An étude in modeling the definability of equilibrium

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ABSTRACT: We analyse the object assignment model enriched with a set of orderings over the set of agents. These orderings provide potential criteria for determining the suitability of agents to be assigned to an object. A candidate for a *definable equilibrium* is an assignment of the agents to the objects and an attachment of a single criterion to each object. In equilibrium, each agent is better-suited to his assigned object than any agent who envies him, according to the criterion attached to that object. We analyze the equilibrium notion and provide some examples.

KEYWORDS: Definability, definable equilibrium, object assignment model.
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1. Introduction

An étude is a usually short instrumental musical composition of considerable difficulty, which is designed to provide practice material for perfecting a musical skill (Wikipedia (2020)). What follows is analogous to an étude –it is a short exercise in modeling that is designed to provide practice material for economic theorists.

The étude builds on the object assignment model. Consider a society consisting of equal numbers of agents and objects. The objective is to uniquely assign each agent to an object. An agent has preferences over the objects and there are no externalities. The novel feature of the model is the inclusion of a language that is a set of orderings over the set of agents. We think of the orderings as potential criteria for determining whether an agent is better-suited to an object than other agent who envies him. The main idea of the paper is that in equilibrium the assignment of an agent to an object should be supported by a statement that is expressible using the language. Specifically, a statement that can support the assignment of an agent i to a particular object, where the set of candidates to be assigned to the object is I, should be "definable" in the following manner: "Agent i is the best-suited agent in I according to the ordering $\geq_\lambda$" (where $\geq_\lambda$ is one of the language's orderings).

The approach adopted is not descriptive nor do we attempt to solve any practical economic problem. Nonetheless, the basic idea of applying different criteria in order to allocate different types of objects is a real-world phenomenon. For example, a university may allocate some seats to local students while other will be allocated according to academic abilities. Another example might be a public housing project, which assigns some apartments according to socio-economic status and others according to willingness-to-pay.

The proposed solution concept is *definable equilibrium* (D-equilibrium). A candidate for D-equilibrium is an assignment of the agents to the objects and an attachment of a single criterion to each object. In D-equilibrium, each agent is the *unique* best-suited agent within the group of agents that includes himself and every agent who envies him, according to the criterion attached to his assigned object. In other words, there is no agent who both envies another agent's assignment and is at least equally suited as him according to the criterion attached to the assigned object.

Later, we present several interpretations of D-equilibrium. For some of them, we have in mind a "decentralized economy" in which a behind-the-scenes process –an "invisible hand–attaches a criterion to each object. For other interpretations, we have in mind a central planner who justifies an assignment by declaring –possibly cynically–that the assigned agent is better-suited to the object than any other agent who prefers the object to the one he is assigned to.

Of special interest is the class of *dichotomous languages* in which each criterion partitions the agents into those who satisfy a certain property and those who do not. Given such a language, an agent i can be singled out from a group of agents I by a statement of the following form: Agent i is the only agent in I who satisfies a certain property. In this case, a D-equilibrium is an assignment of the agents to the objects and an attachment of a property to each object, such that if an agent i envies agent j, then agent j has the property attached to the object while agent i does not.

For societies with a language of strict orderings, a D-equilibrium assignment is identical to a *stable* assignment of a *marriage problem* à la Gale and Shapley (1962). Specifically, an assignment is a D-equilibrium assignment if and only if there it is a stable assignment in the associated marriage problem in which the two sides of the market are the agents and the objects, such that each agent follows his given preference relation over the objects and each object ranks agents according to its attached ordering.

In what follows, we define the notion of D-equilibrium, discuss its interpretations, suggest a refinement of the notion, and present some examples. We also prove several simple propositions on its existence and efficiency.

Our goal is to demonstrate an equilibrium concept in a social situation that does not involve trade, but rather requires that assigning any agent to an object can be supported by a certain type of statement expressed in a given language. This approach stems from the view that solution concepts in economic theory (whether they refer to markets, games or decision scenarios) should be expressed in the language of the participants. Our view is that the sensitivity of the outcome to the underlying language is a merit of a model. This is in line with Rubinstein (1978, 2000) who argues that an agent's preference relation in an economic model should be *definable* in a given langauge.

