareto-efficient profiles.

Claim G *For the near-average economy:*

(i) There is a maximal convex normative equilibrium.

(ii) Every maximal convex NorE outcome is Pareto-efficient.

(iii) There can be Pareto-efficient profiles that are not (even) NorE outcomes.

Proof. (i) For any $a \in X$, let $x^i(a)$ denote agent i's most preferred location in $(-\infty, a]$. An agent i chooses either $x^i(a) = peak^i$ or $x^i(a) = a$. Let $\Phi(a) = \max_i d(x^i(a), \sum x^i(a)/n)$. If $a \leq \min(peak^i)$, then $x^i(a) = a$ for every i and $\Phi(a) = 0$. The function Φ is continuous and strictly increasing on the interval $(\min(peak^i), \max(peak^i))$. If $\Phi(\max(peak^i)) \leq 1$, then $\langle Y = \mathbb{R}, (y^i = peak^i)\rangle$ is a maximal convex NorE. If $\Phi(\max(peak^i)) > 1$, then let b be the unique real number such that $\Phi(b) = 1$. Then, $\langle Y = (-\infty, b], (y^i = x^i(b))\rangle$ is a maximal convex NorE.

(ii) Let $\langle Y, (y^i)\rangle$ be a maximal convex NorE, and let $a = \min(y^i)$ and $b = \max(y^i)$. Denote by $L = \{j : peak^j < a\}$ the set of individuals with peaks to the left of a (those agents choose a) and similarly, denote by R the set of individuals with peaks to the right of b (those agents choose b). The remaining "middle" agents, $M = N - L - R$, choose their peaks. If $L = R = \emptyset$, then all agents are at their peaks and there is no possibility of

Pareto improvement. If $L = \emptyset$ and $R \neq \emptyset$, then the analysis is similar to that in part (a) below.

We now consider the case in which both $L \neq \emptyset$ and $R \neq \emptyset$. Suppose that (z^i) Pareto-dominates (y^i). It must be that $z^i = y^i$ for every $i \in M$, $z^i \leq y^i$ for every $i \in L$ and $z^i \geq y^i$ for every $i \in R$, with at least one strict inequality. Define $\delta_L = \sum_{i\in L}(y^i - z^i)$ and $\delta_R = \sum_{i\in R}(z^i - y^i)$.

Consider three cases:

(a) $0 = \delta_L < \delta_R$. Suppose that the agents face the NorE set $[a, b']$ instead of $Y = [a, b]$ where $b' > b$. In this case, an agent $i \in R$ would optimally choose $w^i(b') = \min\{peak^i, b'\}$ and all other agents would optimally choose $w^i(b') = y^i = z^i$. We can assume that for every $i \in R$, $z^i < peak^i$, because if not, the feasible profile $((1-\epsilon)y^i + \epsilon z^i)$ is also Pareto-improving. Let β be the number such that $\sum_{i\in R}\min\{peak^i, \beta\} = \sum_{i\in R} z^i$ and therefore $Avg(w^i(\beta)) = Avg(z^i)$. Notice that $\beta \leq \max(z^i)$. Therefore, $\beta - Avg(w^i(\beta)) \leq \max_j\{z^j - Avg(z^i)\} \leq 1$ and thus $(w^i(\beta)) \in F$. The pair $\langle [a, \beta], (w^i(\beta))\rangle$ is a larger convex NorE, contradicting the maximality of $\langle Y, (y^i)\rangle$.

(b) $0 < \delta_L = \delta_R$. In this case, $Avg(w^i) = Avg(z^i) = Avg(y^i)$. Let $x^i = b + \varepsilon/|R|$ for $i \in R$, $x^i = y^i = peak^i$ for every $i \in M$ and $x^i = a - \varepsilon/|L|$ for every $i \in L$ where $\varepsilon > 0$ is a number small enough such that it is (i) smaller than both $\min_{i\in R}(peak^i - b)$ and $\min_{i\in L}(a - peak^i)$ and (ii) smaller than both $\max_{i\in R}(z^i - b)$ and $\max_{i\in L}(a - z^i)$. Then, $\langle [a - \varepsilon/|L|, b + \varepsilon/|R|], (x^i)\rangle$ is a larger convex NorE, contradicting the maximality of $\langle Y, (y^i)\rangle$.