2. Definable equilibrium

2.1 The model and the D-equilibrium notion

A *society* is a tuple $\langle N, X, (\succsim^i)_{i\in N}, \mathscr{L}\rangle$. The set of *agents* is $N = \{1,\ldots,n\}$ and the set X consists of *n objects*. Each agent i has a strict preference relation $\succsim^i$, which is a complete, transitive, and anti-symmetric binary relation over X. Up to this point, the model is the familiar object assignment (housing economy) model without initial endowments. The additional feature is the **language** $\mathscr{L}$, which is a set of complete and transitive binary relations over the set of agents N. We write $\mathscr{L} = \{\geq_\lambda\}_{\lambda\in\Lambda}$ where Λ is the index set of $\mathscr{L}$'s members. The set $\mathscr{L}$ is the stock of criteria that can be used to evaluate the choice of an agent from within a **group** that is a nonempty subset of agents.

A candidate for a definable equilibrium is a pair $\langle (x^i)_{i\in N}, p\rangle$ where $(x^i)_{i\in N}$ is an **assignment** that associates each agent with an exclusive object, and $p : X \to \Lambda$ is a **labeling function** that attaches a criterion $\geq_{p(x)}\in \mathscr{L}$ to each object x. Note that the same label can be attached to different objects. The **label** $p(x)$ is the criterion used to evaluate assigning an agent to the object x. For the sake of brevity, we write (x^i) instead of $(x^i)_{i\in N}$. For each assignment (x^i), an agent j **envies** agent i if $x^i \succ^j x^j$. In a definable equilibrium, each agent i is "definable" in the following sense: "i is the best-suited agent according to the criterion $\geq_{p(x^i)}$ from among the group consisting of himself and all agents who envy him". That is, every agent who envies an agent i must be inferior to i according to the criterion $\geq_{p(x^i)}$, which is attached to the object that i is assigned to.

Definition 1 A **definable equilibrium** (D-equilibrium) is a pair $\langle (x^i), p\rangle$ where (x^i) is an assignment and $p : X \to \Lambda$ is a labeling function, such that for every pair of agents i and j if j envies i then $i >_{p(x^i)} j$.

Example A (A language with a single strict ordering) Consider a society with a language $\mathscr{L}$ consisting of a single strict ordering $\geq$ over N. Then, in the unique D-equilibrium, $\geq$ is attached to all objects and the assignment is the one obtained by running the serial dictatorship according to $\geq$.

Example B (Identical preferences) Assume that all agents share the same preferences $a_1 \succ a_2 \succ \cdots \succ a_n$. Let $\mathscr{L}$ be a set of strict orderings. For every labelling function p, there is a unique D-equilibrium with that labelling function. To see this, inductively pick a sequence of agents such that i_l is the unique maximizer of $\geq_{p(a_l)}$ in $N \setminus \{i_1, \ldots, i_{l-1}\}$. Then, the assignment of a_l to i_l combined with p is a D-equilibrium.

2.2 Interpretations of D-equilibrium

The structure of a D-equilibrium is similar to that of a competitive equilibrium. In various versions of the object assignment model, harmony is obtained in the society by means of endogenously determined prices. In contrast, in a D-equilibrium, it is not a price but an ordering over the set of agents that is attached endogenously to each object in order to achieve harmony in the society. The model does not explicitly specify initial endowments but there is an analogy implicit within the language. Thus, an agent's "wealth" is partially determined by his rankings according to the language's orderings.

The different interpretations of D-equilibrium stem from the various interpretations of the language's orderings.

(i) Criteria for suitability

According to the main interpretation described above, each ordering represents a criterion that can be used to determine whether an agent is best-suited to be assigned to an object. In a D-equilibrium, a criterion is assigned to each object and the assignment satisfies the property that each agent is the uniquely best-suited agent (according to the attached criterion) within the group of agents that includes himself and every agent who envies him. For example, the criteria might rank the agents according to income, age, or academic achievement.

(ii) Payments using different assets

The set Λ consists of different assets. The agents differ in their ability to pay as defined by the amounts of assets they own. In a D-equilibrium, an asset is attached to each object and if an agent j envies agent i, then j is not able to pay as much for x^i as i is in terms of the asset attached to x^i, namely $p(x^i)$. Proposition 1 can be thought as a formal expression of this interpretation for the case in which the assets are indivisible and an agent can own at most one unit of each asset.

(iii) Assignment of a power relation to each object

The ordering attached to an object by means of a labeling function can be thought of as a power relation used to determine the "winner" whenever some agents "compete" for the object. In equilibrium, whenever an agent j envies agent i, then agent i is stronger than agent j according to the power relation attached to the object to which i is assigned. As shown in Example A, when the language contains a single ordering, the D-equilibrium boils down to the jungle equilibrium à la Piccione and Rubinstein (2007).

3. Dichotomous languages

A **dichotomous language** consists of properties (unary relations) that an agent may or may not satisfy. Thus, each $\lambda \in \Lambda$ can be thought of as a nonempty $N_\lambda \subseteq N$ with the interpretation that it is the set of agents who satisfy the property λ. Formally, each N_λ can be identified by means of the ordering $\geq_\lambda$ which has two indifference sets: N_λ as the top set and $N \setminus N_\lambda$ as the bottom set. A dichotomous language can also be represented as a profile $(\phi^i)_{i \in N}$ where ϕ^i is a subset of propositions in Λ that are valid for agent i. In the case of a dichotomous language, there are two additional interpretations.

(iv) Endogenous formation of consideration sets

Given a society with a dichotomous language, consider a D-equilibrium such that $p(x^i) \in \phi^i$ for every agent i. Then, we can interpret the label $p(x)$ as a trigger that attracts the attention of every agent i for whom $p(x) \in \phi^i$. An agent considers only the objects with a label that attracts his attention, and then chooses his most preferred object in this "consideration set" (see Masatlioglu, Nakajima, and Ozbay, 2012). Under this interpretation, consideration sets are endogenously formed –via the labeling function–and every agent is assigned to the object he demands.

(v) Market clearing with object-specific payment methods

The set Λ can also be interpreted as a set of indivisible assets. An agent can own at most one unit of each asset. Then, ϕ^i can be thought as the set of assets held initially by agent i. A labeling function p can be thought of as a price system in which $p(x)$ is the specific asset attached to the object x. Based on this interpretation, the following proposition shows that in a D-equilibrium $\langle (x^i), p \rangle$ each agent i is assigned to the object x^i, the object he most prefers in his "budget set", which includes any object that is paid for with an asset he owns (that is $p(x) \in \phi^i$) or is a "free good" in the sense that it is assigned to an agent who is not able to pay with the required asset. That is, in a D-equilibrium an agent might be assigned to an object he cannot pay for unless there is another agent who envies him.

Proposition 1 *Let $\langle N, X, (\succsim^i), (\phi^i)\rangle$ be a society with a dichotomous language. Then, $\langle (x^i), p\rangle$ is a D-equilibrium if and only if for every $i \in N$, x^i is the $\succsim^i$-best object in the set* $\{x \mid p(x) \in \phi^i\} \cup \{x^j \mid p(x^j) \notin \phi^j\}$.

Proof. First, assume that $\langle (x^i), p\rangle$ is a D-equilibrium. Let $i \in N$ and $x = x^j$ be an object such that $x \succ^i x^i$. Now, if $p(x) \in \phi^i$ or $p(x^j) \notin \phi^j$, then agent i envies agent j and $i \geq_{p(x^j)} j$, contradicting that the pair is a D-equilibrium.