(c) $0 < \delta_L < \delta_R$. Let (x^i) be defined as $x^i = y^i$ for all $i \in L \cup M$ and $x^i = \lambda y^i + (1-\lambda) z^i$ for all $i \in R$ such that $\delta_R - \delta_L = \sum_{i\in R}(x^i - b) > 0$. Notice that $Avg(x^i) = Avg(z^i)$ and every x^i is between z^i and $Avg(z^i)$, therefore (x^i) is feasible because (z^i) is. The profile (x^i) Pareto-dominates (y^i) and we are back to case (a).

(iii) Let $peak^1 = peak^2 = peak^3 = -1$ and $peak^4 = 2$. The profile $y^1 = y^2 = -1$ and $y^3 = y^4 = 1$ is not a NorE outcome since agent 3 envies agent 1. However, the profile (y^i) is Pareto-efficient. To see this, note that any Pareto-improving profile (z^i) must satisfy $z^1 = z^2 = -1$, $z^3 \leq 1$ and $z^4 \geq 1$. However, $1 \geq z^4 - avg(z^i) = z^4 - \frac{-2+z^3+z^4}{4} \geq \frac{3z^4+1}{4}$ and therefore $z^4 = 1$. Then, $1 \geq z^4 - Avg(z^i) \geq 1$, which implies $avg(z^i) = 0$, and therefore $z^3 = 1$. Thus, $(z^i) = (y^i)$ and no Pareto-improvement of (y^i) is possible. □

The feasibility constraint in this economy requires that all agents make choices which are not too far from the average. These considerations are more complicated than in Example E because an agent's choice affects the average. The NorE notion is particularly appealing here since it offers a decentralized mechanism to achieve harmony without introducing any extraneous medium. The maximal normative convex equilibrium always exists and its outcomes are a strict subset of the Pareto-efficient profiles.

The equilibrium of this economy is also the limit of a natural dynamic process. Given a convex normative set $S = [a, b]$ and a profile of optimal choices from S, the bounds of the normative set change in a systematic way in response to the "pressures" caused by the distance between the extreme choices and the average point. If the distance from the furthest right choice to the average is no more than 1, then the right bound is increased at some constant rate and if the inequality is strictly reversed, then the right bound is correspondingly decreased. The same applies to the left bound. Together with continuous adjustment of the agents' choices this defines a dynamic process which ends in a maximal convex NorE from any initial starting set.

5. Discussion

This paper is a part of our grand project exploring the logic of "price-like" institutions that can bring order to "general equilibrium" environments. In Richter and Rubinstein (2015), we investigated one such institution which more closely resembles the standard competitive equilibrium notion. There, an *abstract equilibrium* consists of a public ordering and a profile of choices. The public ordering has the interpretation of a prestige ranking on the space of alternatives. The profile is required to be feasible and each agent's choice is required to be personally optimal from among the set of alternatives that are weakly less prestigious than the one assigned to him. For the main solution concept, *primitive equilibrium*, the ordering is required to be a primitive ordering, i.e. a member of a set of basic orderings that all agents use in the formation of their preferences. Thus, in equilibrium agents make choices from individual choice sets induced by a common public ranking. This is fundamentally different from the normative equilibrium setup in which all agents choose from a common normative set.

It is tempting to think about the normative equilibrium concept as a degenerate case of primitive equilibrium. This would involve defining a degenerate public ordering such that all admissible alternatives are equally bottom-ranked, all forbidden alternatives are ranked above them and all agents are assigned bottom-ranked alternatives only. However, there are several essential differences: (i) in general, such an ordering is not a primitive ordering; (ii) the constraint that the set of forbidden alternatives be minimal is not present in the primitive equilibrium framework; (iii) in a primitive equilibrium, the public ordering is required to be convex (which is analogous to requiring the set of forbidden elements to be convex) while here we require that the normative set be convex.

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