Conversely, suppose that $\langle (x^i), p\rangle$ is not a D-equilibrium. Then, there are agents i and j such that i envies j and $i \geq_{p(x^j)} j$. If $p(x^j) \notin \phi^j$, then $x^i \not\succsim^i x$ for every $x \in \{x^j \mid p(x^j) \notin \phi^j\}$. If $p(x^j) \in \phi^j$, then since $i \geq_{p(x^j)} j$ we also have $p(x^j) \in \phi^i$. Therefore, $x^i \not\succsim^i x$ for every $x \in \{x \mid p(x) \in \phi^i\}$. □

Example C (A dichotomous language with pairwise intersections) Let $N = \{1, 2, 3\}$ and $X = \{x, y, z\}$, with the dichotomous language $\phi^1 = \{\alpha, \beta\}$, $\phi^2 = \{\beta, \gamma\}$, and $\phi^3 = \{\alpha, \gamma\}$. For each agent i, let b^i be his most preferred object.

If $b^1 = b^2 = b^3 = b$, then no D-equilibrium exists since for every labeling function, there are two agents i and j for whom $p(b) \in \phi^i \cap \phi^j$. In any other case, a D-equilibrium does exist. If the most-preferred objects are distinct, then the assignment $[b^1, b^2, b^3]$ together with the labeling function $p(b^1) = \alpha$, $p(b^2) = \beta$, and $p(b^3) = \gamma$ is a D-equilibrium. If $b^1 = b^2 = x$ and $b^3 = y$, then the assignment $[x, z, y]$ with $p(x) = \alpha$, $p(y) = \alpha$, and $p(z) = \beta$ is a D-equilibrium. Those equilibrium assignments are Pareto efficient. However, Pareto dominated equilibrium assignments might exist if the agents' second-bests are distinct (with the labeling function $p(b^1) = \gamma$, $p(b^2) = \alpha$, and $p(b^3) = \beta$).

4. Existence of D-equilibrium

We say that a language is **evaluation-friendly** if for every group of agents there is an ordering for which there exists an agent who is the *unique* maximizer within the group. A dichotomous language is evaluation-friendly if for each group of agents there is a particular proposition satisfied by a unique agent in the group. We next verify that if the language is evaluation-friendly, then for every preference profile there exists a D-equilibrium with a Pareto efficient assignment.

Proposition 2 *Let $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ be a society with an evaluation-friendly $\mathscr{L}$. Then, the society has a D-equilibrium with a Pareto efficient assignment.*

Proof. Since $\mathscr{L}$ is evaluation-friendly there exists an agent, without loss of generality assume him to be agent 1, who is the $\geq_{\lambda^1}$-best-suited agent in N. Continuing with $N \setminus \{1\}$ and using a similar process, we obtain a sequence $(\lambda^i)_{i\in\{1,\dots,n\}}$ such that $i >_{\lambda^i} j$ for every $j > i$. By applying the serial dictatorship in ascending order, we obtain an assignment (x^i) such that for every agent i, x^i is $\succsim^i$-best within $X \setminus \{x^1, \dots, x^{i-1}\}$. Define $p(x^i) = \lambda^i$. To see that $\langle (x^i), p\rangle$ is a D-equilibrium, note that for every distinct $i, j \in N$, if $x^i \succ^j x^j$ then $i < j$, which implies that $i >_{\lambda^i} j$. □

Without assuming that the language is evaluation-friendly it is not guaranteed that a D-equilibrium exists for every preference profile. Notice that if every agent has the same preferences $\succsim$ and $\langle (x^i), p\rangle$ is a D-equilibrium, then the language must be evaluation-friendly. To see this, let I be a group of agents and let $i \in I$ be the agent who is assigned to the $\succsim$-best object from among the objects assigned to the members of I. It must be that $i >_{p(x^i)} j$ for every $j \in I$, and therefore the language is evaluation-friendly.

4.1 The case of languages of strict orderings

A special case of an evaluation-friendly language is a language of strict orderings over N. In such societies, a D-equilibrium assignment is identical to a *stable* assignment of a specific *marriage problem* à la Gale and Shapley (1962). Specifically, $\langle (x^i), p\rangle$ is a D-equilibrium if and only if (x^i) is a *stable* assignment in the associated marriage problem, where the two sides of the market are N and X such that each $i \in N$ has the preference relation $\succsim^i$ over X and each $x \in X$ has the preference relation $\geq_{p(x)}$ over N. This directly follows from the observation that there exists an agent j who envies i in (x^i) with $j >_{p(x^i)} i$ if and only if (j, x^i) is a *blocking pair* in the marriage problem (Rubinstein and Yıldız (2021) first observed the connection between stability and what is called *justifiable assignment*, which is a definable equilibrium assignment).

Gale and Shapley (1962) showed the existence of a stable assignment (x^i) in every marriage problem, which is obtained by running the *deferred acceptance algorithm* and that for every other stable assignment (y^i), we have $x^i \succsim^i y^i$ for every $i \in N$. Moreover, in their Theorem 3, Gale and Sotomayor (1985) showed that the assignment (x^i) is weakly Pareto efficient in the sense that is there is no assignment (z^i) (whether stable or not) such that $z^i \succ^i x^i$ for every $i \in N$. This is stated in the following proposition:

Proposition 3 *Let $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ be a society with a language $\mathscr{L}$ of strict orderings. For every labeling function p, there exists a weakly Pareto efficient assignment (x^i) such that $\langle (x^i), p\rangle$ is a D-equilibrium and (x^i) Pareto dominates every other assignment (y^i), such that $\langle (y^i), p\rangle$ is a D-equilibrium.*

4.2 The case of languages of weak orderings

In the case of a language of weak orderings, the notion of a D - equilibrium assignment is no longer equivalent to the notion of a stable assignment à la Gale and Shapley. When agent i envies agent j, the stability simply requires that the object which j is assigned to does not prefer i over j. The notion of the D-equilibrium requires that the object assigned to j *strictly* prefers j over i.

This difference can be expressed by means of the following observation: While in the case of a language of strict orderings, every labeling function is part of a D-equilibrium, In the case of a language of weak orderings a D-equilibrium does not necessarily exist (Example C). It follows that a counterpart of Proposition 3 can hold only for labelling functions that are part of a D-equilibrium. Proposition 4 presents a result along these lines. A statement, analogous to Proposition 4 is not valid regarding stable assignments in the corresponding marriage problem (see Abdulkadiroğlu, Pathak, and Roth, 2009).

Proposition 4 *Let $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ be a society and let p be a labeling function such that a D-equilibrium with p exists. Then, there exists an assignment (x^i) such that $\langle (x^i), p\rangle$ is a*

D-equilibrium and (x^i) Pareto dominates every other assignment (y^i) such that $\langle(y^i),p\rangle$ is a D-equilibrium.

Proof. Let H be the set of all assignments (x^i) such that $\langle(x^i),p\rangle$ is a D-equilibrium. By assumption, H is not empty. Therefore, there exists $(x^i)\in H$ which is Pareto efficient in H. We show that (x^i) Pareto dominates every other assignment in H.

By contradiction, suppose that there exists $(y^i)\in H$ such that (x^i) and (y^i) are not Pareto comparable. The assignment (y^i) is obtained from (x^i) by a *disjoint collection of minimally sized sets of agents*, denoted by $I_1, I_2, \dots I_K$, such that for each k the members of I_k form a trade cycle among themselves. That is, for every k there is a permutation σ_k of I_k of order $|I_k|$ such that $y^i = x^{\sigma_k(i)}$.

For each trade cycle I_k, either $(x^i)_{i\in I_k}$ dominates $(y^i)_{i\in I_k}$, (i.e., for each $i\in I_k$ we have $x^i \succsim^i y^i$) or (y^i) dominates (x^i). If not, then there must be a pair of agents $i,j\in I_k$ such that $y^j = x^i = z$, $x^i \succ^i y^i$ and $y^j \succ^j x^j$. Then, j envies i in (x^i) and i envies j in (y^i) since in that case both $i >_{p(z)} j$ and $j >_{p(z)} i$.

It follows that (y^i) is obtained from (x^i) by a set of disjoint minimally sized trade cycles $I_1, I_2, \dots I_K$ such that for each I_k either $(x^i)_{i\in I_k}$ dominates $(y^i)_{i\in I_k}$ or vice versa. For every I_k, if $(x^i)_{i\in I_k}$ dominates $(y^i)_{i\in I_k}$, then define $z^i = x^i$; if $(y^i)_{i\in I_k}$ dominates $(x^i)_{i\in I_k}$, then define $z^i = y^i$. If $\{i\}$ is a (degenerate) trade cycle, i.e. $x^i = y^i$, then define $z^i = x^i$.

Since (x^i) and (y^i) are not Pareto comparable there is at least one trade cycle in which (x^i) dominates (y^i) and at least one in which (y^i) dominates (x^i). Therefore, the assignment (z^i) Pareto dominates both (x^i) and (y^i).

To see that $\langle(z^i),p\rangle$ is a D-equilibrium, let $i,j\in N$. If j envies i in (z^i), since $z^i \in \{x^i, y^i\}$ and (z^i) Pareto dominates (x^i) and (y^i), then j envies i in either (x^i) or (y^i). Since $\langle(x^i),p\rangle$ and $\langle(y^i),p\rangle$ are D-equilibria, then $i >_{p(z^i)} j$. This contradicts that there is no assignment in H that Pareto dominates (x^i). □

The following example demonstrates the existence of a society with a dichotomous language in which there exists a labeling function p such that there is no weak Pareto efficient D-equilibrium assignment with p, while a D-equilibrium with p does exist.

Example D Let $N = \{1,2,3,4\}$ and $X = \{a,b,c,d\}$. The preference profile $(\succsim^i)$ and the language $(\geq_x)_{x\in X}$ are specified below. Let p be the labelling function such that $p(x) =$

x for every $x \in X$. The assignment $(x^i) = [b,a,d,c]$ is not weakly Pareto efficient, but $\langle (x^i), p\rangle$ is the unique D-equilibrium with labelling function p.

$\succ^1$	$\succ^2$	$\succ^3$	$\succ^4$
a	b	a	a
c	a	c	d
d	c	d	c
b	d	b	b

$\geq_a$	$\geq_b$	$\geq_c$	$\geq_d$
2	1	4	3
1,3,4	2,3,4	1,2,3	1,2,4

5. Maximal-D-equilibrium in the language of all dichotomous orderings

In this section, we suggest a refinement of the D-equilibrium concept in the context of the language of *all* dichotomous orderings. It follows from Proposition 1 that if the language contains all dichotomous orderings, then we can represent a D-equilibrium as a pair $\langle (x^i), (Y^i)_{i\in N}\rangle$ where $Y^i \subseteq X$, such that x^i is the $\succsim_i$-best object in $\{x \mid x \in Y^i\} \cup \{x^j \mid x^j \notin Y^j\}$ for every $i \in N$. The set Y^i can be thought of as the set of objects open to agent i. This relates to the definition of Y-equilibrium in Richter and Rubinstein (2020) and accordingly we define a *maximal D-equilibrium* as a D-equilibrium $\langle (x^i), (Y^i)\rangle$ such that there is no other D-equilibrium $\langle (z^i), (Z^i)\rangle$ where $Y_i \subseteq Z_i$ for every $i \in N$ with strict inclusion for at least one agent. Clearly, in a maximal D-equilibrium $\langle (x^i), (Y^i)\rangle$, we have $x^i \in Y^i$ for every $i \in N$. Proposition 5 shows that for a society with a language consisting of all dichotomous orderings, maximal D-equilibrium assignments coincide with the Pareto efficient assignments.

Proposition 5 *Let $\langle N, X, (\succsim^i), \mathscr{L}\rangle$ be a society where $\mathscr{L}$ is the langauge of all dichotomous orderings. Then, (x^i) is a maximal D-equilibrium assignment if and only if it is Pareto efficient.*

Proof. For a given Pareto efficient assignment (x^i), let N_1 be the set of agents who are assigned to their most preferred objects in X and let X_1 be the set of these objects. Since (x^i) is Pareto efficient, N_1 and X_1 are nonempty. Define $Y^i = X$ for all $i \in N_1$. Proceed inductively by defining N_k as the set of agents in $N \setminus \bigcup_{l=1}^{k-1} N_l$ who are assigned their most preferred objects in $X \setminus \bigcup_{l=1}^{k-1} X_l$. Define $Y^i = X \setminus \bigcup_{l=1}^{k-1} X_l$ for $i \in N_k$. It follows directly that $\langle (x^i),(Y^i)\rangle$ is a D-equilibrium. To see that $\langle (x^i),(Y^i)\rangle$ is also a maximal-D-equilibrium, note that if there is a D-equilibrium $\langle (z^i),(Z^i)\rangle$ with $Y^i \subseteq Z^i$ for every $i \in N$ with at least one strict inclusion, then (z^i) Pareto dominates (x^i).

Conversely, let $\langle (x^i),(Y^i)\rangle$ be a maximal D-equilibrium. If an assignment (z^i) Pareto dominates (x^i), then there is a trading cycle $(i_1,\dots,i_L)$ such that $z^{i_{l+1}} \succ^{i_l} x^{i_{l+1}}$ for every $l \in \{1,\dots,L\}$ (where $L+1$ stands for 1). Now, define $Z^i = Y^i \cup \{x^{i_{l+1}}\}$ for each $l \in \{1,\dots,L\}$ and $Z^i = Y^i$ for all other i. Then, $\langle (z^i),(Z^i)\rangle$ is a D-equilibrium, contradicting the maximality of $\langle (x^i),(Y^i)\rangle$. □

6. The connection to other non-strategic models of equilibrium "without prices"

The current model is part of the family of non-strategic models in which harmony in a society is achieved by the formation of a social entity (such as a price system) which is applied equally to all agents. We will mention here three models in this family. The first two refer to a more general definition of a society in which each agent chooses an element from a grand set of alternatives over which he has a preference relation. However, not all assignments are feasible, and therefore the society needs a mechanism that guides the agents to a feasible assignment.

Richter and Rubinstein (2015) propose a model that includes a set of possible considerations, called "primitive orderings", which are defined over the set of alternatives (rather than the set of agents, as is the case here). Harmony is established by a single primitive ordering interpreted as the prestige ranking of the alternatives. A *primitive equilibrium* is a feasible assignment in which each agent's assigned alternative is one he most prefers from the set of alternatives that are not more prestigious than the one assigned to him.

Richter and Rubinstein (2020) propose a model as in the previous paper, but without the additional element of language. Harmony is established using a single set of alternatives, called a *permissible set*, which is common to all agents. A *para-Y-equilibrium* is a feasible assignment and a permissible set such that the choice made by an agent is optimal for him within the permissible set. A *Y-equilibrium* is a para-Y-equilibrium such that there is no other para-Y-equilibrium with a larger permissible set (in the sense of inclusion). This concept is not interesting in the object assignment model with equal numbers of agents and objects, since it exists only if all agents have distinct most-preferred alternatives.

Rubinstein and Yıldız (2021) proposed a model of a society with a language (like the one presented here) and an additional priority ordering over the set of agents that is interpreted as the power relation prevailing in the society. The defined equilibrium concept, called C-equilibrium, can be thought of as a refinement of D-equilibrium. An agent can "justify" his assignment to an object only if he is the best-suited agent, according to an ordering (not object-specific) in the language, from among the set of agents who wish to be assigned to the object. Harmony is achieved by using the additional priority ordering: If there is more than one agent who can justify being assigned to an object, then the agent actually assigned to the object must have the highest priority.

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